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Symmetry Groups, Conservation Laws and Group–invariant Solutions of the Membrane Shape Equation

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Abstract

The 6-parameter group of motions in 3-dimensional Euclidean space is recognized as the largest group of point (geometric) transformations admitted by the membrane shape equation – the Euler-Lagrange equation associated with the Helfrich functional in Mongé representation.

The conserved currents of six linearly independent conservation laws, which correspond to the variational symmetries of the membrane shape equation and hold on its smooth solutions, are obtained.

All types of non-equivalent group-invariant solutions of the membrane shape equation are identified via an optimal system of one-dimension subalgebras of rhe symmetry algebra. The reduced equations determining these group-invariant solutions are derived.

Membrane Shape Equation

The equilibrium shape of a lipid membrane (bilayer) is supposed to be determined by the extremals of the Helfrich curvature free energy (shape energy)

$$\begin{aligned} \mathcal{F} &= \frac{k_c}{2}\int (2H + c_0)^2 \, \mathrm{d}A + \bar{k}\int K \mathrm{d}A \\ & \quad +\lambda \int \mathrm{d}A + p\int \mathrm{d}V \end{aligned}$$

Here k_c, $\bar{k}$, c_0, λ and p are real constants representing the bending and Gaussian rigidity of the membrane, the spontaneous curvature, tensile stress and osmotic pressure difference between the outer and inner media; $\mathrm{d}A$ is the area element of the membrane surface $\mathcal{S}$, H and K are the mean and Gaussian curvatures of $\mathcal{S}$, $\mathrm{d}V$ is the volume element.

The corresponding Euler-Lagrange equation

$$\begin{gathered} 2k_c \Delta H + k_c (2H + c_0)(2H^2 - c_0 H - 2K) \\ -2\lambda H + p = 0 \end{gathered}$$

is referred to as the membrane shape equation. Here Δ is the Laplace-Beltrami operator on the surface $\mathcal{S}$.

Mongé representation

Let (x^1, x^2, x^3) be a fixed right-handed rectangular Cartesian coordinate system in the 3D Euclidean space $\mathbb{R}^3$ in which the surface $\mathcal{S}$ is immersed, let this surface be given by the equation

$$\mathcal{S} : x^3 = w(x^1, x^2), \quad (x^1, x^2) \in \mathbf{\Sigma} \subset \mathbb{R}^2$$

and let us take x^1, x^2 to serve as Gaussian coordinates on the surface $\mathcal{S}$.

In the above Mongé representation, the membrane shape equation is to be regarded as a fourth-order partial differential equation in two independent variables x^1, x^2 and one dependent variable w.

Symmetries of the Shape Equation

Group of motions in $\mathbb{R}^3$

Generators	**Characteristics**
translations	
$\mathbf{v}_1 = \dfrac{\partial}{\partial x^1}$	$Q_1 = -w_1$
$\mathbf{v}_2 = \dfrac{\partial}{\partial x^2}$	$Q_2 = -w_2$
$\mathbf{v}_3 = \dfrac{\partial}{\partial w}$	$Q_3 = 1$
rotations	
$\mathbf{v}_4 = -x^2\dfrac{\partial}{\partial x^1} + x^1\dfrac{\partial}{\partial x^2}$	$Q_4 = x^2 w_1 - x^1 w_2$
$\mathbf{v}_5 = -w\dfrac{\partial}{\partial x^1} + x^1\dfrac{\partial}{\partial w}$	$Q_5 = x^1 + w w_1$
$\mathbf{v}_6 = -w\dfrac{\partial}{\partial x^2} + x^2\dfrac{\partial}{\partial w}$	$Q_6 = x^2 + w w_2$

Six linearly independent conservation laws

$$D_\alpha P^\alpha_j = Q_j E(L), \quad \alpha = 1,2, \quad j = 1,\ldots,6$$

exist that hold on the solutions of the shape equation. The respective conserved currents P^α_j are

$$P^\alpha_j = N^\alpha_j L$$

where

$$\begin{aligned} N^\alpha_j &= \xi^\alpha_j - \frac{1}{2} Q_j D_\mu \frac{\partial}{\partial w_{\alpha\mu}} - \frac{1}{2} Q_j D_\mu \frac{\partial}{\partial w_{\mu\alpha}} \\ &\quad + \frac{1}{2}\left(D_\mu Q_j\right)\frac{\partial}{\partial w_{\alpha\mu}} + \frac{1}{2}\left(D_\mu Q_j\right)\frac{\partial}{\partial w_{\mu\alpha}} \end{aligned}$$

are the so-called Noether operators, corresponding to the vector fields $\mathbf{v}_j$ with characteristics Q_j; E is the Euler operator:

$$E = \frac{\partial}{\partial w} - D_\mu \frac{\partial}{\partial w_\mu} + D_\mu D_\nu \frac{\partial}{\partial w_{\mu\nu}} - \cdots$$

D_α are the total derivative operators:

$$D_\alpha = \frac{\partial}{\partial x^\alpha} + w_\alpha \frac{\partial}{\partial w} + w_{\alpha\mu}\frac{\partial}{\partial w_\mu} + \cdots$$

L denotes the Lagrangian density of the Helfrich shape energy functional.

