Singularities and Geometry in Discrete Integrable Systems

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Singularities play an important role in the study of integrable systems. Since the time of Kowalewskaya who showed that movable poles of solutions of nonlinear ordinary differential equations (ODE) are compatible with a regular (integrable) dynamics, the study of singularities developed gradually toward a deeper understanding of their role mainly in the topic of discrete integrable systems which, using tools from algebraic geometry of rational/elliptic surfaces, culminated in the discovery and classification of discrete, elliptic and q-Painlevé equations. In these lectures we intend to give a gradual introduction to the role played by the singularities in discrete integrable systems. We start from a brief description of continuos systems and the connection with Hirota bilinear formalism and then we introduce the main tools used in the study of discrete systems.

1: *General Aspects of Continuous Systems*

- Liouville integrability
- Fixed, movable critical singularities
- Painleve property and Hirota bilinear form
- Bilinear integrability

2: *Integrability Detectors in Discrete Systems*

- Singularity confinement for lattice soliton equations
- Singularity confinement for mappings
- Connection between singularity patterns and discrete Hirota bilinear forms
- Complexity index/algebraic entropy
- e)Mixed Painleve analysis and singularity confinement; semi-discrete equations

3: *QRT Mapping - A Paradigm*

- General form; elliptic curves
- Bi-quadratic invariant and solutions
- Deautonomisation from singularity confinement
- Algebraic entropy form singularity patterns; Express method

4: *Singularities and Algebraic Surfaces*

- Singularity confinement as blow-ups of projective varieties
- Divisors, rational-elliptic surfaces, Halphen surfaces
- Action on the Picard group, linearisation of discrete dynamical systems
- Invariant elliptic fibrations, conservation laws of discrete dynamical systems.

5: *Geometry and Discrete Integrability*

- Fiber-exchanging actions on Picard lattices
- Symmetries from affine Weyl groups
- Singular fiber deautonomisations and construction of discrete Painlevé equations
- Higher order mappings/Painlevé equations

References

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