Nonlinear Connections and Description of Photon-like Objects

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Abstract

The notion of of photon-like object (PhLO) is introduced and briefly discussed. The nonlinear connection view on the Frobenius integrability theory on manifolds is considered as a frame in which appropriate description of photon-like objects to be developed

1.The Notion of PhLO

PhLO are real massless time-stable physical objects with a consistent translational-rotational dynamical structure

Remarks:

a/"real"

-*necessarily* carries energy-momentum,
-can be *created* and *destroyed*,
-spatially *finite*, finite values of physical quantities,
-*propagation* and (NOT motion) .

b/"massless"

-$E = cp$, isotropic vector field $\bar{\zeta} = (0, 0, -\varepsilon, 1)$
-$T_{\mu\nu}T^{\mu\nu} = 0$

c/"translational-rotational"

-the propagation has 2 components: *translational* and *rotational*
-both exist *simultatiously* and *consistently*

d/"dynamical structure"

- *internal energy-momentum redistribution*
- may have *interacting subsystems*

2. Non-linear connections

2.1.Projections: Linear maps P in a linear space W^n satisfying: $P.P = P$. If $(e_1, \ldots, e_n)$ and $(\varepsilon_1, \ldots, \varepsilon_n)$ are two dual bases in W then $n = dim(KerP) + dim(ImP)$. If $dim(KerP) = p$ and $dim(ImP) = n - p$ then P is represented by

$$P = \varepsilon^a \otimes e_a + (N_i)^a \varepsilon^i \times e_a, \quad i = 1, \ldots, p; \quad a = p+1, \ldots, n\,.$$

2.2 Nonlinear connections
Let M^n be a smooth (real) manifold with $(x^1, \ldots, x^n)$ be local coordinate system. We have the corresponding local frames $\{dx^1, \ldots, dx^n\}$ and $\{\partial_{x^1}, \ldots, \partial_{x^n}\}$. Let for each $x \in M$ we are given a projection P_x of constant rank p in the tangent space $T_x(M)$. Under this condition we say that a nonlinear connection is given on M. The space $Ker(P_x) \subset T_x(M)$ is called P-*horizontal*, and the space $Im(P_x) \subset T_x(M)$ is called P-*vertical*. Thus, we have two distributions on M. The corresponding integrabilities can be defined in terms of P by means of the Nijenhuis bracket $[P,P]$ given by :

$$[P,P](X,Y) = 2\big\{[P(X),P(Y)]+P[X,Y]-P[X,P(Y)]-P[P(X),Y$$

Now we add and subtract the term $P[P(X),P(Y)]$, so, the right hand expression can be represented by

$$[P,P](X,Y) = \mathcal{R}(X,Y) + \bar{\mathcal{R}}(X,Y),$$

where

$$\mathcal{R}(X,Y) = P\big([(id-P)X,(id-P)Y]\big) = P\big([P_H X, P_H Y]\big)$$

and

$$\bar{\mathcal{R}}(X,Y) = [PX,PY] - P\big([PX,PY]\big) = P_H[PX,PY].$$

Since P projects on the vertical subspace $Im\,P$, then $(id-P)=P_H$ projects on the horizontal subspace. Hence, $\mathcal{R}(X,Y)\neq 0$ measures the nonintegrability of the corresponding horizontal distribution, and $\bar{\mathcal{R}}(X,Y)\neq 0$ measures the nonintegrability of the vertical distribution.

If the vertical distribution is given before-hand and is integrable, then $\mathcal{R}(X,Y)=P\big([P_H X,P_H Y]\big)$ is called *curvature* of the nonlinear connection P if there exist at least one couple of vector fields (X,Y) such that $\mathcal{R}(X,Y)\neq 0$.

Physics + Mathematics.

Any physical system with a dynamical structure is characterized with some internal energy-momentum redistributions, i.e. energy-momentum fluxes, during evolution. Any system of energy-momentum fluxes (as well as fluxes of other interesting for the case physical quantities subject to change during evolution, but we limit ourselves just to energy-momentum fluxes here) can be considered mathematically as generated by some system of vector fields. A *consistent* and *interelated time-stable* system of energy-momentum fluxes can be considered to correspond to an integrable distribution Δ of vector fields according to the principle *local object generates integral object*. An integrable distribution Δ may contain various *nonintegrable* subdistributions $\Delta_1,\Delta_2,\dots$ which subdistributions may be interpreted physically as interacting subsytems. Any physical interaction between 2 subsystems is necessarily accompanied with available energy-momentum exchange between them, this could be understood mathematically as nonintegrability of each of the two subdistributions of Δ and could be naturally measured by the corresponding curvatures. For example, if Δ is an integrable 3-dimensional distribution spent by the vector fields (X_1, X_2, X_3) then we may have, in general, three non-integrable 2-dimensional subdistributions $(X_1, X_2), (X_1, X_3), (X_2, X_3)$. Finally, some interaction with the outside world can be described by curvatures of nonintegrable distributions in which elements from Δ and vector fields outside Δ are involved (such processes will not be considered in this paper).