Group-invariant solutions

The vector fields

$$\langle \mathbf{v}_1 \rangle, \quad \langle \mathbf{v}_4 + a\mathbf{v}_3 \rangle, \quad a = \text{const}$$

constitute an optimal system of one-dimensional subalgebras of the symmetry algebra. So, the essentially different group-invariant solutions correspond to the groups generated by the vector fields $\mathbf{v}_1$ and $\mathbf{v}_4 + a\mathbf{v}_3$.

The $G(\mathbf{v}_1)$-invariant solutions are sought in the form:

$$w = W\left(x^1\right)$$

The $G(\mathbf{v}_4 + a\mathbf{v}_3)$-invariant solutions – of the form:

$$w = W(r) + a \arctan\left(\frac{x^2}{x^1}\right), \quad r = \left[\left(x^1\right)^2 + \left(x^2\right)^2\right]^{\frac{1}{2}}$$

Reduced equation for $G(\mathbf{v}_1)$-invariant solutions

$$R_{\mathbf{v}_1} = const$$

$$\begin{aligned} R_{\mathbf{v}_1} &= k_c \left[-\frac{1}{\left(v^2+1\right)^{5/2}} v_{11} + \frac{5v}{2\left(v^2+1\right)^{7/2}} v_1^2 \right] \\ &\quad + \left(\frac{1}{2} k_c c_0^2 + \lambda \right) \frac{v}{\left(v^2+1\right)^{1/2}} \end{aligned}$$

$$v = \mathrm{d}W/\mathrm{d}x^1$$

Lagrangian for $G(\mathbf{v}_1)$-invariant solutions

$$\begin{aligned} L &= k_c \frac{v_1^2}{2\left(v^2+1\right)^{5/2}} + k_c c_0 \frac{v_1}{\left(v^2+1\right)} \\ &\quad + \left(\frac{1}{2} k_c c_0^2 + \lambda \right) \left(v^2+1\right)^{1/2} \end{aligned}$$

Reduced equations for $G\,(\mathbf{v}_4 + a\mathbf{v}_3)$-invariant solutions

$$R = \text{const}$$

$$R = 2k_c R_1 + 2k_c c_0 R_2 + \left(\tfrac{1}{2}k_c c_0^2 + \lambda\right) R_3$$

$$\begin{aligned}
R_1 = \; & -\frac{r^2\left(r^2+a^2\right)^2}{\left[r^2(v^2+1)+a^2\right]^{5/2}} v_{11} \\
& +\frac{5}{2}\,\frac{r^4 v\left(r^2+a^2\right)^2}{\left[r^2(v^2+1)+a^2\right]^{7/2}} v_1^2 \\
& -\frac{r\left(r^2+a^2\right)\left[2\,a^4-3\,r^2\left(v^2-1\right)a^2+\left(v^2+1\right)r^4\right]}{\left[r^2(v^2+1)+a^2\right]^{7/2}} v_1 \\
& +\frac{1}{2}\frac{r^6 v^7}{\left[r^2(v^2+1)+a^2\right]^{7/2}} + \frac{1}{2}\frac{\left(6\,r^4a^2+4\,r^6\right)v^5}{\left[r^2(v^2+1)+a^2\right]^{7/2}} \\
& +\frac{1}{2}\,\frac{\left(18\,r^4a^2+5\,r^6+14\,r^2a^4\right)v^3}{\left[r^2(v^2+1)+a^2\right]^{7/2}} \\
& +\frac{1}{2}\,\frac{\left(4\,a^6+14\,r^2a^4+12\,r^4a^2+2\,r^6\right)v}{\left[r^2(v^2+1)+a^2\right]^{7/2}}
\end{aligned}$$

$$R_2 = \frac{r^2v^2+a^2}{r^2(v^2+1)+a^2} \qquad R_3 = \frac{2r^2v}{\left[r^2(v^2+1)+a^2\right]^{1/2}}$$

$$v = \mathrm{d}W/\mathrm{d}r$$

Lagrangians for $G(\mathbf{v}_4 + a\mathbf{v}_3)$-invariant solutions

$$L = 2k_c L_1 + 2k_c c_0 L_2 + \left(\tfrac{1}{2}k_c c_0^2 + \lambda\right) L_3$$

where

$$L_1 = \frac{\left[r\left(r^2 + a^2\right)v_1 + r^2 v^3 + \left(r^2 + 2\,a^2\right)v\right]^2}{2\left[r^2\left(1 + v^2\right) + a^2\right]^{5/2}}$$

$$L_2 = \frac{r\left(r^2 + a^2\right)v_1 + r^2 v^3 + \left(r^2 + 2\,a^2\right)v}{r^2\left(v^2 + 1\right) + a^2}$$

$$L_3 = 2\left[r^2\left(v^2 + 1\right) + a^2\right]^{1/2}$$