EXTENDED HAMILTONIAN FORMALISM OF FIELD THEORIES:
Variational aspects and other topics

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A. ECHEVERRÍA-ENRÍQUEZ, M. DE LEÓN, M.C. MUÑOZ-LECANDA, N. ROMÁN-ROY: "Hamiltonian systems in multisymplectic field theories". *math-ph/0506003* (2005).

1 INTRODUCTION

Structure of autonomous Hamiltonian dynamical systems are especially suitable to analyze problems such as: symmetries and related topics (existence of conservation laws and reduction), integrability (including numerical methods), and quantization.

Geometrically, many of their characteristics arise from the existence of a "natural" geometric structure in the phase space: the *symplectic form*.

The dynamic information is carried out by the *Hamiltonian function*, which is 'independent' of the geometry.

We wish to generalize Hamiltonian systems in autonomous mechanics to first-order multisymplectic field theories.

In these models, multisymplectic forms play the same role than symplectic forms in autonomous mechanics.

There are two multimomentum bundles:

The *restricted multimomentum bundle* has not a canonical multisymplectic form. Hamiltonian systems are introduced by means of Hamiltonin sections (carrying the physical information), which allows us to construct the geometric structure.

The *extended multimomentum bundle* is endowed with a canonical multisymplectic form. On it, Hamiltonian systems can be introduced as in autonomous mechanics, by means of suitable closed 1-forms (and certain kinds of Hamiltonian multivector fields). The resultant *extended Hamiltonian formalism* is the generalization to field theories of the extended formalism of non-autonomous mechanical systems.

C. Paufler, H. Römer, "Geometry of Hamiltonian n-vector fields in multisymplectic field theory", *J. Geom. Phys.* **44**(1) (2002) 52-69.

Contents

2 PRELIMINARIES

2.1 Multivector fields in multisymplectic manifolds

$(\mathcal{M},\Omega)$ multisymplectic manifold,

$\Omega \in Z^{m+1}(\mathcal{M})$, 1-nondegenerate $(2 \leq m+1 \leq \dim \mathcal{M})$.

Sections of $\Lambda^k(\mathrm{T}\mathcal{M})$ are called *k-multivector fields* in $\mathcal{M}$

(contravariant skew-symmetric tensors of order k in $\mathcal{M}$).

$\mathfrak{X}^k(\mathcal{M})$ is the set of k-multivector fields in $\mathcal{M}$.

$\mathcal{X} \in \mathfrak{X}^k(\mathcal{M})$ is a *locally decomposable multivector field* if $\mathcal{X}|_U = X_1 \wedge \ldots \wedge X_k$

for $U \subset \mathcal{M}$, and $X_1, \ldots, X_k \in \mathfrak{X}(U)$.

$\forall \mathcal{X} \in \mathfrak{X}^k(\mathcal{M})$ loc. dec. multivector field $\exists$ *associated distribution* $\mathcal{D}_{\mathcal{X}} \subset \mathrm{T}\mathcal{M}$

$\mathcal{D}_{\mathcal{X}}|_U = span\{X_1, \ldots, X_k\}$

$\mathcal{X}$ is *integrable* if $\mathcal{D}_{\mathcal{X}}$ is involutive.

$\mathcal{X} \in \mathfrak{X}^k(\mathcal{M})$ is a *locally Hamiltonian k-multivector field* if $i(\mathcal{X})\Omega \in Z^{m+1-k}(\mathcal{M})$.

$\forall\, x \in M$, $\exists U \subset \mathcal{M}$ and $\exists \zeta \in \mathit{\Omega}^{m-k}(U)$ such that $i(\mathcal{X})\Omega = \mathrm{d}\zeta$ (on U)

ζ locally defined modulo closed $(m-k)$-forms.

Every $\zeta \in \mathit{\Omega}^{m-k}(U)$ is a *local Hamiltonian form* for $\mathcal{X}$.

$\mathcal{X} \in \mathfrak{X}^k(\mathcal{M})$ is a *Hamiltonian k-multivector field* if $i(\mathcal{X})\Omega = \mathrm{d}\zeta$, for $\zeta \in \mathit{\Omega}^{m-k}(\mathcal{M})$.

