Direct Scattering Problems

Will model each Jost system, in each interval, after plane waves:

$$\text{as } x \to \pm\infty \text{ and } v \to e^{\pm ikx}, \text{ then } \lambda = \frac{2\kappa}{4k^2+1}$$

In each interval, we define a ϕ and a ψ solution, satisfying:

On left:

$$\phi(x_\ell, t; k) = e^{-ikx_\ell}$$

$$\partial_x \phi(x,t;k)|_{x_\ell} = -i\,k\,e^{-ikx_\ell}$$

$$\overline{\phi}(x,t;k) = \phi(x,t;k^*)^*$$

On right:

$$\psi(x_r, t; k) = e^{ikx_r}$$

$$\partial_x \psi(x,t;k)|_{x_r} = i\,k\,e^{ikx_r}$$

$$\overline{\psi}(x,t;k) = \psi(x,t;k^*)^*$$

Then we take:

$$\begin{pmatrix} \phi \\ \overline{\phi} \end{pmatrix} = \underbrace{\begin{pmatrix} a(k,t) & b(k,t) \\ \overline{b}(k,t) & \overline{a}(k,t) \end{pmatrix}}_{\text{Scattering Coefficients}} \begin{pmatrix} \overline{\psi} \\ \psi \end{pmatrix}$$

Analytical Properties

In the complex k – plane:

In interval #1 (left): $\phi_1 e^{ikx}\,,\ \psi_1 e^{-ikx}\,,\ a_1 \text{ and } \underline{b_1}$ -- analytic in UHP

In interval #2 (middle): $\phi_2 e^{ikx}\,,\ \psi_2 e^{-ikx}\,,\ a_2 \text{ and } b_2$ -- entire functions

In interval #3 (right): $\phi_3 e^{ikx}\,,\ \psi_3 e^{-ikx}\,,\ a_3 \text{ and } \underline{\bar{b}_3}$ -- analytic in UHP

In all intervals:

$$\phi_j e^{ikx}\,,\ \psi_j e^{-ikx} \text{ and } a_j \to 1 \ \text{ as } |k| \to \infty$$ -- in UHP (at fixed *x*)

Similar for the conjugate quantities.

This will have important consequences later.

Time Evolution of Scattering Coefficients

This equation is now satisfied. →

$$v_{xx} - \frac{1}{4} v + \frac{m+\kappa}{2\lambda} v = 0\,,$$

Lax evolution operator:

$$v_t + \underline{u} v_x + \lambda v_x + (\underline{\alpha} - \frac{1}{2} \underline{u_x}) v = 0\,,$$

1. Determine α from ϕ Jost function at left end.
2. Express ϕ in terms of scattering coefficients and ψ's.
3. Drop result into evolution operator at right end.
4. The flow at x_b determines the evolution.

$$\partial_t a_1 = i\eta_{0b} \frac{\kappa}{2k\lambda} a_1 + \frac{1}{2}(\eta_{1b} + \frac{i}{2k}\eta_{0b}) b_1 e^{2ikx_b}$$

$$\partial_t b_1 = \frac{1}{2}(\eta_{1b} - \frac{i}{2k}\eta_{0b}) a_1 e^{-2ikx_b} + \frac{i}{2k}\left[\lambda - 2\kappa - \frac{\eta_{0b}\kappa}{\lambda}\right] b_1$$

Evolutions of a's and b's

$$\partial_t a_1 = i\eta_{0b}\frac{\kappa}{2k\lambda}\,a_1 + \frac{1}{2}(\eta_{1b} + \frac{i}{2k}\eta_{0b})b_1 e^{2ikx_b}$$

$$\partial_t b_1 = \frac{1}{2}(\eta_{1b} - \frac{i}{2k}\eta_{0b})a_1 e^{-2ikx_b} + \frac{i}{2k}\left[\lambda - 2\kappa - \frac{\eta_{0b}\kappa}{\lambda}\right] b_1$$

$$\partial_t a_2 = i(\eta_{0c} - \eta_{0b})\frac{\kappa}{2k\lambda}\,a_2 + \frac{2\eta_{1c}k + i\eta_{0c}}{4k}e^{2ikx_b}b_2 - \frac{2\eta_{1b}k - i\eta_{0b}}{4k}e^{-2ikx_b}\bar{b}_2$$

$$\partial_t b_2 = \frac{1}{2}(\eta_{1c} - \frac{i}{2k}\eta_{0c})a_2 e^{-2ikx_c} - \frac{1}{2}(\eta_{1b} - \frac{i}{2k}\eta_{0b})\bar{a}_2 e^{-2ikx_b} + \frac{i}{2k}\left[\lambda - 2\kappa - \kappa\frac{\eta_{0b} + \eta_{0c}}{\lambda}\right] b_2$$

$$\partial_t \bar{a}_2 = -i(\eta_{0c} - \eta_{0b})\frac{\kappa}{2k\lambda}\,\bar{a}_2 - \frac{2\eta_{1b}k + i\eta_{0b}}{4k}e^{2ikx_b}b_2 + \frac{2\eta_{1c}k - i\eta_{0c}}{4k}e^{-2ikx_c}\bar{b}_2$$

$$\partial_t \bar{b}_2 = -\frac{1}{2}(\eta_{1b} + \frac{i}{2k}\eta_{0b})a_2 e^{2ikx_b} + \frac{1}{2}(\eta_{1c} + \frac{i}{2k}\eta_{0c})\bar{a}_2 e^{2ikx_c} - \frac{i}{2k}\left[\lambda - 2\kappa - \kappa\frac{\eta_{0b} + \eta_{0c}}{\lambda}\right] \bar{b}_2$$

