Topology and Geometry of Coadjoint Orbits of Semisimple Lie Groups

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We consider the following problems

1) explicit parameterization of an orbit in terms of 'good' complex coordinates (generalized stereographic projection),
2) Kählerian structure (Kirillov-Kostant-Souriau differential form) and G-invariant basis of cohomology groups.

As an example we use coadjoint orbits (general and degenerate) of group $SU(n)$.

Geometry of coadjoint orbits

Let G be a compact semisimple classical Lie group with Lie algebra $\mathfrak{g}$; $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$.

Definition. The set $\mathcal{O}_\mu = \{\mathrm{Ad}_g^* \mu, \forall g \in G\}$ is called a **coadjoint orbit** of the group G through $\mu \in \mathfrak{g}^*$.

In the case of **classical Lie group** $\mathrm{Ad}_g^* \mu = g^{-1}\mu g$. ***Coajoint orbit coincides with adjoint one***, which we define by $\mathrm{Ad}_g X = gXg^{-1}$, $X \in \mathfrak{g}$.

Theorem (R. Bott). *For each $\mu \in \mathfrak{g}^*$, the coadjoint orbit $\mathcal{O}_\mu$ intersects $\mathfrak{h}^*$ in a finite non-empty set of points, which is an orbit of the Weyl group $W(G)$.*

A **Weyl group** is a finite group generated by reflectings w_α across the hyperplanes orthogonal to simple roots α:

$$w_\alpha(\mu) = \mu - 2\tfrac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle}\, \alpha, \qquad \mu \in \mathfrak{h}^*.$$

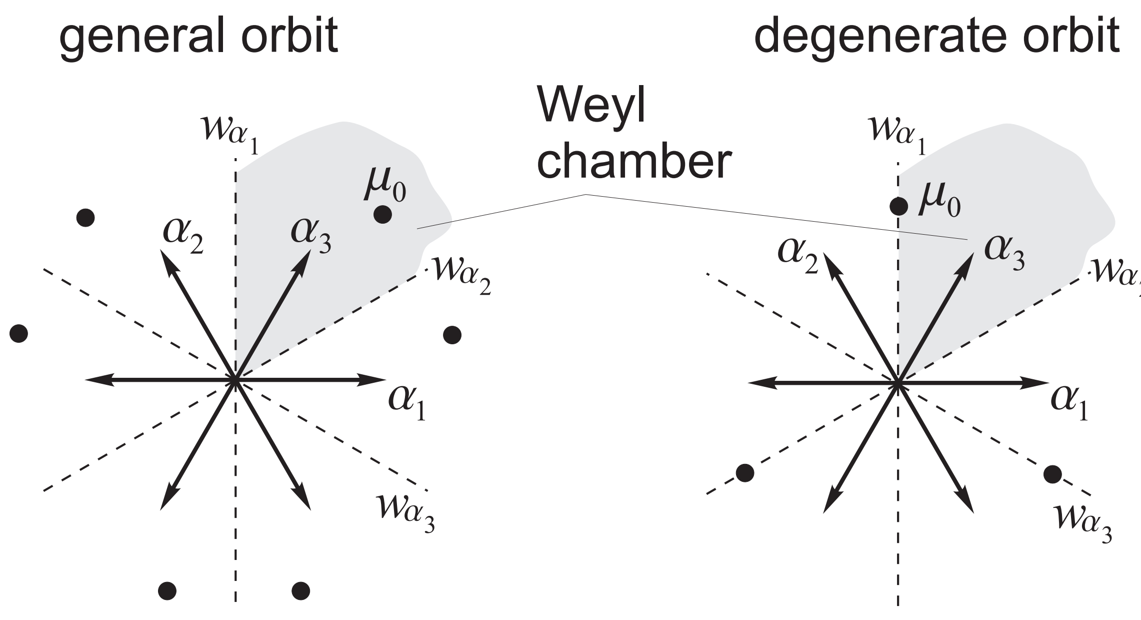
A **Weyl chamber** is an open domain in $\mathfrak{h}^*$ such that $C = \{\mu \in \mathfrak{h}^* : \langle \mu, \alpha \rangle > 0, \forall \alpha \in \Delta^+\}$.

