Interpolation by ZMC and CMC surfaces and Related Problems

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$XXVI^{th}$ International Conference on Geometry, Integrabilty and Quantization
4th-11th June, 2026, Varna, Bulgaria

10th June, 2026

Interpolation Problem

- The classical problem of Plateau asks the following question: "Given a Jordan curve γ in $\mathbb{E}^3$, does there exist a minimal surface having its boundary as γ". In 1930 a solution was obtained by J. Douglas and simultaneously by T. Radó, [25].
- Given two real analytic curves in $\mathbb{R}^3$ or $\mathbb{L}^3$, one can ask if there exists a minimal or maximal surface which contains both of them. For minimal surfaces this problem has been solved by Jesse Douglas.
- In fact, in [10], Douglas also solved the problem of Plateau for the case of two closed curves (non-intersecting) in $\mathbb{E}^n$.
- In ([27]), I.N. Vekua uses implicit function theorem for Banach spaces in order to solve Plateau's problem for all curves which lies sufficiently close to a given plane curve in $\mathbb{E}^3$.
- In ([15]), in example (5.4.13) Hamilton solves the Plateau's problem for minimal surfaces when the surface is a graph using Nash-Moser inverse function theorem.
- The problem of "interpolating" curves by maximal surfaces for specific examples have been considered for instance by Fujimori, Rossman, Umehara, Yang and Ya-

mada, ([13]).

- The existence of spacelike constant mean curvature surfaces spanning two circular contours has been considered by Lopez in ([23]).
- The Plateau's problem for maximal surfaces has been considered for instance in pseudo-hyperbolic spaces by Labourie, Toulisse and Wolf, ([22]) where the boundary data is at infinity.
- In ([4]), we had solved the problem of interpolating a given space-like closed curve (with certain conditions) and a point by a maximal surface in $\mathbb{L}^3$, such that this point is a singularity.

In this talk, we mainly focus on 5 interpolation problems:

1. "Interpolation of two curves by Maximal and Minimal surfaces" – R.D., Rahul Kumar Singh, [2]

2. "Existence of Maximal Surface containing Given Curve and Special Singularity" – R.D., Pradip Kumar, Rahul Kumar Singh, [4].

3. Interpolation by CMC surfaces and applications – work in progress with Anu Dhochak, Sam Mathew and Savita Rani

4. Finite Decomposition of Minimal surfaces, Maximal surfaces, Time-like Minimal surfaces and Born-Infeld solitons; J. Ramanujan Math. Soc. 38, No. 3 (2023) 225-236– arXiv:2010.04405–R. D, K. Ghosh, S. Soundararajan., [3].

Recently, R. K. Singh and his students S. Paul, P. Vasu, S. Panigrahi have explored and got many more results.

5. Decomposition of Zero Mean Curvature Surfaces in Riemannian and Pseudo-Riemannia (R^3) Spaces – P. Vasu, S. Mathew, R. K. Singh, R. Dey, (communicated).

Minimal and Maximal surfaces

Minimal surfaces:

A minimal surface is the zero mean curvature surface in $\mathbb{E}^3$. Locally, if the minimal surface can be written as a graph of a function of two variables, namely $(x, y, \phi = \phi(x, y))$, then mean curvature $= 0$ implies minimal surfaces are solutions of the following non-linear equation:

$$(1 + \phi_x^2)\phi_{yy} - 2\phi_x\phi_y\phi_{xy} + (1 + \phi_y^2)\phi_{xx} = 0.$$

The surface is locally given by $X(x, y) = (x, y, \phi(x, y))$.

In isothermal parametrization this just becomes harmonicity of coordinates! The general solution is given by the Weierstrass-Enneper representation.

Maximal surfaces:

Vector space $\mathbb{R}^3$ with the metric $dx^2 + dy^2 - dz^2$, denoted by $\mathbb{L}^3$, is know as the Lorentz-Minkowski space.

A maximal surface is a spacelike surface with zero mean curvature in $\mathbb{L}^3$.

