FANTASTIC SYMMETRIES AND WHERE TO FIND THEM

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Abstract

The Lie symmetry method, along with variational principles and their various generalizations have become a standard but indispensable tool in the geometrical and structural analysis of dynamical systems, the study of its conservation or evolution laws, as well as problems of integrability and superintegrability. Beyond its original conception, symmetries of differential equations often present puzzling features that have given rise to further generalizations or developments of alternative methods, resulting in some unexpected applications.

These lectures will be concerned with Lie and Noether symmetries of ordinary and partial differential equations, their fundamental properties and features, as well as their various applications to problems in both Classical and Quantum Mechanics.

As tersely stated in [1]: *We are embarked on a new stage of exploration of fundamental laws of nature, a voyage guided largely by the search for and the discovery of new symmetries.*

Outline

Lecture 1: Lie and Noether Symmetries of Differential Equations.

- Lie symmetry method: a brief review [2].
- The complete symmetry group [3], [4], [5].
- Taming chaotic systems by means of Lie symmetry method: an example [6].
- Lagrangian equations and Noether's first theorem [7].
- Missed Lie and Noether symmetries: examples [5, 8].

Lecture 2: Jacobi Last Multiplier and Its Properties

- The Jacobi last multiplier and its connection to first integrals and Lagrangians [9–12]
- The Jacobi last multiplier and its connection to Lie symmetries [13–15].
- Nonlocal symmetries as hidden symmetries: the role of Jacobi last multiplier [16–18].

Lecture 3: Noether Symmetries I.

- Hidden linearity of nonlinear equations [5, 19, 20]
- Inequivalent Lagrangians and their Noether symmetries [14, 21].
- Quantization of classical mechanics problem by means of the preservation of Noether symmetries: the method [21–23]
- Quantization of classical mechanics problem by means of the preservation of Noether symmetries: examples [5, 24–28].

Lecture 4: Noether Symmetries II.

- The Ostrogradsky's method for constructing Lagrangians for equations of order greater than two [9, 29].
- Ghost-free quantization via symmetry preservation: Pais–Uhlenbeck model and its "ghosts" [30, 31].
- Ghost-free quantization via symmetry preservation: Higgs model with a complex ghost pair [22].

Lecture 5: Heir-Equations for Evolution Systems.

- Classical Lie symmetries of partial differential equations [32]: an example [33].
- Nonclassical symmetries of partial differential equations [34]: an example [35].
- Iteration of the nonclassical symmetry method: heir-equations [36].
- Conditional Lie-Bäcklund symmetries and heir-equations [37], [38].
- Nonclassical symmetries as special solutions of heir-equations [39].
- More symmetry solutions than expected with heir-equations [40, 41].

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