Painlevé analysis and Exact Solutions of Nonintegrable Systems

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1 INTRODUCTION

Let us consider an autonomous differential equation:

$$F(y^{(n)}, y^{(n-1)}, \ldots, y) = 0, \tag{1}$$

where F is a polynomial function.

Our goal is to find the elementary or elliptic solutions. There exist a few ways to do this.

1. Let y are some polynomial or rational function of the known function, for example, $\tanh(t)$, then we using

$$\frac{d\tanh(t)}{dt} = 1 - \tanh(t)^2,$$

we transform our system into system of nonlinear algebraic equations.

2. We assume that

$$y = \sum_{k=1}^{N} P_k \rho(t)^k,$$

where

$$\left(\frac{d\rho}{dt}\right)^2 = \sum_{s=1}^{M} A_s \rho(t)^s,$$

substitute it into (1) and obtain a nonlinear algebraic system.

Two steps: transform (1) into nonlinear algebraic equation and solve the obtained equation.

3. The use of the Laurent series solutions of system (1) to seek elliptic and elementary solutions (R. Conte, M. Musette, Physica D **181** (2003) 70–76; nlin.PS/0302051).

The Ablowitz–Ramani–Segur algorithm of the Painlevé test is very useful for obtaining the solutions as formal Laurent series (S.V. math-ph/0209063).

We will use only a finite number of the first coefficients of the obtained Laurent series, so we don't need in its convergence.

The necessary form of a polynomial autonomous first order ODE with the single-valued general solution is

$$\sum_{k=0}^{m} \sum_{j=0}^{2m-2k} h_{jk}\, y^j y_t^k = 0, \qquad h_{0m} = 1, \tag{2}$$

in which m is a positive integer number and h_{jk} are constants.

The general solution of (2) is either an elliptic function, or a rational function of $e^{\gamma t}$, γ being some constant, or a rational function of t. Note that the third case is a degeneracy of the second one, which in its turn is a degeneracy of the first one.

2 THE HÉNON–HEILES HAMILTONIAN

$$H = \frac{1}{2}\Big(x_t^2 + y_t^2 + \lambda_1 x^2 + \lambda_2 y^2\Big) + x^2 y - \frac{C}{3}\, y^3 + \frac{\mu}{2x^2}$$

The system of the motion equations:

$$\left\{ \begin{array}{rcl} x_{tt} & = & -\,\lambda_1 x - 2xy + \dfrac{\mu}{x^3}, \\ y_{tt} & = & -\,\lambda_2 y - x^2 + Cy^2, \end{array} \right. \qquad (3)$$

where $x_{tt} \equiv \frac{d^2x}{dt^2}$ and $y_{tt} \equiv \frac{d^2y}{dt^2}$, λ_1, λ_2, μ and C are arbitrary numerical parameters. Note that if $\lambda_2 \neq 0$, then one can put $\lambda_2 = sign(\lambda_2)$ without loss of generality. If $C = 1$, $\lambda_1 = 1$, $\lambda_2 = 1$ and $\mu = 0$, then (3) is the initial Hénon–Heiles system.

The function y satisfies the following fourth-order equation, which does not include μ:

$$\begin{array}{rcl} y_{tttt} & = & (2C-8)y_{tt}y - (4\lambda_1 + \lambda_2)y_{tt} + 2(C+1)y_t^2 + \\ & + & \frac{20C}{3}y^3 + (4C\lambda_1 - 6\lambda_2)y^2 - 4\lambda_1\lambda_2 y - 4H. \end{array} \qquad (4)$$

We note that the energy of the system H is not an arbitrary parameter, but a function of initial data: y_0, y_{0t}, y_{0tt} and y_{0ttt}. The form of this function depends on μ.

Due to the Painlevé analysis the following integrable cases have been found:

$$\begin{array}{lll} \text{(i)} & C = -1, & \lambda_1 = \lambda_2, \\ \text{(ii)} & C = -6, & \lambda_1,\ \lambda_2 \text{ arbitrary}, \\ \text{(iii)} & C = -16, & \lambda_1 = \lambda_2/16. \end{array}$$