$$\partial_t a_3 = -i\eta_{0c}\frac{\kappa}{2k\lambda}\,a_3 - \frac{1}{2}(\eta_{1c} - \frac{i}{2k}\eta_{0c})\bar{b}_3 e^{-2ikx_c}$$

$$\partial_t \bar{b}_3 = -\frac{1}{2}(\eta_{1c} + \frac{i}{2k}\eta_{0c})a_3 e^{2ikx_c} - \frac{i}{2k}\left[\lambda - 2\kappa - \frac{\eta_{0c}\kappa}{\lambda}\right] \bar{b}_3$$

$$\lambda = \frac{2\kappa}{4k^2 + 1}$$

λ has poles

Analytical Properties

Solution depends on η_0 and η_1 at stationary points.

These have been unspecified and arbitrary up to now.

General solution of b's will have essential singularities in UHP and LHP.

All scattering coefficients are analytic in either UHP or LHP, or both.

Therefore, η_0 and η_1 must evolve such that no essential singularities will ever appear, as long as *m* does not "break".

Requiring all b(λ)'s to be regular, when they should, in the limit of λ approaching infinity, uniquely determines all η_0's and η_1's.

$$I_{ij}^{(\gamma)} = \int_{x_i}^{x_j} m(x,t) e^{\gamma x} dx \,, \quad \gamma = \pm 1 \text{ or } 0 \,, \quad x_a = -\infty \,, \quad x_d = +\infty$$

Near the Singular Points

k = + i/2

$$a_1 \to 1 - \frac{1}{2\lambda} I_{ab}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad b_1 \to \frac{1}{2\lambda} I_{ab}^{(+1)} + \mathcal{O}(\lambda^{-2}),$$

$$a_2 \to 1 - \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad b_2 \to \frac{1}{2\lambda} I_{bc}^{(+1)} + \mathcal{O}(\lambda^{-2}),$$

$$\overline{a}_2 \to 1 + \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad \overline{b}_2 \to -\frac{1}{2\lambda} I_{bc}^{(-1)} + \mathcal{O}(\lambda^{-2}),$$

$$a_3 \to 1 - \frac{1}{2\lambda} I_{cd}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad \overline{b}_3 \to -\frac{1}{2\lambda} I_{cd}^{(-1)} + \mathcal{O}(\lambda^{-2}),$$

k = - i/2

$$\overline{a}_1 \to 1 - \frac{1}{2\lambda} I_{ab}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad \overline{b}_1 \to \frac{1}{2\lambda} I_{ab}^{(+1)} + \mathcal{O}(\lambda^{-2}),$$

$$a_2 \to 1 + \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad b_2 \to -\frac{1}{2\lambda} I_{bc}^{(-1)} + \mathcal{O}(\lambda^{-2}),$$

$$\overline{a}_2 \to 1 - \frac{1}{2\lambda} I_{bc}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad \overline{b}_2 \to \frac{1}{2\lambda} I_{bc}^{(+1)} + \mathcal{O}(\lambda^{-2}),$$

$$\overline{a}_3 \to 1 - \frac{1}{2\lambda} I_{cd}^{(0)} + \mathcal{O}(\lambda^{-2}), \quad b_3 \to -\frac{1}{2\lambda} I_{cd}^{(-1)} + \mathcal{O}(\lambda^{-2}).$$

Conditions and Solution:

$$\eta_{1b} - \eta_{0b} + I_{ab}^{(+1)} e^{-x_b} = 0$$

$$(\eta_{1c} - \eta_{0c}) e^{x_c} - (\eta_{1b} - \eta_{0b}) e^{x_b} + I_{bc}^{(+1)} = 0$$

$$(\eta_{1c} + \eta_{0c}) e^{-x_c} - (\eta_{1b} + \eta_{0b}) e^{-x_b} + I_{bc}^{(-1)} = 0$$

$$\eta_{1c} + \eta_{0c} - I_{cd}^{(-1)} e^{x_c} = 0$$

$$\eta_{0b} = \frac{1}{2} e^{-x_b} I_{ab}^{(+1)} + \frac{1}{2} e^{x_b} I_{bd}^{(-1)}$$

$$\eta_{1b} = -\frac{1}{2} e^{-x_b} I_{ab}^{(+1)} + \frac{1}{2} e^{x_b} I_{bd}^{(-1)}$$

$$\eta_{0c} = \frac{1}{2} e^{-x_c} I_{ac}^{(+1)} + \frac{1}{2} e^{x_c} I_{cd}^{(-1)}$$

$$\eta_{1c} = -\frac{1}{2} e^{-x_c} I_{ac}^{(+1)} + \frac{1}{2} e^{x_c} I_{cd}^{(-1)}$$

$$I_{ij}^{(\gamma)} = \int_{x_i}^{x_j} m(x,t) e^{\gamma x} dx \,, \quad \gamma = \pm 1 \text{ or } 0 \,, \quad x_a = -\infty \,, \quad x_d = +\infty$$

SUMMARY

- Non-uniform asymptotics require multiple IST’s.
- Also applies to the OTSI and DTPP problems.
- Location of stationary points of flow determine the intervals.
- Set up a universal set of Jost functions and scattering coefficients. (same for all intervals)
- Evolution of scattering coefficients found.
- The η’s are determined by derivative of scattering coefficients at k = +- i/2.
- The η’s are also found to be related to simple integrals over *m(x,t).* (dynamics?)
- Much more remains to be done.