# On the Geometry of Pseudo-Euclidean Spaces

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Let $\mathbb{R}^{p,q}$ be the pseudo-Euclidean space with indefinite inner product of signature $(p, q)$, $p, q \in \mathbb{N}$, and let $SO(p, q)$ be the group of proper pseudo-orthogonal transformations, that is, linear transformations of $\mathbb{R}^{p,q}$ that leave the inner product invariant and that can be reached continuously from the identity transformation of $\mathbb{R}^{p,q}$ [1]. The group $SO(1, 3)$ is the proper Lorentz transformation group of special relativity theory. Hence, the group $SO(p, q)$ is called the (generalized) Lorentz transformation group of order $(p, q)$. The author's parametrization of the Lorentz group $SO(1, n)$, $n \in \mathbb{N}$, in 1988 [2] led to the discovery that a parameter space of the proper Lorentz transformation group possesses a novel, nonassociative group-like algebraic structure that became known as a *gyrogroup*. The non-associativity of gyrogroups is controlled by special automorphisms called *gyrations*. The gyration, in turn, is a mathematical abstraction of the relativistic effect known as *Thomas precession*.

The nongroup parameter space of $SO(1, n)$ turns out to form a gyrogroup. Gyrogroups give rise to gyrovector spaces which, in turn, form the setting for $n$-dimensional analytic hyperbolic geometry, just as vector spaces form the setting for $n$-dimensional analytic Euclidean geometry [3, 4, 5, 6, 7, 8, 9, 10]. In this exposition we present a parametrization of the Lorentz group $SO(p, q)$ for any $p, q \in \mathbb{N}$, along with its resulting novel theory that extends the theory of gyrogroups into the so called *bi-gyrogroups*.

## References

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