The Classical Magnetized Kepler Problems in Higher Dimensions

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Abstract

It is demonstrated that, for the recently introduced classical magnetized Kepler problems in dimension $2k+1$, the non-colliding orbits in the "external configuration space" $\mathbb{R}^{2k+1}\setminus\{\mathbf{0}\}$ are all conics, moreover, a conic orbit is an ellipse, a parabola, and a branch of a hyperbola according as the total energy is negative, zero, and positive. It is also demonstrated that the Lie group $\mathrm{SO}^+(1,2k+1)\times\mathbb{R}_+$ acts transitively on both the set of oriented elliptic orbits and the set of oriented parabolic orbits.

Introduction

The Kepler problem for planetary motion is a two-body dynamic problem with an attractive force obeying the inverse square law. Mathematically it can be reduced to the one-body dynamic problem with the equation of motion

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3}, \tag{0.1}$$

where $\mathbf{r}$ is a function of t taking value in $\mathbb{R}^3_* := \mathbb{R}^3\setminus\{\mathbf{0}\}$, $\mathbf{r}''$ is the acceleration vector and r is the length of $\mathbf{r}$.

A surprising discovery due to D. Zwanziger [1] and to H. McIntosh and A. Cisneros [2] independently in the late 1960s is that there exist magnetized companions for the Kepler problem. These extra dynamic problems plus the Kepler problem, referred to as *MICZ-Kepler problems*, are indexed by the magnetic charge μ, with $\mu = 0$ for the Kepler problem. The parameter μ can take any real number at the classical mechanics level, a half of any integer at the quantum mechanics level.

Since the Kepler problem has long been known to exist in all dimensions, one naturally wonders whether there are magnetized Kepler problems in higher dimensions. By realizing [3] that the MICZ-Kepler problems are the $\mathrm{U}(1)$-symmetric reductions of the isotropic oscillator in space $\mathbb{R}^4=\mathbb{C}^2$, T. Iwai [4] obtained the magnetized Kepler problems in dimension five (referred to as the $\mathrm{SU}(2)$-Kepler problems), as the $\mathrm{SU}(2)$-symmetric reductions of the isotropic oscillator in space $\mathbb{R}^8=\mathbb{H}^2$. For quite a while the magnetized Kepler problems were thought to exist only in dimensions three, five and (possibly) nine, corresponding to the division algebras $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ respectively.

This talk is about the recently introduced [5] classical magnetized Kepler problems in dimension $2k+1$ ($k=1,2,\cdots$) and a description of their non-colliding orbits. These magnetized Kepler problems are the MICZ-Kepler problems in dimension three and Iwai's $\mathrm{SU}(2)$-Kepler problems in dimension five.

1 Equation of Motion

The equation of motion for the magnetized Kepler problems in odd dimension $n \geq 3$ is the n-dimensional analogue of the equation of motion

$$\mathbf{r}'' = -\frac{\mathbf{r}}{r^3} + \mu^2 \frac{\mathbf{r}}{r^4} - \mathbf{r}' \times \mu \frac{\mathbf{r}}{r^3} \tag{1.1}$$

for the MICZ-Kepler problems [1, 2], where the parameter μ is the magnetic charge.

It turns out that the high dimensional analogue of Eq. (1.1) is far from straightforward. The reason is that, if $k > 1$, instead of governing motions on $\mathbb{R}^{2k+1}_* := \mathbb{R}^{2k+1} \setminus \{\mathbf{0}\}$, the equation of motion governs motions on a manifold P_μ which fibers over $\mathbb{R}^{2k+1}_*$.

To describe the fiber bundle $P_\mu \to \mathbb{R}^{2k+1}_*$, we let $G = \mathrm{SO}(2k)$ and consider the canonical principal G-bundle over S^{2k}:

$$\begin{array}{c} \mathrm{SO}(2k+1) \\ \downarrow \\ \mathrm{S}^{2k}. \end{array}$$

This bundle comes with a natural connection

$$\omega(g) := \mathrm{Pr}_{\mathfrak{so}(2k)}\left(g^{-1}dg\right),$$

where $g^{-1}dg$ is the Maurer-Cartan form for $\mathrm{SO}(2k+1)$, so it is an $\mathfrak{so}(2k+1)$-valued differential one form on $\mathrm{SO}(2k+1)$, and $\mathrm{Pr}_{\mathfrak{so}(2k)}$ denotes the orthogonal projection of $\mathfrak{so}(2k+1)$ onto $\mathfrak{g} := \mathfrak{so}(2k)$.

Under the map

$$\begin{array}{rcl} \pi : \mathbb{R}^{2k+1}_* & \to & \mathrm{S}^{2k} \\ \mathbf{r} & \mapsto & \frac{\mathbf{r}}{r}, \end{array} \tag{1.2}$$

the above bundle and connection are pulled back to a principal G-bundle

$$\begin{array}{l} P \\ \downarrow \\ X := \mathbb{R}^{2k+1}_* \end{array} \tag{1.3}$$

with a connection which is usually referred to as the generalized Dirac monopole [6]. Now

$$P_\mu \to \mathbb{R}^{2k+1}_*$$

is the associated fiber bundle with fiber being a certain co-adjoint orbit $\mathcal{O}_\mu$ of G, the so-called magnetic orbit with magnetic charge $\mu \in \mathbb{R}$.

