Geometry of the Shilov Boundary of Bounded Symmetric Domains

Jean-Louis Clerc

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The first part will be devoted to a presentation of several approaches to bounded symmetric domains.

- Hermitian symmetric spaces of the noncompact type

  In this perspective, $M = G/K$ is a Riemannian symmetric space of the noncompact type, together with an appropriate complex structure. If $\mathfrak{p}$ is the tangent space at the origin $o = eK$, then its complexification $\mathfrak{p}_{\mathbb{C}}$ splits as $\mathfrak{p}_+ \oplus \mathfrak{p}_-$ (holomorphic/antiholomorphic tangent space). One major result is the Harish Chandra imbbedding : M can be analytically embedded as the unit ball for some complex norm in $\mathfrak{p}_+$.

- Bounded symmetric domains

  Here one considers a bounded domain $\mathcal{D}$ in some complex vector space $\mathbb{V} \simeq \mathbb{C}^N$, such that, for each point $z \in \mathcal{D}$ there exists an involutive holomorphic transformation s_z of $\mathcal{D}$ which admits z as isolated fixed point. The introduction of the *Bergmann metric* gives a connection with the first point of view.

- Positive Hermitian Jordan triple systems

  One analyses the holomorphic vector fields in $\mathcal{D}$ associated to elements of $\mathfrak{p}$: they are quadratic vector fields. The fact that $\mathfrak{p}$ is closed under triple bracket ($[[\mathfrak{p}, \mathfrak{p}], \mathfrak{p}] \subset \mathfrak{p}$) is reflected in the *Jordan triple product* on $\mathbb{V}$. It is then possible to develop a spectral analysis in $\mathbb{V}$, and in particular, to define a *spectral norm* on $\mathbb{V}$. The domain $\mathcal{D}$ turns out to be the unit ball for this norm. Conversely, one may start with a positive Hermitian Jordan triple, use the Koecher-Kantor-Tits construction to recover the group G and the domain $\mathcal{D}$.

- Yet another realization of the domain will be useful : there is a notion of *Cayley transform* (generalizing the transformation from the unit disc in $\mathbb{C}$ to the upper half-plane), which maps the bounded domain $\mathcal{D}$ into a *Siegel domain* (in general of type II). It makes very clear the distinction between *tube-type domains* (the Siegel domain is a tube over a symmetric cone) and the *non tube-type domains.*

These various presentations allow a fine description of the group G of analytic transformations of the domain $\mathcal{D}$, together with a very explicit description of the boundary of $\mathcal{D}$ in $\mathbb{V}$ as a union of G-orbits.

The second part will be concerned with a specific orbit in the boundary, namely the *Shilov boundary* S. It is the only closed G-orbit, and it is a projective variety, as $S \simeq G/P$, where P is a parabolic subgroup of G. We will prove the following results.

- G has a finite number of orbits in $S \times S$: a consequence of Bruhat's theory, which can be obtained by convex analysis.

- Construction of an invariant on $S \times S \times S$: if z_1, z_2, z_3 are three points in $\mathcal{D}$, one can consider the geodesic triangle $T(z_1, z_2, z_3)$ and define without ambiguity its *Kåhler (or symplectic) area* $A(z_1, z_2, z_3)$, and it is possible to find an explicit expression for it in terms of the *Bergman kernel* of $\mathcal{D}$. Letting z_1 (resp. z_2, z_3) tend to some point σ_1 (resp. σ_2, σ_3) in S, one shows that the area $A(z_1, z_2, z_3)$ has a limit, thus defining an invariant $\iota(\sigma_1, \sigma_2, \sigma_3)$ on $S \times S \times S$. Moreover by construction, the invariant ι satisfies a cocycle relation. If the domain $\mathcal{D}$ happens to be the unit ball is $\mathbb{C}^2$ for the usual inner product, this coincides with an invariant constructed in 1932 by E. Cartan.

- When $\mathcal{D}$ is of tube-type (and only in this case), there is a finite number of G-orbits in $S \times S \times S$, and the invariant takes only integral values (after normalization). If $\mathcal{D}$ happened to be the *Siegel disc*, then S is diffeomorphic to the *Lagrangian manifold* and the invariant coincides with the *Maslov index*. A classification of the G-orbits in $S \times S \times S$ is obtained.

- In the tube-type situation, it is possible to construct a *Maslov index for paths* on S, generalizing the classical approach to the Maslov index, due to Maslov, Arnold, Leray, Souriau, and to make connection with the previous triple index.

Some connected questions and further problems could be considered: bounded cohomology and rigidity for locally symmetric Hermitian spaces, metaplectic representation, extension to the infinite dimensional setting, ...

Selective Bibliography

On Hermitian symmetric spaces

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The first is a classical presentation, using Lie groups and Riemannian symmetric space as background. The second uses mostly Jordan triple systems. The third is a mixed approach. The fourth presentation is based on examples.

The results in the second part are mainly due to the author and collaborators.

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