3 The nonlinear algebraic system

To find a special solution of eq. (4) we used the substitution (E.I. Timoshkova, S.V., math-ph/0402049):

$$y(t) = \varrho(t)^2 + P_0,$$

$$\begin{gathered}\varrho_{ttttt}\varrho = -4\varrho_{ttt}\varrho_t - 3\varrho_{tt}^2 + 2(C-4)\varrho_{tt}\varrho^3 + (2P_0(C-4) - 4\lambda_1 - \\ - \lambda_2)\varrho_{tt}\varrho + (6C-4)\varrho_t^2\varrho^2 + (2CP_0 - 4\lambda_1 - 8P_0 - \lambda_2)\varrho_t^2 - 2H + \\ + \tfrac{10}{3}C\varrho^6 + (2C\lambda_1 + 10CP_0 - 3\lambda_2)\varrho^4 + 2(2\lambda_1 CP_0 + 5CP_0^2 - \\ - \lambda_1\lambda_2 - 3P_0\lambda_2)\varrho^2 + \tfrac{10}{3}CP_0^3 + 2\lambda_1 CP_0^2 - 3P_0^2\lambda_2 - 2\lambda_1\lambda_2 P_0.\end{gathered} \tag{5}$$

Let

$$\varrho_t^2 = \frac{1}{4}(A_4\varrho^4 + A_3\varrho^3 + A_2\varrho^2 + A_1\varrho + A_0). \tag{6}$$

then

$$\left\{\begin{array}{l}(3A_4+4)(-3A_4+2C) = 0, \\ A_3(-21A_4 + 9C - 16) = 0, \\ 96A_4CP_0 - 240A_4A_2 - 192A_4\lambda_1 - 384A_4P_0 - 48A_4\lambda_2 - \\ \quad - 105A_3^2 + 128A_2C - 192A_2 + 128C\lambda_1 + \\ \quad + 640CP_0 - 192\lambda_2 = 0, \\ 40A_3CP_0 - 90A_4A_1 - 65A_3A_2 - 80A_3\lambda_1 - 160A_3P_0 - \\ \quad - 20A_3\lambda_2 + 56CA_1 - 64A_1 = 0, \\ 16A_2CP_0 - 36A_4A_0 - 21A_3A_1 - 8A_2^2 - 32A_2\lambda_1 - \\ \quad - 64A_2P_0 - 8\lambda_2A_2 + 24CA_0 + 64\lambda_1CP_0 + 160CP_0^2 - \\ \quad - 16A_0 - 32\lambda_1\lambda_2 - 96P_0\lambda_2 = 0, \\ 10A_3A_0 + (5A_2 + 8CP_0 - 16\lambda_1 - 32P_0 - 4\lambda_2)A_1 = 0.\end{array}\right. \tag{7}$$

and

$$H = (-48A_2A_0 + 96CA_0P_0 + 384C\lambda_1P_0^2 + 640CP_0^3 - 9A_1^2 - \\ - 192A_0\lambda_1 - 384A_0P_0 - 48A_0\lambda_2 - 384\lambda_1\lambda_2P_0 - 576\lambda_2P_0^2)/384.$$

This system has been solved by REDUCE using the standard function **solve** and the Groebner basis method. The method, which uses the Laurent series solution, allows to obtain some solutions of (7) solving only the linear system and nonlinear equation in one variable.

We can not use this method for arbitrary C, because the Laurent series solutions are different for different C. We will consider the case $A_5 \neq 0$. In this case from two first equations of system (7) we obtain:

$$C = -\frac{4}{3} \quad \text{and} \quad A_4 = -\frac{4}{3} \quad \text{or} \quad C = -\frac{16}{5} \quad \text{and} \quad A_4 = -\frac{32}{15}.$$

4 Construction of linear algebraic system

Let choose $C = -4/3$ and construct the Laurent series solutions for eq. (5) using the Ablowitz–Ramani–Segur algorithm of the Painlevé test.

Solutions of eq. (5) with $C = -4/3$ have singularities proportional to $1/t$ and the values of resonances are -1 (correspond to t_0), 1, 4 and 10. We obtain the following Laurent series solutions (functions $\tilde{\rho}$ and $-\tilde{\rho}$ correspond to one and the same $\tilde{y}$):

$$\tilde{\rho} = \pm\left(\frac{i\sqrt{3}}{t} + c_0 + \frac{i\sqrt{3}}{24}(3\lambda_2 - 2\lambda_1 + 4P_0 + 62c_0^2)t + \ldots\right)$$

where

$$c_0 = \frac{\pm\sqrt{161700\lambda_1 - 121275\lambda_2 \pm \sqrt{1155(5481\lambda_2^2 - 12768\lambda_1\lambda_2 + 8512\lambda_1^2)}}}{2310}.$$

Two signs " $\pm$ " are independent, so we obtain four different Laurent series solutions. The coefficients c_3 and c_9 are arbitrary.