ζ is a *Hamiltonian form* for $\mathcal{X}$.

2.2 Multimomentum bundles

$\pi\colon E \longrightarrow M$ (*Configuration fibre bundle*) $\quad \dim\, M = m$, $\dim\, E = n+m$.

$\omega \in \Omega^m(M)$ volume form on M.

Natural coordinates in E adapted to the bundle: (x^ν, y^A) $(\nu = 1,\ldots,m;\ A = 1,\ldots,n)$

$$\omega = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^m \equiv \mathrm{d}^m x$$

$\mathcal{M}\pi \equiv \Lambda_2^m \mathrm{T}^*E \simeq \mathrm{Aff}(J^1\pi, \Lambda^m\mathrm{T}^*M)$ (*Extended multimomentum bundle*).
$\Lambda_2^m\mathrm{T}^*E \hookrightarrow \Lambda^m\mathrm{T}^*E$.

Canonical forms in $\mathcal{M}\pi$: $\quad \Theta \in \Omega^m(\mathcal{M}\pi)$, $\Omega := -\mathrm{d}\Theta \in \Omega^{m+1}(\mathcal{M}\pi)$.

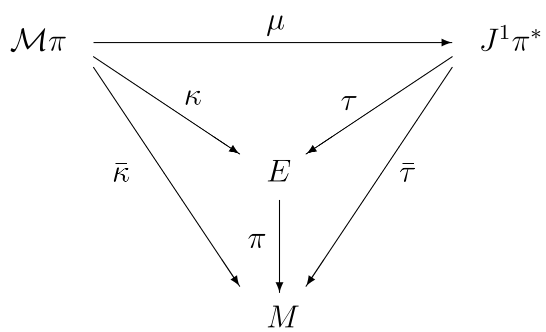
Natural coordinates in $\mathcal{M}\pi$: (x^ν, y^A, p_A^ν, p).

$$\begin{aligned} \Theta &= p_A^\nu \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + p\mathrm{d}^m x \qquad (\text{where } d^{m-1}x_\nu := i\left(\tfrac{\partial}{\partial x^\nu}\right)\mathrm{d}^m x) \\ \Omega &= -\mathrm{d}p_A^\nu \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{d}p \wedge \mathrm{d}^m x \end{aligned} \tag{1}$$

$J^1\pi^* \equiv \Lambda_2^m\mathrm{T}^*E/\pi^*\Lambda^m\mathrm{T}^*E$ (*Restricted multimomentum bundle*).

$\mu\colon \mathcal{M}\pi \longrightarrow J^1\pi^*$.

Natural coordinates in $J^1\pi^*$: (x^ν, y^A, p_A^ν).



2.3 Hamiltonian systems in $J^1\pi^*$

Definition 1 *A section* $h\colon J^1\pi^* \to \mathcal{M}\pi$ *of the projection* μ *is called a* Hamiltonian section. *The differentiable forms* $\Theta_h := h^*\Theta$ *and* $\Omega_h := -\mathrm{d}\Theta_h = h^*\Omega$ *are called the* Hamilton-Cartan m and $(m+1)$ forms *of* $J^1\pi^*$ *associated with the Hamiltonian section* h.

$(J^1\pi^*, h)$ *is a* restricted Hamiltonian system.

In natural coordinates: $h(x^\nu, y^A, p^\nu_A) \equiv (x^\nu, y^A, p^\nu_A, p = -\mathrm{h}(x^\gamma, y^B, p^\eta_B))$.