✦ A graph $(x, y, f(x, y))$ in Lorentz-Minkowski space $\mathbb{L}^3 := (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$ is maximal if it satisfies

$$(1 - f_x^2)f_{yy} + 2f_x f_y f_{xy} + (1 - f_y^2)f_{xx} = 0, \tag{1}$$

for some smooth function $f(x, y)$ satisfying $f_x^2 + f_y^2 < 1$.

It is the zero mean curvature equation in Lorentz-Minkowski space.

Maximal surfaces also have Weierstrass-Enneper representation analogous to minimal surfaces.

Generalized maximal surfaces have singularities (conical, cuspidal, swallowtail....)

Examples of maximal surfaces:(Image from publicly availble domain in the web)

1. Spacelike Elliptic Catenoid:

$X(u, v) = (cos(u)sinh(v), sin(u)sinh(v), v)$

2. Spacelike Parabolic Catenoid:

$X(u, v) = (v + \frac{v^3}{3} - u^2v, 2uv, v - \frac{v^3}{3} + u^2v)$

3. Spacelike Hyperbolic Catenoid:

$X(u, v) = (v, sin(v)sinh(u), sin(v)cosh(u))$

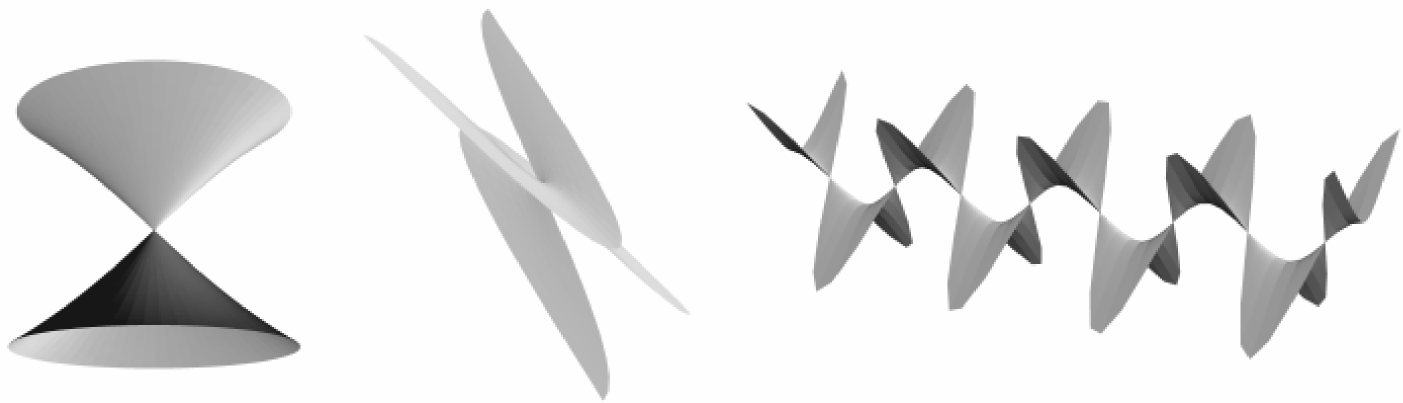


Figure 1: Elliptic, Parabolic and Hyperbolic catenoids

Interpolation of two curves by Maximal and Minimal surfaces

In the next few slides we give a sketch of our method of interpolation of two spacelike real analytic curves in $\mathbb{L}^3$ by maximal surfaces.