The algorithm is the following:

- Choose a positive integer m and define the first order ODE (2), which contains unknown constants h_{jk}.
- Compute coefficients of the Laurent series $\tilde{\rho}$.
- Substituting the obtained coefficients, transform eq. (2) into a linear and overdetermined system in h_{jk}.
- Exclude h_{jk} and solve the obtained the nonlinear system in parameters of the Laurent-series solutions.

On the first step we choose eq. (2), which coincides with eq. (6). It means that $m = 2$, all h_{j1} are equal to zero and all $h_{j0} = -A_j/4$. We have five unknowns coefficients A_j and six parameters: λ_1, λ_2, P_0 H, c_3 and c_9, so we need of twelve first coefficients of the Laurent series to obtain the overdetermined system. After the third steps we obtain a linear system in A_j. This system has the triangular form and linear in H, c_3 and c_9 as well. From the first equation we obtain,

$$A_4 = -4/3.$$

From the second equation follows

$$A_3 = \frac{16}{3} c_0,$$

and so on:

$$A_2 = -70c_0^2 - 3\lambda_2 + 2\lambda_1 - 4P_0,$$

$$A_1 = \left(\frac{40}{3}P_0 - 60\lambda_1 + 50\lambda_2 + 1300c_0^2\right) c_0,$$

$$A_0 = -40i\sqrt{3}c_3 - \frac{21535}{12}c_0^4 + \left(\frac{565}{6}\lambda_1 - \frac{405}{4}\lambda_2 - \frac{245}{3}P_0\right) c_0^2 + \\ + \frac{7}{4}\lambda_1\lambda_2 - \frac{21}{16}\lambda_2^2 - \frac{7}{12}\lambda_1^2 + \frac{7}{3}\lambda_1 P_0 - \frac{7}{2}\lambda_2 P_0 - \frac{7}{3}P_0^2.$$

From the next equation we obtain c_3 and finally we obtain

$$A_0 = \frac{15645}{4}c_0^4 + \left(-465P_0 - \frac{1495}{2}\lambda_1 + \frac{1545}{4}\lambda_2\right) c_0^2 + \\ + \frac{537}{20}\lambda_1^2 - \frac{663}{20}\lambda_1\lambda_2 + \frac{729}{80}\lambda_2^2 + 19\lambda_1 P_0 - \frac{37}{2}\lambda_2 P_0 - \frac{17}{3}P_0^2.$$

Substituting the values of A_k, which correspond to one of the possible values of c_0, in the system (7) we obtain that this system is satisfied for all values of λ_1, λ_2 and P_0. Therefore we find solutions of the nonlinear algebraic system solving only linear equations and nonlinear equation in one variable.

5 The proof of non-existence of elliptic solutions of the cubic complex Ginzburg–Landau equation

The one-dimensional cubic complex Ginzburg–Landau equation (CGLE)

$$\mathrm{i}A_t + pA_{xx} + q|A|^2A - \mathrm{i}\gamma A = 0, \tag{8}$$

$A_t \equiv \frac{\partial A}{\partial t}$, $A_{xx} \equiv \frac{\partial^2 A}{\partial x^2}$, $p \in \mathbb{C}$, $q \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ is not integrable if $pq\gamma \neq 0$. In the case $q/p \in \mathbb{R}$, $\gamma = 0$ the CGLE coincides with the well-known nonlinear Schrödinger equation. The CGLE is one of the most-studied nonlinear equations (I. Aranson, L. Kramer, *The World of the Complex Ginzburg–Landau Equation*, *Rev. Mod. Phys.* **74** (2002) 99–143, (cond-mat/0106115)).

Let us consider travelling wave reduction:

$$A(x,t) = \sqrt{M(\xi)}e^{\mathrm{i}(\varphi(\xi) - \omega t)}, \quad \xi = x - ct, \quad c, \omega \in \mathbb{R},$$

which defines the following third order system

$$\begin{cases} \dfrac{M''}{2M} - \dfrac{M'^2}{4M^2} - \left(\psi - \dfrac{cs_r}{2}\right)^2 - \dfrac{cs_iM'}{2M} + d_rM + g_i = 0, \\ \psi' + \left(\psi - \dfrac{cs_r}{2}\right)\left(\dfrac{M'}{M} - cs_i\right) + d_iM - g_r = 0, \end{cases} \tag{9}$$

where $\psi \equiv \varphi' \equiv \frac{d\varphi}{d\xi}$, $M' \equiv \frac{dM}{d\xi}$.