$\mathrm{h} \in \mathrm{C}^\infty(U)$, $U \subset J^1\pi^*$, is a *local Hamiltonian function*.

$$\Theta_h = p^\nu_A \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu - \mathrm{h}\mathrm{d}^m x \quad , \quad \Omega_h = -\mathrm{d}p^\nu_A \wedge \mathrm{d}y^A \wedge \mathrm{d}^{m-1}x_\nu + \mathrm{dh} \wedge \mathrm{d}^m x$$

Definition 2 *Let* $(J^1\pi^*, h)$ *be a restricted Hamiltonian system. Let* $\Gamma(M, J^1\pi^*)$ *be the set of sections of* $\bar\tau$. *Consider the map*

$$\begin{array}{cccccc} \mathbf{H} & : & \Gamma(M, J^1\pi^*) & \longrightarrow & \mathbb{R} \\ & & \psi & \mapsto & \int_M \psi^*\Theta_h \end{array}$$

The variational problem *for this system is the search of the critical sections of the functional* $\mathbf{H}$, *with respect to the variations of* ψ *given by* $\psi_t = \sigma_t \circ \psi$, *where* $\{\sigma_t\}$ *is a local one-parameter group of every* $Z \in \mathfrak{X}^{\mathrm{V}(\bar\tau)}(J^1\pi^*)$ *($\bar\tau$-vertical vector fields in $J^1\pi^*$).*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \int_M \psi_t^*\Theta_h = 0$$

This is the so-called Hamilton-Jacobi principle *of the Hamiltonian formalism.*

Theorem 1 *The following assertions on a section* $\psi \in \Gamma(M, J^1\pi^*)$ *are equivalent:*

1. ψ *is critical section for the variational problem posed by the Hamilton-Jacobi principle.*
2. $\psi^*\, i(X)\Omega_h = 0$, $\forall\, X \in \mathfrak{X}(J^1\pi^*)$.
3. ψ *is an integral section of an integrable connection* ∇_h *satisfying the equation*
$$i(\nabla_h)\Omega_h = (m-1)\Omega_h$$
4. ψ *is an integral section of an integrable multivector field* $\mathcal{X}_h \in \mathfrak{X}^m(J^1\pi^*)$ *satisfying that*
$$i(\mathcal{X}_h)\Omega_h = 0 \quad , \quad i(\mathcal{X}_h)(\bar\tau^*\omega) = 1 \quad (\bar\tau\text{-transversality}) \tag{2}$$
5. *If* $(U; x^\nu, y^A, p^\nu_A)$ *is a system of coordinates in* $J^1\pi^*$, *then* ψ *satisfies in* U
$$\frac{\partial(y^A\circ\psi)}{\partial x^\nu} = \frac{\partial \mathrm{h}}{\partial p^\nu_A}\Big|_\psi \quad ; \quad \frac{\partial(p^\nu_A\circ\psi)}{\partial x^\nu} = -\frac{\partial \mathrm{h}}{\partial y^A}\Big|_\psi \tag{3}$$
which are the Hamilton-De Donder-Weyl equations *of the rest. Hamiltonian system.*

Definition 3 $\mathcal{X}_h \in \mathfrak{X}^m(J^1\pi^*)$ *is a* Hamilton-De Donder-Weyl (HDW) multivector field *for the system* $(J^1\pi^*, h)$ *if it is locally decomposable and verifies equations (2).*

3 HAMILTONIAN SYSTEMS IN $\mathcal{M}\pi$

3.1 Extended Hamiltonian systems

Definition 4 *$(\mathcal{M}\pi, \Omega, \alpha)$ is an* extended Hamiltonian system *if:*

1. *$\alpha \in Z^1(\mathcal{M}\pi)$.*
2. *There exists a locally decomposable multivector field $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ satisfying that*

$$i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha \quad ; \quad i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = 1 \quad (\bar{\kappa}\text{-transversality}) \tag{4}$$

If α is exact, $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended global Hamiltonian system*. Then $\exists\, \mathrm{H} \in \mathrm{C}^\infty(\mathcal{M}\pi)$ such that $\alpha = \mathrm{dH}$, which are called* Hamiltonian functions *of the system. (For an extended Hamiltonian system, H exist only locally, and they are called* local Hamiltonian functions*).*

In addition, the integrability of $\mathcal{X}_\alpha$ must be also considered.

