The Kähler Calculus

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ABSTRACT

This lecture course is meant to initiate participants in the use of the Kähler calculus (KC), which supersedes Cartan's exterior calculus. It is based on Kähler's 1960-62 papers, written in German and for which neither textbook nor English translation exists. Its great potential appears to have been overlooked.

The underlying algebra of KC is Clifford algebra (CA), or true Euclidean algebra for arbitrary dimension. Just try to get a vector product in arbitrary dimension to realize how artificial vector algebra is. Dirac's is a poor version of CA, and the Dirac matrices themselves are irrelevant; only the structure of that set of matrices matters.

Given all that, it is clear that learning KC is no easy task for most physicists and even mathematicians. This lecture course will first deal with CA and exterior calculus in order to bridge the gap existing between the usual mathematical methods and the KC, thus effecting a smooth transition to the latter. The course will be as interactive as possible and customized in situ to meet the needs of the audience.

From that platform, we shall enter the main topics of the Kähler calculus that are of interest for the applications. Those topics are: the concept of differential forms in Kähler, not only scalar-valued but also tangent and cotangent valued; his approach to Lie differentiation; his unifying concept of differentiation for all those types of differential forms; his concepts of harmonic and strict harmonic differentials; his concept and properties of constant differentials; his Dirac-like equations; his approach to the problem of symmetry in solutions of such equations.

As applications, we shall deal with the emergence in his work of the concept of spin at par rather as attachment to the concept of orbital angular momentum, and the emergence of two types of charge from his Dirac-like equation with electromagnetic coupling. To any extent possible, we shall also deal with applications developed by this lecturer to real and complex analysis, to theoretical high energy physics and to the foundations of quantum mechanics.

PROGRAM

I. EXTERIOR CALCULUS

1. Motivation and connection with the vector calculus
2. The Cartan-Kähler-Rudin concept of differential forms
3. Mathematical and physical examples of differential forms
4. Exterior algebra
5. Exterior differentiation, interpretation and properties
6. Significant operators and commutation-anticommutation relations
7. Stokes theorem
8. Summary of applications: structure of Euclidean space, potentials of differential forms (Poincaré's inverse theorem), computations in spacetime (specifically with Maxwell's equations) and use of orthonormal bases of differential forms.

II. CLIFFORD ALGEBRA

1. Motivation and connection with widely known algebra
2. Several ways to approach Clifford algebra
3. Products where one factor is a vector
4. Subspaces of a Clifford algebra
5. General exterior, interior and mixed products
6. Hodge duality and its applications
7. Superefficient treatment of rotations, including spinors.

III. KÄHLER CALCULUS

1. Motivation and relation to "exterior algebra cum Hodge duality"
2. Kähler's unified exterior-interior differentiation
3. Differentiation of products
4. Harmonic and strict harmonic differential forms
5. Constant differentials, idempotents and the ideals they define
6. Kähler-Dirac (KD) equations
7. Spinors for spacetime symmetry

IV. KÄHLER'S OWN APPLICATIONS OF HIS CALCULUS

1. Kähler's version of quantum mechanics
2. Kähler's conceptually simple concept of Lie differentiation (LD)
3. Angular momentum components from LD, spin included

4. Concept of operator-valued total angular momentum differential form
5. Sophisticated treatment of spacetime symmetries
6. Emergence of two types of charge.
7. Kähler's efficient solving for the fine structure of the H atom

V. OTHER BASIC APPLICATIONS OF KÄHLER'S CALCULUS

1. Application to the calculus of complex variable
2. Application to relativistic quantum mechanics
3. Clifford-valued differential forms
4. Application to the foundations of quantum mechanics
5. Unification of Helmoltz' and Hodge's theorems
6. Application to high energy physics
7. Brief idea of specialized applications by other authors

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