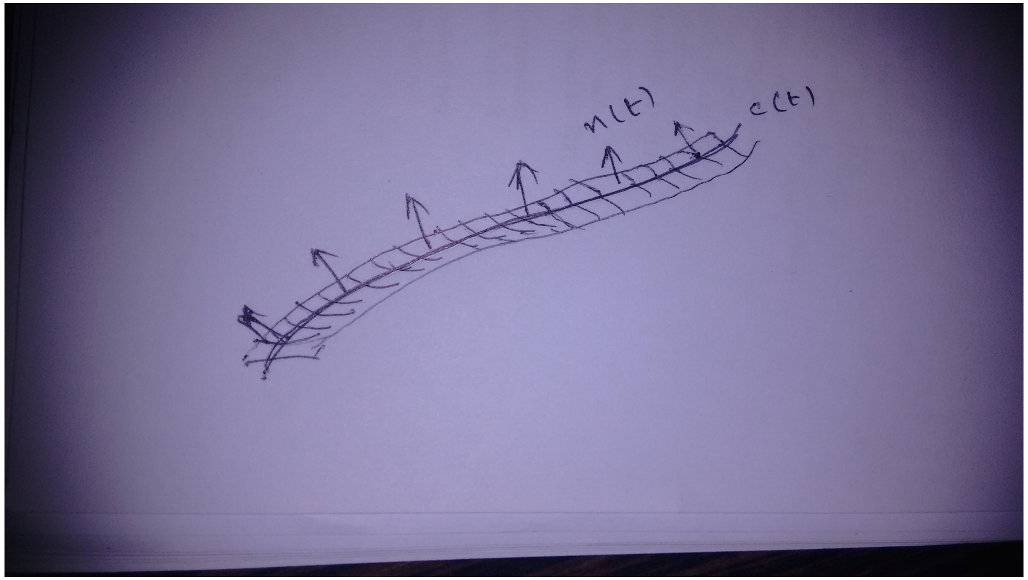
Interpolation by Maximal surfaces

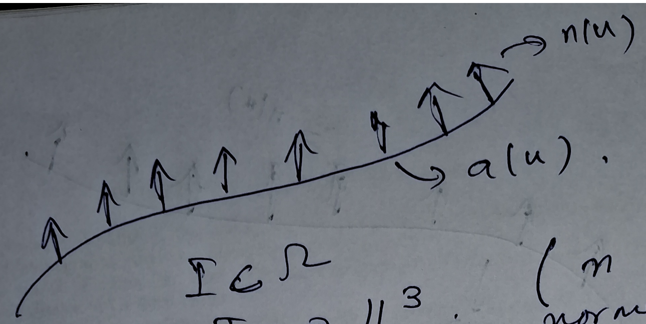
✦ *The Björling problem:* Recall that $\mathbb{L}^3$ is defined as $(\mathbb{R}^3, \langle\cdot,\cdot\rangle_L)$, where $\langle v, w\rangle_L = v_1w_1 + v_2w_2 - v_3w_3$ for $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ and $w = (w_1, w_2, w_3) \in \mathbb{R}^3$. The symbol $\times_L$ denotes the cross product of $\mathbb{L}^3$. Let $a : I \to \mathbb{L}^3$ be a real analytic spacelike curve with $a'(u) \neq 0$ for all $u \in I$ which can be extended to Ω.

Also, consider $n : I \to \mathbb{L}^3$ as a real analytic map such that $\langle n(u), a'(u)\rangle_L = 0$ for all $t \in I$ and $n_1^2(u) + n_2^2(u) - n_3^2(u) = -1$ for all $u \in I$, where $n = (n_1, n_2, n_3)$. The map n is referred to as the unit normal vector field along a, and it is timelike. The Björling problem poses the question of whether there exists a maximal surface $X : \Omega \to \mathbb{L}^3$ with $I \subset \Omega \subset \mathbb{C}$, where Ω is simply connected, and the surface X satisfies the following conditions: $X(u, 0) = a(u)$ and $N(u, 0) = n(u)$ for all $u \in I$. Here, $N : \Omega \to \mathbb{L}^3$ is a normal to the surface X. The solution to Björling's problem is given by

$$X(u, v) = \mathrm{Re}\{a(w) + i\int_{u_0}^{w} n(\widetilde{w}) \times_L a'(\widetilde{w})d\widetilde{w}\} : w = u + iv, u_0 \in I, \tag{2}$$

where $n(w)$ and $a(w)$ are the complex analytic extensions of $n(u)$ and $a(u)$ over Ω.


n(t)
c(t)



$I \subset \Omega$

$a : I \to \mathbb{L}^3$.