A **wall** of the Weyl chamber is the set

$\Gamma_\alpha = \{\mu \in \mathfrak{h}^* : \langle \mu, \alpha \rangle = 0\}$.

Obviously, $\mathfrak{h}^* = \bigcup_w w \cdot \overline{C}, \quad w \in W(G)$.

Example. Group $SU(3)$.



$$\alpha_1 \mapsto (1, -1, 0), \quad \alpha_2 \mapsto (0, 1, -1), \quad \alpha_3 \mapsto (1, 0, -1)$$

Statement. *Each orbit $\mathcal{O}$ of G is uniquely defined by an element μ_0 of the closed Weyl chamber $\overline{C}$.*

If $\mu_0 \in C$, then the orbit is **general** (flag manifold).

If $\mu_0 \in \Gamma_\alpha$, $\alpha \in \Delta^+$, then the orbit is **degenerate**.

Coadjoint orbit as a bundle

For the **general orbit** ($\mu_0 \in C$)

$$\mathcal{O}_{\mu_0} = T\backslash G, \quad T \simeq G_{\mu_0},$$

T is the maximal torus of G (the Cartan subgroup),

$G_{\mu_0} = \{g \in G : \ \mathrm{Ad}_g^* \mu_0 = \mu_0\}$ is a stability subgroup.

For the **degenerate orbit** ($\mu_0 \in \Gamma_\alpha$, $\alpha \in \Delta^+$)

$$\mathcal{O}_{\mu_0} = G_{\mu_0}\backslash G, \quad G \supset G_{\mu_0} \supset T.$$

Proposition 1. *Suppose $\mathcal{O}_{\mu_0} = G_{\mu_0}\backslash G$ is not the maximal degenerate orbit in G.*
Then a subgroup K such that $G \supset K \supset G_{\mu_0}$ exists, and $\mathcal{O}_{\mu_0}$ is a holomorphic bundle over $K\backslash G$ with fibre $G_{\mu_0}\backslash K$:

$$\mathcal{O}_{\mu_0} \simeq G_{\mu_0}\backslash K \ltimes K\backslash G.$$

Example. Group $SU(n)$.

The only orbit of $SU(2)$ is
$\mathcal{O}^{SU(2)} = \frac{SU(2)}{U(1)} \simeq \mathbb{C}\mathrm{P}^1$.

The orbits of $SU(3)$:
$\mathcal{O}^{SU(3)} = \frac{SU(3)}{U(1)\times U(1)}$,
$\mathcal{O}^{SU(3)}_{\text{degen}} = \frac{SU(3)}{SU(2)\times U(1)} \simeq \mathbb{C}\mathrm{P}^2$.

$$\mathcal{O}^{SU(3)} \simeq \mathcal{O}^{SU(3)}_{\text{degen}} \rtimes \mathcal{O}^{SU(2)} \simeq \mathbb{C}\mathrm{P}^2 \rtimes \mathbb{C}\mathrm{P}^1$$

The orbits of $SU(4)$:
$\mathcal{O}^{SU(4)} = \frac{SU(4)}{U(1)\times U(1)\times U(1)}$,
$\mathcal{O}^{SU(4)}_1 = \frac{SU(4)}{SU(2)\times U(1)\times U(1)}$,
$\mathcal{O}^{SU(4)}_2 = \frac{SU(4)}{S(U(2)\times U(2))}$,
$\mathcal{O}^{SU(4)}_3 = \frac{SU(4)}{SU(3)\times U(1)} \simeq \mathbb{C}\mathrm{P}^3$.

$$\mathcal{O}^{SU(4)} \simeq \mathcal{O}^{SU(4)}_3 \rtimes \mathcal{O}^{SU(3)} \simeq \mathbb{C}\mathrm{P}^3 \rtimes \mathbb{C}\mathrm{P}^2 \rtimes \mathbb{C}\mathrm{P}^1$$

Examples. Classical Lie groups

The maximal tori are

$$SU(n) \qquad T = \underbrace{U(1) \times U(1) \times \cdots \times U(1)}_{n-1},$$

$$SO(2n) \qquad T = \underbrace{SO(2) \times SO(2) \times \cdots \times SO(2)}_{n},$$

$$SO(2n+1) \quad T = \underbrace{SO(2) \times SO(2) \times \cdots \times SO(2)}_{n},$$

$$Sp(n) \qquad T = \underbrace{U(1) \times U(1) \times \cdots \times U(1)}_{n-1}.$$