Proposition 1 *If $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system, then $i(Y)\alpha \neq 0$, $\forall Y \in \mathfrak{X}^{\mathrm{V}(\mu)}(\mathcal{M}\pi)$, $Y \neq 0$. In particular, for every system of natural coordinates $(x^\nu, y^A, p^\nu_A, p))$ in $\mathcal{M}\pi$ adapted to the bundle structure (with $\omega = \mathrm{d}^m x$),*

$$i\left(\frac{\partial}{\partial p}\right)\alpha = 1$$

Proposition 2 *If $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system, locally $\alpha = \mathrm{d}p + \beta$, where β is a closed and μ-basic local 1-form in $\mathcal{M}\pi$.*

$$\alpha = \mathrm{d}p + \mathrm{d}\tilde{\mathrm{h}}(x^\nu, y^A, p^\nu_A) = \mathrm{d}(p + \tilde{\mathrm{h}}(x^\nu, y^A, p^\nu_A)) \equiv \mathrm{dH} \tag{5}$$

where $\tilde{\mathrm{h}} = \mu^*\mathrm{h}$, for some $\mathrm{h} \in \mathrm{C}^\infty(\mu(U))$, $(U \subset \mathcal{M})$.

Theorem 2 *Let $\alpha \in Z^1(\mathcal{M}\pi)$ satisfying the condition stated in Proposition 1. Then there exist locally decomposable multivector fields $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ (not necessarily integrable) satisfying equations (4) (and hence $(\mathcal{M}\pi, \Omega, \alpha)$ is an extended Hamiltonian system). In a local system the above solutions depend on $n(m^2-1)$ arbitrary functions.*

Local expressions: $\mathcal{X}_\alpha = \bigwedge_{\nu=1}^{m} \left(\frac{\partial}{\partial x^\nu} + \tilde{F}_\nu^A \frac{\partial}{\partial y^A} + \tilde{G}_{A\nu}^\rho \frac{\partial}{\partial p_A^\rho} + \tilde{g}_\nu \frac{\partial}{\partial p} \right)$, where

$$\tilde{F}_\nu^A = \frac{\partial \mathrm{H}}{\partial p_A^\nu} = \frac{\partial \tilde{\mathrm{h}}}{\partial p_A^\nu} \qquad (A = 1, \ldots, n, \ \nu = 1, \ldots, m) \tag{6}$$

$$\tilde{G}_{A\mu}^\mu = -\frac{\partial \mathrm{H}}{\partial y^A} = -\frac{\partial \tilde{\mathrm{h}}}{\partial y^A} \qquad (A = 1, \ldots, n) \tag{7}$$

$$\tilde{g}_\nu = -\frac{\partial \tilde{\mathrm{h}}}{\partial x^\nu} + \frac{\partial \tilde{\mathrm{h}}}{\partial p_A^\nu} \tilde{G}_{A\eta}^\eta - \frac{\partial \tilde{\mathrm{h}}}{\partial p_A^\eta} \tilde{G}_{A\nu}^\eta \qquad (A = 1, \ldots, n; \ \eta \neq \nu) \tag{8}$$

Definition 5 *$\mathcal{X}_\alpha \in \mathfrak{X}^m \mathcal{M}\pi)$ is an* extended Hamilton-De Donder-Weyl multivector field *for $(\mathcal{M}\pi, \Omega, \alpha)$ if it is a solution to eqs. (4).*

Integrability of $\mathcal{X}_\alpha$ makes that the number of arbitrary functions $\leq n(m^2-1)$.

If $\mathcal{X}_\alpha$ integrable and $\tilde{\psi}(x)$ is an integral section of $\mathcal{X}_\alpha$, then

$$\frac{\partial(y^A \circ \tilde{\psi})}{\partial x^\nu} = \tilde{F}_\nu^A \circ \tilde{\psi} \quad ; \quad \frac{\partial(p_A^\nu \circ \tilde{\psi})}{\partial x^\nu} = \tilde{G}_{A\nu}^\nu \circ \tilde{\psi} \quad ; \quad \frac{\partial(p \circ \tilde{\psi})}{\partial x^\nu} = \tilde{g}_\nu \circ \tilde{\psi}$$

and equations (6), (7) and (8) give PDE's for the integral sections of $\mathcal{X}_\alpha$.