$n : I \to \mathbb{L}^3$.

(n is a normal distribution which coincides with surface normal on I)

$\langle a', n \rangle_L = 0, \quad \langle n', n' \rangle_L = -1.$

Solution to a maximal strip is given by Björling - Schwarz formula.

Our idea:- a be a real analytic curve, l is another " " curve 'close by'. Then, $\exists$ a maximal surface passing through a & l obtained by finding a suitable 'normal distribution' on a.

- We invert the Bjorling-Schwartz formula to get the correct normal on the first curve, such that the Bjorling-Schwartz surface passes through the second curve. We need to use Inverse Function Theorem for Banach spaces. So we finally have a theorem of how close the two curves should be such that there is a guaranteed surface of the Bjorling-Schwartz type which interpolates the two curves.

Interpolation by CMC surfaces, Born-Infeld soltions of two curves in E^3

For CMC surfaces, the Sym-Bobenko formula can be inverted (using inverse function theorem) and one can obtain CMC surfaces of interpolatuon between the two curves if they are close enough. Some applications can be found.

For Born-Infeld solitons, one can find the most general solution (using Barbishov-Cherniko theory) and use inverse function theorem again.

(Work in progress jointly with Anu Dhochak, Sam Mathew and Savita Rani)

"Existence of Maximal Surface containing Given Curve and Special Singularity"

The Main Question Analogous to the Bjorling problem, we have the singular Bjorling problem, which we use to solve this problem. We have the following theorem.

- In general we can ask the following: Given a real analytic curve $\gamma(\theta)$, does there exists $F : A(r, R) \to \mathbb{L}^3$, a generalised maximal surface and $r_0 \neq 1$ such that $F(r_0 e^{i\theta}) = \gamma(\theta)$ and F has a special singularity at $|z| = 1$.
- For curve $\gamma(r_0 e^{i\theta}) = (f(r_0 e^{i\theta}), g(r_0 e^{i\theta}))$, $r_0 \neq 1$, we define the following modified fourier constants of f and g as

$$c = \frac{1}{2\pi \log r_0} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) d\theta; \quad d = \frac{1}{2\pi \log r_0} \int_{-\pi}^{\pi} g(r_0 e^{i\theta}) d\theta, \tag{3}$$

for $n \neq 0$;

$$c_n = \frac{r_0^n}{2\pi(r_0^{2n} - 1)} \int_{-\pi}^{\pi} f(r_0 e^{i\theta}) e^{-in\theta} d\theta; \;\; d_n = \frac{r_0^n}{2\pi(r_0^{2n} - 1)} \int_{-\pi}^{\pi} g(r_0 e^{i\theta}) e^{-in\theta} d\theta. \tag{4}$$

- We see that if γ is real analytic then $\limsup |c_{-n}|^{\frac{1}{n}} = 0$ and $\limsup |c_n|^{\frac{1}{n}} = 0$, therefore following series converges for all $|z| \neq 0$,

$$h(z) = \sum_{-\infty}^{\infty} c_n \left(z^n - \frac{1}{\bar{z}^n} \right) + c \log |z|. \tag{5}$$

$$w(z) = \sum_{-\infty}^{\infty} d_n \left(z^n - \frac{1}{\bar{z}^n} \right) + d \log |z|, \tag{6}$$

Now we state the main theorem of our paper:

✦ **Theorem:** Let $\gamma_1(\theta)$ be a closed real analytic space like curve. Then there exists $s_0 \neq 1$ and a generalised maximal surface $F : \mathbb{C} - \{0\} \to \mathbb{L}^3$ such that $F(s_0 e^{i\theta}) := \gamma_1(\theta)$ and having a special singularity at $|z| = 1$ if and only if there exists $r_0 \neq 1$ and constants $c, c_n' s, d, d_n' s$ for the curve $\gamma(r_0 e^{i\theta}) := \gamma_1(\theta) = (f(r_0 e^{i\theta}), g(r_0 e^{i\theta}))$, as in equations 3,4 satisfies the relation:

$$\forall\ k \neq 0; \ \sum_{-\infty}^{\infty} 4n(n-k)(c_n \bar{c}_{n-k} - d_n d_{n-k}) + 2k(c_k \bar{c} - c\bar{c}_{-k} - 2d_k d) = 0 \tag{7}$$

$$\text{and} \quad \sum 4n^2(c_n \bar{c}_n - d_n^2) + c\bar{c} - d^2 = 0. \tag{8}$$

Interpolation of surfaces by minimal surfaces

As a consequence of an Euler- Ramanujan's identity:

$$\tan^{-1}(\tanh(y)\cot(x)) = \sum_{k=-\infty}^{k=\infty} \tan^{-1}\left(\frac{y}{x+k\pi}\right).$$

The left hand side is the height function of Scherk's first surface and the right hand side is sum of height functions of shifted helicoids – all minimal surfaces!

In fact, $z = \tan^{-1}(\tanh(y)\cot(x))$ is the Scherk's first surface and $z = \tan^{-1}(\frac{y}{x+k\pi})$ is a shifted helicoid.

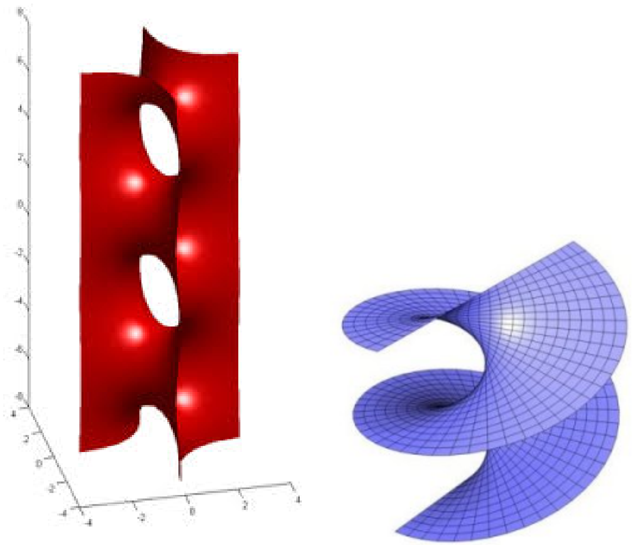


Figure 2: Scherk's first surface and helicoid

This last fact was discovered and used by a condensed matter physicist and his collaborators –Randall Kamien et. al. , U . Penn.

Finite Decompositions

It is possible to decompose the height functions into scaled and translated versions of itself. Instance of finite decomposition [3]:

- Let $z(x,y) = \ln\left(\frac{\cos(y)}{\cos(x)}\right)$ be the height function of the Scherk's second minimal surface. Then we have $z(x,y) = \sum_{m=0}^{n-1} z\left(\frac{x}{n} - c(m), \frac{y}{n} - c(m)\right)$, where n is any integer and $c(m) = \frac{(2m-n+1)}{2n}\pi$.

- This particular one is proved using an Euler-Ramanujan Identity: We have Ramanujan's identity, Ramanujan Notebook I, Example (1) page 38, where X, A are complex, A is not an odd multiple of $\pi/2$:
$\frac{\cos(X+A)}{\cos(A)} = \Pi_{k=1}^{\infty}\left(\left(1 - \frac{X}{(k-\frac{1}{2}\pi)-A}\right)\left(1 + \frac{X}{(k-\frac{1}{2}\pi)+A}\right)\right)$.
Then we take ln on both sides, to get:

$$\ln\left(\frac{\cos(X+A)}{\cos(A)}\right) = \sum_{k=1}^{\infty} \ln\left(\left(\frac{(k-\frac{1}{2}\pi)-(X+A)}{(k-\frac{1}{2})\pi - A}\right)\left(\frac{(k-\frac{1}{2})\pi + (X+A)}{(k-\frac{1}{2})\pi + A}\right)\right)$$