For the general orbits

$$\mathcal{O}^{SU(n)} = \mathbb{C}\mathrm{P}^{n-1} \rtimes \mathcal{O}^{SU(n-1)},$$

$$\mathcal{O}^{SO(2n)} = G_{2n;2} \rtimes \mathcal{O}^{SO(2n-2)},$$

$$\mathcal{O}^{SO(2n+1)} = G_{2n-1;2} \rtimes \mathcal{O}^{SO(2n-1)},$$

$$\mathcal{O}^{Sp(n)} = \mathbb{H}\mathrm{P}^{n-1} \rtimes \mathcal{O}^{Sp(n-1)}.$$

$G_{2n;2}$, $G_{2n-1;2}$ are real Grassman manifolds,

$\mathbb{H}$ is the quaternionic space.

1. Complex parameterization of an orbit

To **introduce complex structure** we complexify G:

$$G^{\mathbb{C}} = \text{exp}\{\mathfrak{g} + i\mathfrak{g}\}$$

and use the diffeomorphism (D. Montgomery)

$$\mathcal{O} = T\backslash G \simeq P_0\backslash G^{\mathbb{C}}, \qquad (1)$$

P_0 denotes the minimal parabolic subgroup in $G^{\mathbb{C}}$.

(1) takes place for the **general orbit**.

Indeed, $G^{\mathbb{C}}$ admits an **Iwasawa decomposition**

$$G^{\mathbb{C}} = NAG,$$

N corresponds to a nilpotent subalgebra of $\mathfrak{g}$,

A corresponds to an abelian subalgebra of $\mathfrak{g}$,

G is the maximal compact subgroup in $G^{\mathbb{C}}$.

In this connection, $P_0 = NAT$, then (1) is evident.

For the **degenerate orbit** instead of (1) we have

$$G_{\mu_0}\backslash G \simeq P_M\backslash G^{\mathbb{C}},$$

$P_M = NAM$ is a non-minimal parabolic subgroup.

Generalized stereographic projection

Holod P.I., Skrypnyk T.V. *Ukrainian Mathematical Journal* **50** (1998), 1504–1512.

Gauss decomposition gives

$$G^{\mathbb{C}} = NT^{\mathbb{C}}Z,$$

$T^{\mathbb{C}}$ is the Cartan subgroup of $G^{\mathbb{C}}$, $T^{\mathbb{C}} = AT$;

N and $Z = N^*$ are nilpotent subgroups of $G^{\mathbb{C}}$ normalized by $T^{\mathbb{C}}$.

It is obvious that $\mathcal{O} = \frac{NAG}{NAT} = \frac{NATZ}{NAT} = Z.$

Proposition 2. *Z gives complex parameters of stereographic projection for the orbit $\mathcal{O}$.*

Gauss decomposition is local and covers one map.

Gauss-Bruhat decomposition is global

$$G^{\mathbb{C}} = \bigcap_{w \in W(G)/W(T)} P_0 Z w,$$

$$Z = \exp\left\{\sum_{\alpha\in\Delta^+} z_\alpha X_{-\alpha}\right\},$$

$X_{-\alpha}$ are negative root vectors,

z_α denotes complex coordinates, $\alpha \in \Delta^+$.

Example. Complexification of $SU(n)$ is $SL(n,\mathbb{C})$.

Decomposition gives

$$\widehat{z} = \widehat{n}\widehat{a}\widehat{u}, \qquad \widehat{u} \in G = SU(n),$$

$Z \ni \widehat{z}$ — low triangular matrix (1s on the diagonal)

$N \ni \widehat{n}$ — upper triangular matrix (1s on the diagonal)

$A \ni \widehat{a}$ is a real diagonal matrix

$\widehat{a} = \mathrm{diag}(a_1, a_2, a_3), \quad a_i > 0, \quad a_1 a_2 a_3 = 1.$