Six real parameters d_r, d_i, g_r, g_i, s_r and s_i are given in terms of c, p, q, γ and ω as

$$d_r+\mathrm{i}d_i=\frac{q}{p},\quad s_r-\mathrm{i}s_i=\frac{1}{p},\quad g_r+\mathrm{i}g_i=\frac{\gamma+\mathrm{i}\omega}{p}+\frac{1}{2}c^2s_is_r+\frac{\mathrm{i}}{4}c^2s_r^2.$$

Using (9) one can express ψ in terms of M and its derivatives:

$$\psi=\frac{cs_r}{2}+\frac{G'-2cs_iG}{2M^2(g_r-d_iM)}, \tag{10}$$

where

$$G\equiv\frac{1}{2}MM''-\frac{1}{4}M'^2-\frac{cs_i}{2}MM'+d_rM^3+g_iM^2, \tag{11}$$

and obtain the third order equation in M:

$$(G'-2cs_iG)^2-4GM^2(d_iM-g_r)^2=0. \tag{12}$$

We will consider the case

$$\frac{p}{q}\notin\mathbb{R}. \tag{13}$$

System (9) includes seven arbitrary constants, some of them can be fixed without loss of generality. First of all one can fix s_r and s_i,

$$s_r=\,-\frac{1}{10}\qquad\text{and}\qquad s_i=\,-\frac{3}{10}. \tag{14}$$

and also d_r or d_i using the transformations

$$\tilde{M}=\mu M,\qquad \tilde{d}_i=\frac{d_i}{\mu},\qquad \tilde{d}_r=\frac{d_r}{\mu}. \tag{15}$$

The known facts about the CGLE:

- In this case equation (12) is not integrable, which means that the general solution (which should depend on three arbitrary integration constants) is not known. Using the Painlevé analysis it has been shown that single-valued solutions can depend on only one arbitrary parameter.
- Equation (12) is autonomous, so this parameter is ξ_0: if $M = f(\xi)$ is a solution, then $M = f(\xi - \xi_0)$, where $\xi_0 \in \mathbb{C}$ has to be a solution.
- All known exact solutions of the CGLE are elementary (rational, trigonometric or hyperbolic) functions.
- In 2003 R.Conte and M.Musette developed a new method to search single-valued particular solutions. Rather than looking for an explicit, closed form expression, the look for the first order polynomial autonomous ODE for $M(\xi)$. This method allows to find either elliptic or elementary solutions. It is based on the Painlevé analysis and uses the formal Laurent-series solutions. It has been shown that the CGLE has two different Laurent-series solutions.
- In 2004 A.N.W. Hone has proved that a necessary condition for eq. (12) to admit elliptic solutions is $c = 0$.

Our goal is to prove that eq. (12) does not admit elliptic solutions in the case $c = 0$ as well. In other words, neither travelling nor standing wave solutions are elliptic functions.

6 Elliptic functions

The function $\varrho(z)$ of the complex variable z is a doubly-periodic function if there exist two numbers ω_1 and ω_2 with $\omega_1/\omega_2 \notin \mathbb{R}$, such that for all $z \in \mathbb{C}$

$$\varrho(z) = \varrho(z + \omega_1) = \varrho(z + \omega_2). \tag{16}$$

A double-periodic meromorphic function is called an elliptic function.

The classical theorems for elliptic functions prove that

- **If an elliptic function has no poles then it is a constant.**
- **The number of elliptic function poles within any finite period parallelogram is finite.**
- **The sum of residues within any finite period parallelogram is equal to zero (the residue theorem).**
- **If $\varrho(z)$ is an elliptic function then any rational function of $\varrho(z)$ and its derivatives is an elliptic function as well.**

If M is an elliptic function then ψ has to be an elliptic function. Therefore, if we prove that ψ can not be an elliptic function, we prove that M can not be an elliptic function as well.

7 Nonexistence of the standing wave elliptic solutions

7.1 The Laurent-series solutions

We denote the different Laurent series for the function ψ as

$$\psi_1 = \sum_{k=-1}^{\infty} C_k(\xi - \xi_0)^k \quad \text{and} \quad \psi_2 = \sum_{k=-1}^{\infty} D_k(\xi - \tilde{\xi}_0)^k, \quad (17)$$

with $C_{-1} \neq 0$ and $D_{-1} \neq 0$. A nonconstant elliptic function should have poles. Let $\psi(\xi)$ in some parallelogram of periods has $N_1 + N_2$ poles, its Laurent series expansions are ψ_1 in the neighbourhood of N_1 poles and are ψ_2 in the neighbourhood of N_2 poles. If $\psi(\xi)$ is an elliptic function then the sum of its residues in some parallelogram of periods has to be zero, therefore, this function has both types of the Laurent series expansions (17) and

$$N_1 = -\frac{D_{-1}}{C_{-1}} N_2. \quad (18)$$

If $\psi(\xi)$ is an elliptic function then powers ψ^k have to be elliptic functions as well.