✦ Let $z(x,y) = \ln\left(\frac{\cos(y)}{\cos(x)}\right)$ be the height function of the Scherk's second minimal surface. Then we have $z(x,y) = \sum_{m=0}^{n-1} z\left(\frac{x}{n} - c(m), \frac{y}{n} - c(m)\right)$, where n is any natural number ≥ 2 and $c(m) = \frac{(2m-n+1)}{2n}\pi$.

Proof: Let $z(x,y) = \ln\left(\frac{\cos(y)}{\cos(x)}\right)$.

We have an Euler-Ramanujan identity : X, A are complex, A is not an odd multiple of $\pi/2$:

$\frac{\cos(X+A)}{\cos(A)} = \Pi_{k=1}^{\infty}\left(\left(1 - \frac{X}{(k-\frac{1}{2}\pi)-A}\right)\left(1 + \frac{X}{(k-\frac{1}{2}\pi)+A}\right)\right)$.

As in [7] we take ln on both sides, to get:

$$\ln\left(\frac{\cos(X+A)}{\cos(A)}\right) = \sum_{k=1}^{\infty} \ln\left(\left(1 - \frac{X}{(k-\frac{1}{2})\pi - A}\right)\left(1 + \frac{X}{(k-\frac{1}{2})\pi + A}\right)\right)$$
$$= \sum_{k=1}^{\infty} \ln\left(\left(\frac{(k-\frac{1}{2}\pi) - (X+A)}{(k-\frac{1}{2})\pi - A}\right)\left(\frac{(k-\frac{1}{2})\pi + (X+A)}{(k-\frac{1}{2})\pi + A}\right)\right)$$

The Scherk's second surface is given by $z = \ln\left(\frac{\cos(y)}{\cos(x)}\right)$.

Let $X + A = y$ and $A = x$ in Ramanujan's identity.

Then, if x is not an odd multiple of $\frac{\pi}{2}$, we have,

$$\ln\left(\frac{\cos(y)}{\cos(x)}\right) = \sum_{k=1}^{\infty} \ln\left(\left(\frac{y-(k-\frac{1}{2})\pi)}{x-(k-\frac{1}{2})\pi}\right)\left(\frac{y+(k-\frac{1}{2})\pi}{x+(k-\frac{1}{2})\pi}\right)\right)$$

For x not an odd multiple of $\frac{\pi}{2}$, we have (from Euler-Ramanujan's identity)

$$z(x,y) = \sum_{k=0}^{\infty} \ln\left(\frac{(2y-(2k+1)\pi)(2y+(2k+1)\pi)}{(2x-(2k+1)\pi)(2x+(2k+1)\pi)}\right)$$

For convenience, let $f(y,x,k) = \frac{2y-(2k+1)\pi}{2x-(2k+1)\pi}$ and let $g(y,x,k) = \frac{2y+(2k+1)\pi}{2x+(2k+1)\pi}$, so that

$$z(x,y) = \sum_{k=0}^{\infty} \ln(f(y,x,k)g(y,x,k))$$

Rewriting $k = np + m$ for arbitrary fixed n, $0 \leq m \leq n-1$, we have

$$z(x,y) = \sum_{p=0}^{\infty} \sum_{m=0}^{n-1} \ln \left(\frac{(2y - 2m\pi + (n-1)\pi - n(2p+1)\pi)(2y + 2m\pi - (n-1)\pi + n(2}{(2x - 2m\pi + (n-1)\pi - n(2p+1)\pi)(2x + 2m\pi - (n-1)\pi + n(2}\right.$$