3. Back to PhLO.

The base manifold is the Minkowski space-time $M = (\mathbb{R}^4,\ \eta)$, where η is the pseudometric with $sign_\eta = (-,-,-,+)$, canonical coordinates $(x, y, z, \xi = ct)$, and canonical volume form $\omega_o = dx \wedge dy \wedge dz \wedge d\xi$. We have the corresponding vector field

$$\bar{\zeta} = -\varepsilon\frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}, \quad \varepsilon = \pm 1$$

determining that the straight-line of translational propagation of our PhLO is along the spatial coordinate z.

Let's denote the corresponding to $\bar{\zeta}$ completely integrable 3-dimensiona Pfaff system by $\Delta^*(\bar{\zeta})$. Thus, $\Delta^*(\bar{\zeta})$ is generated by three linearly independent 1-forms $(\alpha_1, \alpha_2, \alpha_3)$ which annihilate $\bar{\zeta}$, i.e.

$$\alpha_1(\bar{\zeta}) = \alpha_2(\bar{\zeta}) = \alpha_3(\bar{\zeta}) = 0; \quad \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \neq 0.$$

Instead of $(\alpha_1, \alpha_2, \alpha_3)$ we introduce the notation (A, A^*, ζ) and define ζ by

$$\zeta = \varepsilon dz + d\xi,$$

Now, since ζ defines 1-dimensional completely integrable Pfaff system we have the corresponding completely integrable distribution $(\bar{A}, \bar{A}^*, \bar{\zeta})$. We specify further these objects according to the following

Definition: We shall call these dual systems *electromagnetic* if they satisfy the following conditions ($\langle , \rangle$ is the coupling between forms and vectors):

1. $\langle A, \bar{A}^* \rangle = 0, \quad \langle A^*, \bar{A} \rangle = 0,$
2. the vector fields $(\bar{A}, \bar{A}^*)$ have no components along $\bar{\zeta}$,
3. the 1-forms (A, A^*) have no components along ζ,
4. $(\bar{A}, \bar{A}^*)$ are η-corresponding to (A, A^*) respectively .

Further we shall consider only PhLO of electromagnetic nature.

From conditions 2,3 and 4 it follows that

$$A = u\,dx + p\,dy, \quad A^* = v\,dx + w\,dy;$$

$$\bar{A} = -u\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}, \quad \bar{A}^* = -v\frac{\partial}{\partial x} - w\frac{\partial}{\partial y},$$

and from condition 1 it follows $v = -\varepsilon u, \quad w = \varepsilon p$, where $\varepsilon = \pm 1$, and (u, p) are two smooth functions on M. Thus we have

$$A = u\,dx + p\,dy, \quad A^* = -\varepsilon\,p\,dx + \varepsilon\,u\,dy;$$

$$\bar{A} = -u\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}, \quad \bar{A}^* = \varepsilon\,p\frac{\partial}{\partial x} - \varepsilon\,u\frac{\partial}{\partial y}.$$

The completely integrable 3-dimensional Pfaff system (A, A^*, ζ) contains three 2-dimensional subsystems: $(A, A^*), (A, \zeta)$ and (A^*, ζ). We have the following

Proposition 1. The following relations hold:

$$\mathbf{d}A \wedge A \wedge A^* = 0; \quad \mathbf{d}A^* \wedge A^* \wedge A = 0;$$

$$\mathbf{d}A \wedge A \wedge \zeta = \varepsilon\big[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)\big]\omega_o;$$

$$\mathbf{d}A^* \wedge A^* \wedge \zeta = \varepsilon\big[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)\big]\omega_o.$$

Proof. Immediately checked.

These relations say that the 2-dimensional Pfaff system (A, A^*) is completely integrable for any choice of the two functions (u, p), while the two 2-dimensional Pfaff systems (A, ζ) and (A^*, ζ) are NOT completely integrable in general, and the same curvature factor

$$\mathbf{R} = u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)$$

determines their nonintegrability.