If we demand that the functions ψ^2, ψ^3, ψ^4 and ψ^5 are elliptic, then, using (18), we obtain the following system on

C_k and D_k:

$$\begin{cases} C_0 = D_0, \\ C_1C_{-1} + C_0^2 = D_1D_{-1} + D_0^2, \\ C_2C_{-1}^2 + 3C_1C_0C_{-1} + C_0^3 = D_2D_{-1}^2 + 3D_1D_0D_{-1} + D_0^3, \\ C_3C_{-1}^3 + 4C_2C_0C_{-1}^2 + 2C_1^2C_{-1}^2 + 6C_{-1}C_0^2C_1 + C_0^4 = \\ \ = D_3D_{-1}^3 + 4D_2D_0D_{-1}^2 + 2D_1^2D_{-1}^2 + 6D_1D_0^2D_{-1} + D_0^4. \end{cases} \quad (19)$$

We have calculated the residues of ψ^k with the help of the procedure **ydegree** from our package of Maple procedures (S.V., nlin.SI/0407062).

7.2 The case $d_r = 0$

Let us consider system (9) with

$$d_r = 0, \qquad s_r = -\frac{1}{10} \qquad \text{and} \qquad s_i = -\frac{3}{10}. \qquad (20)$$

From (13) it follows that $d_i \neq 0$, therefore, there exist two different Laurent-series solutions (we put $\xi_0 = \tilde{\xi}_0 = 0$) of (9):

$$\breve{\psi}_1 = \frac{\sqrt{2}}{\xi} - \frac{c(\sqrt{2}+1)}{20} + \mathcal{O}(\xi), \qquad (21)$$

$$\breve{M}_1 = \frac{3\sqrt{2}}{d_i}\left(\frac{1}{\xi^2} - \frac{1}{10\xi}\right) + \mathcal{O}(1) \qquad (22)$$

and

$$\breve{\psi}_2 = -\frac{\sqrt{2}}{\xi} + \frac{c(\sqrt{2}-1)}{20} + \mathcal{O}(\xi), \qquad (23)$$

$$\breve{M}_2 = -\frac{3\sqrt{2}}{d_i}\left(\frac{1}{\xi^2} - \frac{1}{10\xi}\right) + \mathcal{O}(1). \qquad (24)$$

From (18) it follows that $N_1 = N_2$.

From the first equation of system (19) ($C_0 = D_0$) we obtain that the sum of residues of the function ψ^2 is equal to zero if and only if $c = 0$. So, we prove the absence of the travelling wave solutions. In the case $c = 0$ we have to apply the residue theorem for ψ_3 and ψ_4, so, we have to calculate four coefficients in these series (two of them are zero at $c = 0$)

$$\breve{\psi}_1 = \frac{\sqrt{2}}{\xi} + \frac{0}{\xi} + \frac{1}{21}\left(5\sqrt{2}g_i - g_r\right)\xi + 0\xi^2 + \mathcal{O}(\xi^3) \qquad (25)$$

and

$$\breve{\psi}_2 = -\frac{\sqrt{2}}{\xi} + \frac{0}{\xi} - \frac{1}{21}\left(5\sqrt{2}g_i + g_r\right)\xi + 0\xi^2 + \mathcal{O}(\xi^3). \qquad (26)$$

From the second and the third equations of (19) we obtain that the functions ψ^3 and ψ^4 satisfy the residue theorem if and only if

$$g_i = 0 \qquad \text{and} \qquad g_r = 0. \qquad (27)$$

In this case the Laurent-series solutions give exact solutions

$$\breve{\psi}_1(\xi) = \frac{\sqrt{2}}{\xi}, \qquad \breve{M}_1(\xi) = \frac{3\sqrt{2}}{d_i\xi^2} \qquad (28)$$

and

$$\breve{\psi}_2(\xi) = -\frac{\sqrt{2}}{\xi}, \qquad \breve{M}_2(\xi) = -\frac{3\sqrt{2}}{d_i\xi^2}. \qquad (29)$$

The case $d_r \neq 0$ has been considered in (S.V., nlin.SI/0503009) and the non-existence of elliptic solutions has been proved.

8 Conclusions

- It is easy to find formal Laurent-series solutions of a non-linear differential equation.
- These solutions assist to find elementary and elliptic solutions. (S.V. astro-ph/0502356)
- These solutions assist to prove the nonexistence of elliptic solutions (S.V. nlin.SI/0503009).