Let $c(m) = \frac{(2m-n+1)}{2n}\pi$. Then

$$z(x,y) = \sum_{p=0}^{\infty} \sum_{m=0}^{n-1} \ln \left(\frac{(\frac{2y}{n} - 2c(m) - (2p+1)\pi)(\frac{2y}{n} + 2c(m) + (2p+1)\pi)}{(\frac{2x}{n} - 2c(m) - (2p+1)\pi)(\frac{2x}{n} + 2c(m) + (2p+1)\pi)} \right)$$

$$= \sum_{p=0}^{\infty} \sum_{m=0}^{n-1} \ln \left(f\left(\frac{y}{n} - c(m), \frac{x}{n} - c(m), p\right) g\left(\frac{y}{n} + c(m), \frac{x}{n} + c(m), p\right)\right)$$

Now, $c(n-1-m) = \frac{2(n-1-m)-n+1}{2n}\pi = \frac{2n-2-2m-n+1}{2n}\pi = \frac{n-1-2m}{2n}\pi = -c(m)$

On expanding the finite summation, we get a sum of natural log of f and that of g. We can reverse the order of finite summation for the part involving g and rewrite the same expression as follows.

$$
\begin{aligned}
z(x,y) &= \sum_{p=0}^{\infty}\sum_{m=0}^{n-1} \ln\left(f\left(\frac{y}{n}-c(m),\frac{x}{n}-c(m),p\right) g\left(\frac{y}{n}+c(n-1-m),\frac{x}{n}+c(n-1-m\right.\right.\\
&= \sum_{p=0}^{\infty}\sum_{m=0}^{n-1} \ln\left(f\left(\frac{y}{n}-c(m),\frac{x}{n}-c(m),p\right) g\left(\frac{y}{n}-c(m),\frac{x}{n}-c(m),p\right)\right)\\
&= \sum_{m=0}^{n-1} z\left(\frac{x}{n}-c(m),\frac{y}{n}-c(m)\right)
\end{aligned}
$$

interchanging the order of the infinite and the finite sum. The interchange of the sums is justified as follows. Suppose

$$
A_{pm} = \ln\left(f\left(\frac{y}{n}-c(m),\frac{x}{n}-c(m),p\right) g\left(\frac{y}{n}-c(m),\frac{x}{n}-c(m),p\right)\right).
$$

Then for every m, $\sum_{p=0}^{\infty} A_{pm}$ converges to $z\left(\frac{x}{n}-c(m),\frac{y}{n}-c(m)\right)$.

Then

$$\begin{aligned}\sum_{p=0}^{\infty}\sum_{m=0}^{n-1} A_{pm} &= \lim_{N\to\infty}\sum_{p=0}^{N}\sum_{m=0}^{n-1} A_{pm} = \lim_{N\to\infty}\sum_{m=0}^{n-1}\sum_{p=0}^{N} A_{pm} \\ &= \sum_{m=0}^{n-1}\left(\lim_{N\to\infty}\sum_{p=0}^{N} A_{pm}\right) = \sum_{m=0}^{n-1}\sum_{p=0}^{\infty} A_{pm}.\end{aligned}$$

The limit can be interchanged with a finite sum.

✦ Thus we have a finite decomposition: $z(x,y) = \sum_{m=0}^{n-1} z\left(\frac{x}{n} - c(m), \frac{y}{n} - c(m)\right)$.

Weierstrass Product and Infinite and Finite Decompositions of height functions of ZMC surfaces

Recently we have found a plethora of infinite and finite decompositions of ZMC surfaces in Euclidean, Lorentzian and Isotropic 3-spaces, using the Weierstrass-Product theorem. This is joint work with Priyank Vasu, Sam Mathew, Rahul K Singh.

We give one example.

