Lie Systems and Geometric Structures

Cristina Sardon

Instituto de Ciencias Matematicas, Consejo Superior de Investigaciones Cientifica, Spain
E-mail: cristinasardon@icmat.es

1. Abstract

In short, a Lie system is a system of ODEs describing the integral curves of a t-dependent vector field taking values in a finite-dimensional Lie algebra V of vector fields: a so called Vessiot–Guldberg Lie algebra (VG) for the Lie system. Equivalently, a Lie system is a system of ODEs admitting a superposition principle or superposition rule, i.e., a map allowing us to express the general solution of the system of ODEs in terms of a family of particular solutions and a set of constants related to initial conditions.

From the point of view of their applications, Lie systems play a relevant role in Physics, Mathematics, and other fields of research (see [7, 3] which provide more than 400 references on Lie systems and related topics). Some of the most representative Lie systems are the Riccati equations and their multiple derived versions (matrix Riccati equations, projective Riccati equations, etc.). These latter differential equations frequently appear in Cosmology, Financial Mathematics, Control Theory and other disciplines.

Very surprisingly, it was proven that the found new Lie systems admitted Vessiot–Guldberg algebras of Hamiltonian vector fields with respect to some symplectic or Poisson structure[1, 2]. This led to the study of an important particular case of Lie systems, the so called LieHamilton systems. LieHamilton systems are Lie systems that admit Vessiot–Guldberg Lie algebras of Hamiltonian vector fields with respect to a Poisson structure. Lie-Hamilton systems possess a time-dependent Hamiltonian given by a curve in a finite-dimensional Lie algebra of functions with respect to a Poisson bracket related with the Poisson structure, a Lie–Hamilton algebra. But not every Lie system is a LieHamilton system. The so called no-go theorem shows a very general condition that helps us identify when a Lie system is not a Lie–Hamilton one.

Due to the interest of Lie–Hamilton systems, we will classify all the Vessiot–Guldberg Lie algebras of Hamiltonian vector fields on the plane with respect to a Poisson structure and have analyzed their properties. We obtained twelve different nondiffeomorphic classes of Lie algebras of Hamiltonian vector fields. Our classification permitted the identification of physical and mathematical properties of the Lie–Hamilton systems on the plane. For example, it helped in the understanding of trigonometric Lie systems appearing in the study of integrable systems, diffusion equations, SmorodinskyWinternitz oscillators...

Another notorious example of Lie system compatible with another geometrical structure is that of Dirac–Lie systems [4]. These are Lie systems that possess a Vessiot Guldberg Lie algebra of Hamiltonian vector fields with respect to a Dirac structure. As Dirac structures describe Poisson structures as particular cases, Dirac–Lie systems cover LieHamilton systems. The last type of geometric background that will be employed in this course is the Jacobi as another generalization of Poisson manifolds [5].

To finish, some recent papers have been devoted to apply the theory of Lie systems to Quantum Mechanics. As a result it has been proved that such a theory can be used to treat some types of Schödinger equations, these will be the so called quantum Lie systems. One of the fundamental properties found is that quantum Lie systems can be investigated by means of equations in a Lie group. Through this equation we can analyse the properties of the associated Schrdinger equation, i.e. the type of the associated Lie group allows us to know whether a Schödinger equation can be integrated [6]. We will briefly see the most

important characteristic of all the aforementioned (structure)-Lie systems. Let us give a possible structure of the lectures.

2. Outline

- **Lecture 1: Introduction to Lie systems**
  - Historical introduction
  - Lie systems and superposition rules. Frobenius theorem and particular solutions. An algorithm to derive superposition rules
  - Lie systems on Lie groups
  - Application: the second–order Riccati equation
- **Lecture 2: Lie Hamilton–systems**
  - On the necessity of Lie–Hamilton systems
  - Lie–Hamilton structures
  - Derivation of superposition rules through the coalgebra method for Lie-Hamilton systems.
  - Applications: a superposition rule for Riccati equations, Kummer–Schwarz equations in Hamiltonian form, Smorodinsky–Winternitz oscillators
- **Lecture 3: Classification of Lie Hamilton systems**
  - General definitions and properties
  - Lie–Hamilton algebras
  - Local classification on the plane
  - Applications in $\mathfrak{sl}(2,\mathbb{R})$, Lie–Hamilton biological systems, two-photon Lie–Hamilton systems, etc.
- **Lecture 4: Dirac–Lie systems**
  - Dirac–Lie Hamiltonians
  - Superposition rules for Dirac–Lie systems
  - Bi-Dirac Lie systems
  - Application: Schwarztian-KdV equations
- **Lecture 5: Quantum Lie systems**
  - Quantum Lie systems
  - Spin Hamiltonians
  - Lie structure of an equation of transformation of Lie systems
  - Integrability conditions for $SU(2)$ Schrödinger equations
  - Applications in Physics

References

[1] A. Ballesteros, A. Blasco, F.J. Herranz, J. de Lucas, C. Sardon, Lie–Hamilton systems on the plane: theory, classification and applications, J. Differential Equations **258**, 2873–2907 (2015).

[2] A. Ballesteros, J.F. Carinena, F.J. Herranz J. de Lucas, C. Sardón, From constants of motion to superposition rules of LieHamilton systems, J. Phys. A **46**, 285203 (2013).

[3] J.F. Cariñena, J. de Lucas, Lie systems: theory, generalisations, and applications, Dissertationes Math. (Rozprawy Mat.) **479**, 11–62 (2011).

[4] J.F. Cariñena, J. Grabowski, J. de Lucas, C. Sardón, Dirac–Lie systems and Schwarzian equations, J. Differential Equations **257**, 2303–2340 (2014).

[5] F.J. Herranz, J. de Lucas, C. Sardón, Jacobi–Lie systems: fundamentals and low-dimensional classification, Proceedings AIMS, 2014, Madrid. arXiv:1412.0300, (2014).

[6] J. de Lucas, C. Sardón, Quantum Lie systems and integrability conditions, International Journal of Geometric Methods in Modern Physics **6**(8) (2009).

[7] C. Sardón, Lie systems, Lie symmetries and reciprocal transformations. Arxiv:1508.00726. (2015)