3.2 Geometric properties of extended Hamiltonian systems

Proposition 3 *Let $(\mathcal{M}\pi, \Omega, \alpha)$ be an extended Hamiltonian system, and $\mathcal{D}_\alpha$ the characteristic distribution of α. Then:*

1. *$\mathcal{D}_\alpha$ is a μ-transverse involutive distribution of corank equal to 1.*
2. *The integral submanifolds S of $\mathcal{D}_\alpha$ are 1-codimensional and μ-transverse local submanifolds of $\mathcal{M}\pi$.*
3. *If S is an integral submanifold of $\mathcal{D}_\alpha$, then $\mu|_S \colon S \to J^1\pi^*$ is a local diffeomorphism.*
4. *For every integral submanifold S of $\mathcal{D}_\alpha$, and $\mathbf{p} \in S$, there exists $W \subset \mathcal{M}\pi$, with $\mathbf{p} \in W$, such that $h = (\mu|_{W\cap S})^{-1}$ is a local Hamiltonian section of μ defined on $\mu(W \cap S)$.*

As $\alpha = \mathrm{dH} = \mathrm{d}(p + \mu^*\mathrm{h})$ (locally), every local Hamiltonian function H is a constraint defining locally the integral submanifolds of $\mathcal{D}_\alpha$. Thus the local Hamiltonian sections associated with these submanifolds are expressed as

$$h(x^\nu, y^A, p_A^\nu) = (x^\nu, y^A, p_A^\nu, p = -\mathrm{h}(x^\gamma, y^B, p_B^\eta))$$

Proposition 4 *Every extended HDW $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ for $(\mathcal{M}\pi, \Omega, \alpha)$ is tangent to every integral submanifold of $\mathcal{D}_\alpha$. As a consequence, if $\mathcal{X}_\alpha$ is integrable, then its integral sections are contained in the integral submanifolds of $\mathcal{D}_\alpha$.*

Using the local expressions of α and $\mathcal{X}_\alpha$, and equations (6) and (7), the tangency condition leads to equations (8), which are just consistency conditions. (See also the comment in Remark 1).

3.3 Relation between extended and restricted Hamiltonian systems

Theorem 3 *Let $(\mathcal{M}\pi, \Omega, \alpha)$ be an extended Hamiltonian system, and $(J^1\pi^*, h)$ a restricted Hamiltonian system such that $\mathrm{Im}\, h = S$ is an integral submanifold of $\mathcal{D}_\alpha$. For every $\mathcal{X}_\alpha \in \mathfrak{X}^m(\mathcal{M}\pi)$ solution to the equations (4)*

$$i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha \quad , \quad i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = 1$$

there exists $\mathcal{X}_h \in \mathfrak{X}^m(J^1\pi^*)$, such that $\Lambda^m h_* X_h = X_\alpha|_S$, which is a solution to the eqs. (2)

$$i(\mathcal{X}_h)\Omega_h = 0 \quad , \quad i(\mathcal{X}_h)(\bar{\tau}^*\omega) = 1$$

Furthermore, if $\mathcal{X}_\alpha$ is integrable, then $\mathcal{X}_h$ is integrable too, and the integral sections of $\mathcal{X}_h$ are recovered from those of $\mathcal{X}_\alpha$ as follows: if $\tilde{\psi}\colon M \to \mathcal{M}\pi$ is an integral section of $\mathcal{X}_\alpha$, then $\psi = \mu \circ \tilde{\psi}\colon M \to J^1\pi^*$ is an integral section of $\mathcal{X}_h$.