$SU(3)$:

$$\begin{pmatrix} 1 & 0 & 0 \\ z_1 & 1 & 0 \\ z_3 & z_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & n_1 & n_3 \\ 0 & 1 & n_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a_1} & 0 & 0 \\ 0 & \frac{a_1}{a_2} & 0 \\ 0 & 0 & a_2 \end{pmatrix} \widehat{u}$$

For the **general orbit**

$$a_2^2 = 1 + |z_2|^2 + |z_3|^2, \quad a_1^2 = 1 + |z_1|^2 + |z_3 - z_1 z_2|^2,$$

$$n_1 = \frac{1}{a_1^2}(\bar{z}_1(1 + |z_2|^2) - z_2 \bar{z}_3),$$

$$n_2 = \frac{1}{a_2^2}(\bar{z}_2 + z_1 \bar{z}_3), \quad n_3 = \frac{\bar{z}_3}{a_2^2}$$

For the **degenerate orbit** assign $z_1 = 0$.

A coadjoint orbit is generated by the formula

$$\widehat{\mu} = \widehat{u}^* \widehat{\mu}_0 \widehat{u}, \qquad \widehat{\mu}_0 = \widehat{\mu}(\infty)$$

with the initial point $\widehat{\mu}_0$ in the Weyl chamber.

Suppose $\widehat{\mu} = \begin{pmatrix} \mu_3 + \frac{1}{\sqrt{3}}\mu_8 & \mu_1 - i\mu_2 & \mu_4 - i\mu_5 \\ \mu_1 + i\mu_2 & -\mu_3 + \frac{1}{\sqrt{3}}\mu_8 & \mu_6 - i\mu_7 \\ \mu_4 + i\mu_5 & \mu_6 + i\mu_7 & -\frac{2}{\sqrt{3}}\mu_8 \end{pmatrix}$

Let $\widehat{\mu}_0$ have the following form:

for the degenerate orbit $\qquad \widehat{\mu}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$

for the general orbit (ξ, $\eta \in \mathbb{R}$)

$$\widehat{\mu}_0 = \frac{2}{3}\xi \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{2}{3}\eta \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

If we denote $m = \mu_3(\infty)$, $q = \mu_8(\infty)$, then

$$\eta = -\frac{1}{2}\left(m - \sqrt{3}q\right), \quad \xi = m.$$

Complex parameterization of the **general orbit**

$$
\begin{aligned}
\mu_1 &= -\frac{\eta}{a_2^2}(\bar{z}_2 z_3 + z_2 \bar{z}_3) - \frac{\xi}{a_1^2}(z_1 + \bar{z}_1), \\
\mu_2 &= \quad \frac{i\eta}{a_2^2}(\bar{z}_2 z_3 - z_2 \bar{z}_3) + \frac{i\xi}{a_1^2}(z_1 - \bar{z}_1), \\
\mu_3 &= \quad \frac{\eta}{a_2^2}(|z_2|^2 - |z_3|^2) + \frac{\xi}{a_1^2}(1 - |z_1|^2), \\
\mu_4 &= -\frac{\eta}{a_2^2}(z_3 + \bar{z}_3) - \frac{\xi}{a_1^2}(z_3 - z_1 z_2 + \bar{z}_3 - \bar{z}_1 \bar{z}_2), \\
\mu_5 &= \quad \frac{i\eta}{a_2^2}(z_3 - \bar{z}_3) + \frac{i\xi}{a_1^2}\big(z_3 - z_1 z_2 - (\bar{z}_3 - \bar{z}_1 \bar{z}_2)\big), \\
\mu_6 &= -\frac{\eta}{a_2^2}(z_2 + \bar{z}_2) + \frac{\xi}{a_1^2}\big(\bar{z}_1(z_3 - z_1 z_2) + z_1(\bar{z}_3 - \bar{z}_1 \bar{z}_2)\big), \\
\mu_7 &= \quad \frac{i\eta}{a_2^2}(z_2 - \bar{z}_2) - \frac{i\xi}{a_1^2}\big(\bar{z}_1(z_3 - z_1 z_2) - z_1(\bar{z}_3 - \bar{z}_1 \bar{z}_2)\big), \\
\sqrt{3}\mu_8 &= \frac{\eta}{a_2^2}(2 - |z_2|^2 - |z_3|^2) + \frac{\xi}{a_1^2}(1 + |z_1|^2 - 2|z_3 - z_1 z_2|^2);
\end{aligned}
$$

where

$$
a_2^2 = 1 + |z_2|^2 + |z_3|^2, \quad a_1^2 = 1 + |z_1|^2 + |z_3 - z_1 z_2|^2.
$$

On the **degenerate orbit** $\xi = 0$.