To describe $\mathcal{O}_\mu$, let us use γ_{ab} ($1 \leq a, b \leq 2k$) to denote the element of $i\mathfrak{g}$ such that in the defining representation of $\mathfrak{g}$, $M_{a,b} := i\gamma_{ab}$ is represented by the skew-symmetric real symmetric matrix whose ab-entry is -1, ba entry is 1, and all other entries are 0. For the invariant metric $(,)$ on $\mathfrak{g}$, we take the one such that $M_{a,b}$ ($1 \leq a < b \leq 2k$) form an orthonormal basis for $\mathfrak{g}$. Via this invariant metric, one can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, hence co-adjoint orbits with adjoint orbits. By definition, for any $\mu \in \mathbb{R}$,

$$\mathcal{O}_\mu := \mathrm{SO}(2k) \cdot \frac{1}{\sqrt{k}}(|\mu| M_{1,2} + \cdots + |\mu| M_{2k-3,2k-2} + \mu M_{2k-1,2k}). \tag{1.4}$$

It is easy to see that $\mathcal{O}_\mu = \{0\}$ if $\mu = 0$ and is diffeomorphic to $\frac{\mathrm{SO}(2k)}{\mathrm{U}(k)}$ if $\mu \neq 0$.

We are now ready to describe the equation of motion for the magnetized Kepler problem in dimension $2k+1$. Let $\mathbf{r}$: $\mathbb{R} \to X$ be a smooth map, and ξ be a smooth lifting of $\mathbf{r}$:

$$\begin{array}{ccc} & & P_\mu \\ & \xi \nearrow & \downarrow \\ \mathbb{R} & \xrightarrow{\mathbf{r}} & X \end{array} \tag{1.5}$$

Let Ad_P be the adjoint bundle $P \times_G \mathfrak{g} \to X$, d_∇ be the canonical connection, i.e., the generalized Dirac monopole on $\mathbb{R}^{2k+1}_* = X$. Then the curvature $\Omega := d^2_\nabla$ is a smooth section of the vector bundle $\wedge^2 T^*X \otimes Ad_P$. (With a trivialization of $P \to X$, locally Ω can be represented by $\frac{1}{2}\sqrt{-1}F_{jk}\, dx^j \wedge dx^k$.) The equation of motion is

$$\boxed{\begin{cases} \mathbf{r}'' = -\frac{\mathbf{r}}{r^3} + \frac{\mu^2}{k}\frac{\mathbf{r}}{r^4} + (\xi, \mathbf{r}' \lrcorner \Omega), \\ \\ \frac{D}{dt}\xi = 0. \end{cases}} \tag{1.6}$$

Here $\frac{D}{dt}\xi$ is the covariant derivative of ξ, $(,)$ refers to the inner product on the fiber of the adjoint bundle coming from the invariant inner product on $\mathfrak{g}$, and 2-forms are identified with 2-vectors via the standard euclidean structure of $\mathbb{R}^{2k+1}$. Eq. (1.6) defines a super integrable model, referred to as the **classical Kepler problem with magnetic charge μ in dimension** $2k+1$, which generalize the classical MICZ-Kepler problem. Indeed, in dimension 3, the bundle is topological trivial, $\xi = \mu M_{12}$, and $\Omega = \frac{*(\sum_{i=1}^3 x^i\, dx^i)}{r^3} M_{12}$, then Eq. (1.6) reduces to Eq. (1.1), i.e., the equation of motion for the MICZ-Kepler problem with magnetic charge μ. In dimension 5, it is essentially Iwai's $\mathrm{SU}(2)$-Kepler problem, cf. Ref. [4].

The equation of motion appears to be mysterious, but it doesn't. As demonstrated in Ref. [5], with a key input from the work [7] of Sternberg, Weinstein, and Montgomery, it emerges naturally from the notion of universal Kepler problem in Ref. [8]. As a side remark, we would like to point out that the quantum magnetized Kepler problems [9] were obtained much earlier.

2 Orbits

While the orbits for the magnetized Kepler problems in dimension three have been thoroughly studied from the very beginning [2], in higher dimensions, in view of the fact that the equation of motion is a bit more sophisticated, one might expect that the orbits are a bit more sophisticated, too. That is probably the reason why the orbits for Iwai's $\mathrm{SU}(2)$-Kepler problems were never investigated in Ref. [4] and the subsequent papers [10].

Since the bundle $P_\mu \to X$ has a canonical connection, an orbit inside P_μ is the the lifting (via the canonical connection) of its projection onto X. So it suffices to understand the projection of the orbits onto X. Recently we [11] found that the projection curve is either a part of straight line (colliding orbit) or a conic (non-colliding orbit).

3 Outlook

An interesting further study is to work out the geometric quantization of the classical models introduced here so that one can reproduce the quantum models introduced in Ref. [9]. We expect that the earlier work carried out by I. Mladenov and V. Tsanov [12] for the Kepler problems in higher dimensions or the MICZ Kepler problems shall serve a good guidance in such a study.

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