Generalized Inverses and Applications

Dragan S. Djordjević

Department of Mathematics and Informatics
Faculty of Sciences and Mathematics
University of Niš, Višegradska 33, 18000 Niš, Serbia
E-Mail: dragandjordjevic70@gmail.com

We will present results on matrix generalized inverses with some applications. It is well-known that a square complex matrix A is invertible if and only if $\det(A) \neq 0$. In this case $AA^{-1} = A^{-1}A = I$, where I is the identity matrix of the appropriate dimension. However, in many situations the considerd matrix is not invertible, or not square.
There are several ways to define a generalized inverse of a general complex matrix. If the matrix $A \in \mathbb{C}^{m\times n}$ is given, then the Moore-Penrose of A is the matrix $A^\dagger \in \mathbb{C}^{n\times m}$ satisfying the conditions

$$(1)\ AA^\dagger A = A, \quad (2)\ A^\dagger AA^\dagger = A^\dagger, \quad (3)\ (AA^\dagger)^* = AA^\dagger, \quad (4)\ (A^\dagger A)^* = A^\dagger A.$$

The Moore-Penrose inverse of a complex matrix always exists and it is unique. If A is square and invertible, then $A^\dagger = A^{-1}$, i.e. in this case the ordinary and the Moore-Penrose inverse of A coincide.
Furthermore, if a vector $b \in \mathbb{C}^n$ is also given, we want to solve the equation $Ax = b$. In the most general situation, the obvious candidate for a solution is $x = A^\dagger b$. Such x minimizes the norm $\|Ax - b\|$ and such x has the minimum norm $\|x\|$ among all other minimizers. This is the approximation property of the Moore-Penrose inverse. The Moore-Penrose inverse is also connected with the Singular value decomposition of a complex matrix and linear regression.

Now, is $S \subset \{1, 2, 3, 4\}$, we can define a generalized inverse $A^{(S)}$ of A, such that only equations from S are satisfied. Thus, we obtain the following classes of generalized inverses: $\{1\}$-inverses, $\{2\}$-inverses, $\{1, 2\}$-inverses, etc. This clases are also important in many applications.
If we consider a square matrix $A \in \mathbb{C}^{m\times m}$, then the commutativity conditions can be involved. For previously given A, the Drazin inverse $A^D \in \mathbb{C}^{m\times m}$ satisfy the conditions

$$(2)\ A^D AA^D = A^D, \qquad (5) AA^D = A^D A, \qquad A^{n+1}A^D = A^n$$

for some $n \in \mathbb{N}_0$. The least such n is called the Drazin index of A, denoted by $\mathrm{ind}(A)$. The Drazin inverse A^D of $A \in \mathbb{C}^{m\times m}$ is unique and always exists. We have $\mathrm{ind}(A) = 0$ if and only if A is invertible, and in this case $A^D = A^{-1}$.

If $\text{ind}(A) \leq 1$, then the multiplicative semigroup generated by $\{A, A^D\}$ is actually a group with the unit AA^D. For this special reason, if $\text{ind}(A) \leq 1$, then $A^D = A^\#$ is the group inverse of A. Thus, the group inverse of A, $\text{ind}(A) \leq 1$, is the unique matrix $A^\#$ satisfying

$$(1)\ AA^\# A = A, \qquad (2)\ A^D AA^D = A^D, \qquad (5) AA^D = A^D A.$$

If J is the Jordan normal form of $A \in \mathbb{C}^{m\times n}$, and $J(0)$ is the appropriate Jordan block corresponding to the eigenvalue $\{0\}$ (if $J(0)$ exists), then $\text{ind}(A)$ is the index of nilpotencity of $J(0)$. Of course, $J(0)$ does not exists if and only if $\text{ind}(A) = 0$.

Drazin inverse have applications in finite Markov chains and signular systems of differential equations.

Lectures will be given accorging to the references and with the following containts.

Lecture 1. Moore-Penrose inverse, $\{1\}$-, $\{2\}$- and $\{1,2\}$-generalized inverses and related topics.

Lecture 2. Drazin inverse and group inverse.

Lecture 3. Computation of generalized inverses and solving matrix equations.

Lecture 4 and Lecture 5. Applications of generalized inverses: parallel sums and shorted matrices, linear regression, finite Markov chains, singular systems of differential equations.

References

[1] A. Ben-Israel, T.N.E. Greville, *Generalized Inverses, Theory and Applications*, Springer, New York 2003.

[2] D. S. Djordjević, V. Rakočević, *Lectures on Generalized Inverses*, University of Niš, Faculty of Sciences and Mathematics, Niš 2008.

[3] G. Wang, Y. Wei, S. Qiao, *Generalized Inverses: Theory and Computations*, Springer, Beijing 2018.