Definition 6 *Given an extended Hamiltonian system $(\mathcal{M}\pi, \Omega, \alpha)$, and considering all the Hamiltonian sections $h\colon J^1\pi^* \to \mathcal{M}\pi$ such that $\mathrm{Im}\, h$ are integral submanifolds of $\mathcal{D}_\alpha$, we have a family $\{(J^1\pi^*, h)\}_\alpha$, which is called the* class of restricted Hamiltonian systems associated with *$(\mathcal{M}\pi, \Omega, \alpha)$.*

Proposition 5 *Let $\{(J^1\pi^*, h)\}_\alpha$ be the class of restricted Hamiltonian systems associated with an extended Hamiltonian system $(\mathcal{M}\pi, \Omega, \alpha)$. The submanifolds $\{(S_h, \jmath_{S_h}^*\Omega\})$ are multisymplectomorphic (where $S_h = \mathrm{Im}\, h$, for every Hamiltonian section h in this class, and $\jmath_{S_h}\colon S_h \hookrightarrow \mathcal{M}\pi$ is the natural embedding).*

Proposition 6 *Given a restricted Hamiltonian system $(J^1\pi^*, h)$, let $\jmath_S\colon S = \mathrm{Im}\, h \hookrightarrow \mathcal{M}\pi$ be the natural embedding. Then, there exists a unique local form $\alpha \in \Omega^1(\mathcal{M}\pi)$ such that:*

1. *$\alpha \in Z^1(\mathcal{M}\pi)$ (it is a closed form).*
2. *$\jmath_S^*\alpha = 0$.*
3. *$i(Y)\alpha \neq 0$, for every non-vanishing $Y \in \mathfrak{X}^{\mathrm{V}(\mu)}(\mathcal{M}\pi)$ and, in particular, such that $i\left(\dfrac{\partial}{\partial p}\right)\alpha = 1$, for every system of natural coordinates (x^ν, y^A, p_A^ν, p) in $\mathcal{M}\pi$, adapted to the bundle structure (with $\omega = \mathrm{d}^m x$).*

Definition 7 *Given a restricted Hamiltonian system $(J^1\pi^*, h)$, if $\alpha \in \Omega^1(\mathcal{M}\pi)$ satisfies the above conditions, $(\mathcal{M}\pi, \Omega, \alpha)$ is called the* local extended Hamiltonian system associated with *$(J^1\pi^*, h)$.*

3.4 Variational principle and field equations

$\Gamma_\alpha(M,\mathcal{M}\pi)$: set of sections of $\bar{\kappa}$ which are integral submanifolds of $\mathcal{D}_\alpha$.

$$\mathfrak{X}^{\mathrm{V}(\bar{\kappa})}_\alpha(\mathcal{M}\pi) = \{Z \in \mathfrak{X}(\mathcal{M}\pi) \ \mid \ i(Z)\alpha = 0 \ , \ Z \text{ is } \bar{\kappa}\text{-vertical}\}$$

(Z are $\bar{\kappa}$-vertical vector fields tangent to the integral submanifolds of $\mathcal{D}_\alpha$).

Definition 8 *Let $(\mathcal{M}\pi,\Omega,\alpha)$ be an extended Hamiltonian system. Consider the map*

$$\begin{array}{ccccc} \tilde{\mathbf{H}}_\alpha & : & \Gamma_\alpha(M,\mathcal{M}\pi) & \longrightarrow & \mathbb{R} \\ & & \tilde{\psi} & \mapsto & \int_U \tilde{\psi}^*\Theta \end{array}$$

The variational problem *for this system is the search for the critical sections of the functional $\tilde{\mathbf{H}}_\alpha$, with respect to the variations of $\tilde{\psi} \in \Gamma_\alpha(M,\mathcal{M}\pi)$ given by $\tilde{\psi}_t = \sigma_t \circ \tilde{\psi}$, where $\{\sigma_t\}$ is the local one-parameter group of any compact-supported vector field $Z \in \mathfrak{X}^{\mathrm{V}(\bar{\kappa})}_\alpha(\mathcal{M}\pi)$, that is*

$$\frac{d}{dt}\Big|_{t=0}\int_U \tilde{\psi}_t^*\Theta = 0$$

This is the extended Hamilton-Jacobi principle.

