It is well known that with each Lax operator L

$$L\psi(x,\lambda) \equiv \mathrm{i}\frac{\mathrm{d}\psi}{dx} + (q(x,t) - \lambda J)\psi(x,t,\lambda) = 0 \qquad (1)$$

related to the simple Lie algebra $\mathfrak{g}$ one can relate a hierarchy of integrable non-linear evolution equations (NLEE)

$$i\mathrm{ad}\,_J^{-1}\frac{\partial q}{\partial t} + f(\Lambda)q(x,t) = 0 \qquad (2)$$

solvable through the inverse scattering method (ISM), see the review paper [1]. Here J is a constant element of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the potential $q(x,t)$ takes values in $\mathfrak{g}/\mathfrak{h}$, $\Lambda = (\Lambda_+ + \Lambda_-)/2$ is the generating operator:

$$\Lambda_\pm X = \mathrm{ad}\,_J^{-1}\left(\mathrm{i}\frac{\mathrm{d}X}{\mathrm{d}x} + P_0([q(x), X(x)] + \mathrm{i}\left[q(x), \int_{\pm\infty}^{x} \mathrm{d}y[q(y), X(y)]\right]\right) \qquad (3)$$

and $P_0\cdot\ = \mathrm{ad}\,_J^{-1}\mathrm{ad}\,_J\cdot$. A most effective method to solve the inverse scattering problem for L is based on a local Riemann-Hilbert problem [2]

$$\xi^+(x,\lambda) = \xi^-(x,\lambda)G(x,\lambda), \qquad \lambda \in \Gamma \qquad (4)$$

on the contour Γ in the complex λ-plane; in fact Γ defines the continuous spectrum of L and $G(x,\lambda)$ defines the spectral data of L. We show that to each RHP one can relate generalized exponentials and a recursion operator Λ which play fundamental role in the theory of the NLEE.
We review a class of RHP with additional symmetry conditions which can be used to solve NLEE with additional reductions. This method can be applied to a wide class of Lax operators with vanishing and non-vanishing boundary conditions. For the special case of $\mathbb{Z}_h$ reduction group [3] where h is the Coxeter number of $\mathfrak{g}$ we derive the action-angle variables of the corresponding NLEE, which include the two-dimensional Toda field theories.

References

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