Topological Aspects of Classical and Quantum Gauge Theories

Siye Wu

National Tsing Hua University, Hsinchu, Taiwan

Abstract

Interaction between geometry/topology and gauge theory started in the 1970s. Examples of fruitful developments that benefit both subjects are instanton and 4-manifolds, anomaly and index theory, in the 1980s. Subsequently, gauge theory has witnessed startling progress in directions such as electromagnetic duality, its stringy origin and, gauge/gravity duality, resulting in a much profound involvement of mathematics and physics in each other. The planned Lectures revisit some of the original fundamental problems of gauge theory with the hindsight of these novel experiences and techniques.

We start Lecture 1 by explaining why classical electromagnetism is a gauge theory using the well known Aharonov–Bohm effect and the quantisation of magnetic fluxes through a superconducting ring. We then consider non-Abelian gauge theory and briefly account for its role in the standard model of particle physics. We end with some classical solutions of the Yang-Mills equations that are called monopoles and instantons.

In Lecture 2, we describe the characteristic classes of a principal fibre bundle that classify its non-trivial topology [4]. The first non-trivial class is the discrete flux discovered from the physics point of view by ‘t Hooft [5]. On a 4-dimensional spacetime, there is another familiar quantity called the instanton number. We explain some of the subtleties such as the noncommutativity of discrete fluxes [2, 6] and fractional instanton numbers in the presence of discrete fluxes.

Lecture 3 is on the role of these topological invariants in quantum gauge theory. We show, using both canonical and path integral quantisation, that quantum Hilbert spaces are labelled by a theta angle and discrete fluxes, rectified because of their noncommutativity [6]. With Weyl fermions in the theory, we explain the relation to chiral anomaly and the algebra of the centre symmetry, parity and the anomaly-free part of the chiral symmetry.

In Lecture 4, we investigate quantum gauge theory on a compact space or spacetime. In the Abelian case, we show that the exact partition function exhibits electric-magnetic duality. For non-Abelian gauge theories, by perturbative expansion around flat backgrounds, we explore the energy spectrum and the degeneracy of the theory put on a compact space; when the spatial slice is a torus, this was originally studied by ‘t Hooft [5] and van Baal [1].

In the last Lecture, we return to classical field theory. We recall higher-form symmetries as defined in [3] and explore their classical aspects as an action on the fields by a cohomology group [7]. We then review Noether’s second theorem for local symmetries and its relation to the need of quotient by gauge redundancies. Finally, we explain how to construct higher-form symmetries when there is a redundancy in the description of redundancies.

Outline

1. Abelian and non-Abelian gauge theories

1.1. Electromagnetism as a gauge theory

1.2. Non-Abelian gauge theory: origin and applications

1.3. Classical solutions: monopoles and instantons

2. Topological quantities in gauge theories

2.1. Topology of fibre bundles

2.2. Discrete electric and magnetic fluxes, and their noncommutativity

2.3. Discrete fluxes and fractional instanton numbers

3. Quantised gauge theory

3.1. The theta angle, vacuum and chiral anomaly

3.2. States with definite discrete fluxes

3.3. The algebra of centre symmetry, parity and chiral symmetry

4. Quantum gauge theory on a finite volume

4.1. Abelian gauge theory and duality

4.2. Spectrum of a non-Abelian gauge theory on a compact space

4.3. Dependence on the theta angle and discrete fluxes

5. Higher-form symmetries and Noether's second theorem

5.1. What is a higher-form symmetry?

5.2. Gauge redundancy and Noether's second theorem

5.3. Construction of higher-form symmetries

References

[1] P. van Baal, Twisted boundary conditions: a nonperturbative probe for pure nonabelian gauge theories, PhD thesis, Utrecht University, Utreacht (1984)

[2] D. S. Freed, G. W. Moore, G. Segal, The uncertainty of fluxes, Commun. Math. Phys. 271 (2007) 247–274, `arXiv:hep-th/0605198`

[3] D. Gaiotto, A. Kapustin, N. Seiberg and B. Willett, Generalized global symmetries, J. High Energy Phys. 02 (2015) 172, 61pp., `arXiv:1412.5148 [hep-th]`

[4] N. Steenrod, The topology of fibre bundles, Princeton Univ. Press, Princeton (1951)

[5] G. 't Hooft, A property of electric and magnetic flux in non-abelian gauge theories, Nucl. Phys. B153 (1979) 141–160

[6] S. Wu, Non-orientable surfaces and electric-magnetic duality, J. High Energy Phys. 10 (2018) 169, 61 pp, `arXiv:1811.12571 [hep-th]`

[7] S. Wu, Discrete fluxes, higher-form symmetries and Ward identities, to appear (2024)