Theorem 4 *The following assertions on $\tilde{\psi} \in \Gamma_\alpha(M,\mathcal{M}\pi)$ are equivalent:*

1. *$\tilde{\psi}$ is a critical section for the Hamilton-Jacobi principle.*
2. *$\tilde{\psi}^*\, i(X)\Omega = 0$, for every $X \in \mathfrak{X}_\alpha(\mathcal{M}\pi)$.*
3. *$\tilde{\psi}$ is an integral section of an integrable multivector field $\mathcal{X} \in \mathfrak{X}^m(\mathcal{M}\pi)$ which is a solution to the equations (4)*

$$i(\mathcal{X}_\alpha)\Omega = (-1)^{m+1}\alpha \quad , \quad i(\mathcal{X}_\alpha)(\bar{\kappa}^*\omega) = 1$$

4. *If $(U;x^\nu,y^A,p^\nu_A,p)$ is a natural system of coordinates in $\mathcal{M}\pi$, then $\tilde{\psi}$ satisfies the following system of equations in U*

$$\frac{\partial(y^A\circ\tilde{\psi})}{\partial x^\nu} = \frac{\partial\tilde{\mathrm{h}}}{\partial p^\nu_A}\Big|_{\tilde{\psi}} \quad , \quad \frac{\partial(p^\nu_A\circ\tilde{\psi})}{\partial x^\nu} = -\frac{\partial\tilde{\mathrm{h}}}{\partial y^A}\Big|_{\tilde{\psi}} \quad , \quad \frac{\partial(p\circ\tilde{\psi})}{\partial x^\nu} = -\frac{\partial(\tilde{\mathrm{h}}\circ\tilde{\psi})}{\partial x^\nu} \qquad (9)$$

   where $\tilde{\mathrm{h}} = \mu^*\mathrm{h}$, for some $\mathrm{h} \in \mathrm{C}^\infty(\mu(U))$, is any function such that $\alpha|_U = \mathrm{d}p + \mathrm{d}\tilde{\mathrm{h}}(x^\nu,y^A,p^\nu_A)$. These are the extended Hamilton-De Donder-Weyl equations *of the extended Hamiltonian system.*

Remark 1 The last group of equations (9) are consistency conditions with respect to the hypothesis on the sections $\tilde{\psi}$. In fact, this group of equations leads to $p\circ\tilde{\psi} = -\tilde{\mathrm{h}}\circ\tilde{\psi} + ctn.$, that is, $\tilde{\psi} \in \Gamma_\alpha(M,\mathcal{M}\pi)$. (See also the comment after Proposition 4). The rest of the equations (9) are just the Hamilton-De Donder-Weyl equations (3) of the restricted case.

4 CONCLUSIONS and OUTLOOK

1. $J^1\pi^*$ is the 'natural' multimomentum phase space for field theories, but it has not a natural multisymplectic structure.

   Restricted Hamiltonian systems are defined by Hamiltonian sections $h\colon J^1\pi^* \to \mathcal{M}\pi$, which are also used to construct the multisymplectic form Ω_h in $J^1\pi^*$.

   Both the geometry and the 'physical information' are coupled in Ω_h.

2. $(\mathcal{M}\pi, \Omega)$ is a canonical multisymplectic manifold.

   Extended Hamiltonian systems are defined by closed 1-forms, $\alpha \in Z^1(\mathcal{M}\pi)$, which must be μ-transverse.

   The geometry (Ω) and the 'physical information' (α) are not coupled.

   Field equations are analogous to the dynamical equations of autonomous mechanical Hamiltonian systems.

3. Every extended Hamiltonian system is associated with a family of restricted Hamiltonian systems.

   Every restricted Hamiltonian system is associated with an extended Hamiltonian systems (at least locally).

4. The definitions of restricted and extended Hamiltonian systems for submanifolds of $J^1\pi^*$ and $\mathcal{M}\pi$ (satisfying suitable conditions) can be achieved. Their properties are analogous to the former case.

5. We hope that some problems could be studied in the extended formalism in an easier way than in the restricted case:

   - Multisymplectic reduction by symmetries.
   - Integrability.
   - Quantization.

   In fact, the extended Hamiltonian formalism has already been used for defining Poisson brackets in field theories.