✦ **Scherk's tower and dilated catenoids.**

$$f_{\text{ScherkTower}} = \sqrt{2}\cosh^{-1}\left[\sqrt{\sinh^2\left(\frac{y}{\sqrt{2}}\right) + 2\sin^2\left(\frac{x}{2}\right)}\right]. \tag{9}$$

Dilated catenoids. Their explicit graphical representations are given by:

✦

$$f^{k}_{\text{Dil.Cat}} = \cosh^{-1}\left[\sqrt{\left(\frac{x}{2} + k\pi\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2}\right] \quad \text{and} \quad f^{0}_{\text{Dil.Cat}} = \cosh^{-1}\left[\sqrt{\left(\frac{x}{2}\right)^2 + \left(\frac{y}{\sqrt{2}}\right)^2}\right]. \tag{10}$$

✦ Catenoid-Scherk's Tower Identity

$$\cosh^2(f^0_{\text{Dil.Cat}}) \cdot \prod_{k\neq 0} \frac{\cosh^2\left(f^k_{\text{Dil.Cat}}\right)}{\pi^2 k^2}\Bigg|_{(x,y)} + \cosh^2(f^0_{\text{Dil.Cat}}) \cdot \prod_{k\neq 0} \frac{\cosh^2\left(f^k_{\text{Dil.Cat}}\right)}{\pi^2 k^2}\Bigg|_{(x,0)} \tag{11}$$

$$= \cosh^2\left(\frac{f_{\text{ScherkTower}}}{\sqrt{2}}\right)\Bigg|_{(x,y)} \tag{12}$$

defined on each strip-shaped domains

$$-2(k-1)\pi + 1 < x < -2k\pi - 1 \quad \text{for each } k \in \mathbb{Z}.$$

✦ Second Catenoid-Scherk's Tower Identity

$$\cosh^2(f^0_{\text{Dil.Cat}}) \cdot \prod_{k\neq 0} \frac{\cosh^2\left(f^k_{\text{Dil.Cat}}\right)}{\pi^2 k^2}\Bigg|_{(x,y)} = \frac{1}{2}\Big(\cosh^2\left(\frac{f_{\text{ScherkTower}}}{\sqrt{2}}\right)\Big|_{(x,y)}$$

$$+ \cosh^2\left(\frac{f_{\text{ScherkTower}}}{\sqrt{2}}\right)\Big|_{(0,y)}\Big).$$

defined on the domain

$$|y| > \sqrt{2}.$$

Finite Decomposition

We apply a translation and dilation on the Scherk's tower given by $(x, y) \to (\frac{x+2i\pi}{m}, \frac{y}{m})$ and denote the resulting dilated and translated surface as $f^i_{\text{Dil.ScherkTower}}(x, y)$. Then we get the following theorem.

- On a cerrtain domain $\Omega \subset \mathbb{C}$,

$$\frac{1}{2}\left[\cosh^2\left(\frac{f_{\text{ScherkTower}}(x,y)}{\sqrt{2}}\right) + \cosh^2\left(\frac{f_{\text{ScherkTower}}(0,y)}{\sqrt{2}}\right)\right]$$
$$= m^2 \cdot \prod_{i=1}^{m-1} \operatorname{cosec}^2(\pi i/m) \cdot \prod_{i=0}^{m-1}\left[\tfrac{1}{2}\left(\cosh^2\left(\frac{f^i_{\text{Dil.ScherkTower}}(x,y)}{\sqrt{2}}\right)\right.\right.$$
$$\left.\left. + \cosh^2\left(\frac{f^i_{\text{Dil.ScherkTower}}(0,y)}{\sqrt{2}}\right)\right)\right]. \tag{14}$$

✦ In all these examples, we have crucially used the Weierstrass product formula for $\sin(a)$, as follows.

$$\sin a = a \prod_{k \neq 0} \left(\frac{k\pi + a}{k\pi} \right). \tag{15}$$

Talk based on the following papers:

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- ”Truth comes as conqueror only because we have lost the art of receiving it as guest.” –Rabindranath Tagore (1861-1941) Bengali poet, philosopher, The Fourfold Way of India (1924).
- Often paraphrased: "Truth comes as conqueror only to those who have lost the art of receiving it as friend."
- Mathematicians are loyal old friends of Truth. They search for what is true and receive it with grace.

Thank You