2. Kählerian structure and G-invariant basis of cohomology groups

Proposition. *(A. Borel) Suppose G is a semisimple Lie group. Then each orbit $\mathcal{O}_{\mu_0} = G_{\mu_0}\backslash G$ admits a complex analytic Kählerian structure invariant under the group G (G-invariant).*

Borel A. Kählerian Coset Spaces of Semisimple Lie Groups. *Proceedings of the National Academy of Sciences of the United States of America* **40** (1954), 1147–1151.

The corresponding Kählerian form is generated by a Kählerian potential Φ:

$$\omega = i \sum_{j,k} \frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k} \, dz_j \wedge d\bar{z}_k.$$

The aims:

to find a Kählerian potential for each orbit;

to construct a G-invariant basis of cohomology groups.

Structure of cohomology ring of an orbit

All forms of odd degrees on an orbit are precise

(A. Borel). Moreover, for the **general orbit**

$$b^0 + b^2 + \cdots + b^{2n} = \text{ord}\, W(G),$$

b^k denotes Betti number.

For the **degenerate orbit**

$$b^0 + b^2 + \cdots + b^{2m} = \frac{\text{ord}\, W(G)}{\text{ord}\, W(G_{\mu_0})},$$

where G_{μ_0} is the stability subgroup in μ_0.

Examples.

Group $SU(2)$, $\dim \mathcal{O}^{SU(2)} = 2$

cohomology ring	$H^* = H^0 \oplus H^2$
Betti numbers	$1 + 1 = 2$

Group $SU(3)$

for **general orbit** $\mathcal{O}^{SU(3)}$, $\dim \mathcal{O}^{SU(3)} = 6$

cohomology ring	$H^* = H^0 \oplus H^2 \oplus H^4 \oplus H^6$
Betti numbers	$1 + 2 + 2 + 1 = 6$

for **degenerate orbit** $\mathcal{O}^{SU(3)}_{\text{degen}}$, $\dim \mathcal{O}^{SU(3)}_{\text{degen}} = 4$

cohomology ring	$H^* = H^0 \oplus H^2 \oplus H^4$
Betti numbers	$1 + 1 + 1 = 3$

As shown above almost all orbits are bundles. This fact can be used for constructing cohomology rings.

Let $\mathcal{O}$ be not a maximal degenerate orbit in G, then there exist orbits $\mathcal{O}_1$, $\mathcal{O}_2$ such that

$$\mathcal{O} = \mathcal{O}_1 \rtimes \mathcal{O}_2.$$

By **Leray-Hirsch theorem**

$$H^*(\mathcal{O}) = H^*(\mathcal{O}_1) \otimes H^*(\mathcal{O}_2).$$

Example. For $SU(3)$ we have

$$\mathcal{O}^{SU(3)} = \mathcal{O}^{SU(3)}_{\text{degen}} \rtimes \mathcal{O}^{SU(2)} \simeq \mathbb{C}\mathrm{P}^2 \rtimes \mathbb{C}\mathrm{P}^1 .$$

Then

$$H^*(\mathcal{O}^{SU(3)}) = (H^0 \oplus H^2 \oplus H^4) \otimes (H^0 \oplus H^2) =$$

$$= H^0 \otimes H^0 \oplus \overbrace{H^0 \otimes H^2 \oplus H^2 \otimes H^0}^{H^2(\mathcal{O}^{SU(3)})} \oplus$$

$$\oplus \underbrace{H^2 \otimes H^2 \oplus H^4 \otimes H^0}_{H^4(\mathcal{O}^{SU(3)})} \oplus H^4 \otimes H^2$$

Basis of $H^2(\mathcal{O}^{SU(3)})$

From Leray-Hirsch theorem we obtain

$$H^2(\mathcal{O}^{SU(3)}) = H^0(1) \otimes H^2(2) \oplus H^2(1) \otimes H^0(2),$$

1 denotes $\mathcal{O}^{SU(3)}_{\text{degen}} \simeq \mathbb{C}\mathrm{P}^2$,
2 denotes $\mathcal{O}^{SU(2)} \simeq \mathbb{C}\mathrm{P}^1$.

Then $\omega^{SU(3)}(z_1, z_2, z_3) =$

$$= f_1(\widetilde{z_1}) \cdot \omega^{\mathbb{C}\mathrm{P}^2}(z_2, z_3) + f_2(z_2, z_3) \cdot \omega^{\mathbb{C}\mathrm{P}^1}(\widetilde{z_1}),$$

where $\widetilde{z_1} = \frac{z_1 \sqrt{1+|z_2|^2+|z_3|^2}}{1+|z_3|^2 - z_1 z_2 \bar{z}_3}$.

It is easy to compute the Kählerian potentials corresponding to $\omega^{\mathbb{C}\mathrm{P}^2}(z_2, z_3)$ and $\omega^{\mathbb{C}\mathrm{P}^1}(\widetilde{z}_1)$:

$$\Phi^{\mathbb{C}\mathrm{P}^1}(\widetilde{z}_1) = \ln(1 + |\widetilde{z}_1|^2) = \\ = \ln\left(1 + \frac{|z_1|^2(1+|z_2|^2+|z_3|^2)}{|1+|z_3|^2 - z_1 z_2 \bar{z}_3|^2}\right).$$

$$\Phi^{\mathbb{C}\mathrm{P}^2}(z_2, z_3) = \ln(1 + |z_2|^2 + |z_3|^2).$$

Basis forms on $\mathcal{O}^{SU(3)}$ can be

$$f_1(\widetilde{z_1}) \cdot \omega^{\mathbb{C}\mathrm{P}^2}(z_2, z_3), \quad f_2(z_2, z_3) \cdot \omega^{\mathbb{C}\mathrm{P}^1}(\widetilde{z_1}).$$

Kählerian structure.

Kirillov-Kostant-Souriau differential form

A Kählerian structure can be represented by the Kirillov-Kostant-Souriau differential form.

Besse A. L. Einstein Manifolds. Springer-Verlag, 1987.

Define the Killing form on $\mathfrak{g}$ by

$$\langle \mu, X \rangle = \mathsf{Tr}(\mu \cdot \mathsf{ad}_\mu X), \quad \mu \in \mathfrak{g}^*,\ X \in \mathfrak{g};$$

the Kirillov-Kostant-Souriau differential 2-form by

$$\omega(\mathsf{ad}_\mu X, \mathsf{ad}_\mu Y) = \langle \mu, [X, Y] \rangle, \quad X, Y \in \mathfrak{g}.$$

Statement. *If G is a compact semisimple group, the Kirillov-Kostant-Souriau 2-form coincides with a G-invariant Kählerian form.*

Example. Kählerian potential for $SU(3)$:

$$\Phi = (\boldsymbol{\mu}_0, \boldsymbol{\alpha}_1)\Phi_1 + (\boldsymbol{\mu}_0, \boldsymbol{\alpha}_2)\Phi_2,$$
$$\Phi_1 = \ln(1 + |z_1|^2 + |z_3 - z_1 z_2|^2),$$
$$\Phi_2 = \ln(1 + |z_2|^2 + |z_3|^2),$$

where $\boldsymbol{\mu}_0 = (m, q)$ is an initial value of $\boldsymbol{\mu} = (\mu_3, \mu_8)$; $\boldsymbol{\alpha}_1 = (1, 0)$, $\boldsymbol{\alpha}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$ are simple roots.

We guess that the **Kählerian potential** for each orbit has the following form:

$$\Phi = \sum_k (\boldsymbol{\mu}_0, \boldsymbol{\alpha}_k)\Phi_k.$$

If $\boldsymbol{\mu}_0$ satisfies the **integer condition**

$$2\frac{(\boldsymbol{\mu}_0, \boldsymbol{\alpha}_k)}{(\boldsymbol{\alpha}_k, \boldsymbol{\alpha}_k)} \in \mathbb{Z},$$

then the corresponding orbit can be **quantized**.

In other words, there exists an irreducible unitary representation of the corresponding group in the space of holomorphic functions on the orbit. The function is a section and serves as a quantum state.

The end