Correspondingly, the 3-dimensional completely integrable distribution (or differential system) $\Delta(\zeta)$ contains three 2-dimensional subsystems: $(\bar{A}, \bar{A}^*)$, $(\bar{A}, \bar{\zeta})$ and $(\bar{A}^*, \bar{\zeta})$. We have the

Proposition 2. The following relations hold ($[X, Y]$ denotes the Lie bracket):

$$[\bar{A}, \bar{A}^*] \wedge \bar{A} \wedge \bar{A}^* = 0,$$
$$[\bar{A}, \bar{\zeta}] = (u_\xi - \varepsilon u_z)\frac{\partial}{\partial x} + (p_\xi - \varepsilon p_z)\frac{\partial}{\partial y},$$
$$[\bar{A}^*, \bar{\zeta}] = -\varepsilon(p_\xi - \varepsilon p_z)\frac{\partial}{\partial x} + \varepsilon(u_\xi - \varepsilon u_z)\frac{\partial}{\partial y}.$$

Proof. Immediately checked.

From these last relations and in accordance with **Prop.1** it follows that the distribution $(\bar{A}, \bar{A}^*)$ is integrable, and it can be easily shown that the two distributions $(\bar{A}, \bar{\zeta})$ and $(\bar{A}^*, \bar{\zeta})$ would be completely integrable only if the same curvature factor

$$\mathbf{R} = u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)$$

is zero.

We mention also that the projections

$$\langle A, [\bar{A}^*, \bar{\zeta}]\rangle = -\langle A^*, [\bar{A}, \bar{\zeta}]\rangle = \varepsilon u(p_\xi - \varepsilon p_z) - \varepsilon p(u_\xi - \varepsilon u_z) = \varepsilon\, \mathbf{R}$$

give the same factor $\mathbf{R}$. The same curvature factor appears, of course, as coefficient in the exterior products $[\bar{A}^*, \bar{\zeta}] \wedge \bar{A}^* \wedge \bar{\zeta}$ and $[\bar{A}, \bar{\zeta}] \wedge \bar{A} \wedge \bar{\zeta}$.

In fact, we obtain

$$[\bar{A}^*,\bar{\zeta}]\wedge\bar{A}^*\wedge\bar{\zeta}=-[\bar{A},\bar{\zeta}]\wedge\bar{A}\wedge\bar{\zeta}=-\varepsilon\mathbf{R}\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}+\mathbf{R}\,\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial \xi}.$$

On the other hand, for the other two projections we obtain

$$\langle A,[\bar{A},\bar{\zeta}]\rangle=\langle A^*,[\bar{A}^*,\bar{\zeta}]\rangle=\frac{1}{2}\big[(u^2+p^2)_\xi-\varepsilon(u^2+p^2)_z\big].$$

Clearly, the last relation may be put in terms of the Lie derivative $L_{\bar{\zeta}}$ as

$$\frac{1}{2}L_{\bar{\zeta}}(u^2+p^2)=-\frac{1}{2}L_{\bar{\zeta}}\langle A,\bar{A}\rangle=-\langle A,L_{\bar{\zeta}}\bar{A}\rangle=-\langle A^*,L_{\bar{\zeta}}\bar{A}^*\rangle.$$

Remark. Further in the paper we shall denote $\sqrt{u^2+p^2}\equiv\phi$, and shall assume that ϕ is a *spatially finite* function, so, u and p must also be spatially finite.

Proposition 3. There is a function $\psi(u,p)$ such, that

$$L_{\bar{\zeta}}\psi=\frac{u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)}{\phi^2}=\frac{\mathbf{R}}{\phi^2}.$$

Proof. It is immediately checked that $\psi=\arctan\frac{p}{u}$ is such one.

We note that the function ψ has a natural interpretation of *phase* because of the easily verified now relations $u=\phi\cos\psi$, $p=\phi\sin\psi$, and ϕ acquires the status of *amplitude*. Since the transformation $(u,p)\rightarrow(\phi,\psi)$ is non-degenerate this allows to work with the two functions (ϕ,ψ) instead of (u,p).

From **Prop.3** we have

$$\mathbf{R}=\phi^2L_{\bar{\zeta}}\psi=\ \phi^2(\psi_\xi-\varepsilon\psi_z)\,.$$

Back to Non-linear connections

The above relations show that we can introduce two nonlinear connections: P and $\tilde{P}$. In fact, since the integrable distribution $(\bar{A}, \bar{A}^*)$ lives in the (x, y)-plane we present the coordinates in order (z, ξ, x, y) and the bases $(dz, d\xi, dx, dy)$, $(\partial_z, \partial_\xi, \partial_x, \partial_y)$. We choose the vertical distribution to be generated by (∂_x, ∂_y). The corresponding projections look like:

$$P_V = dx\otimes\frac{\partial}{\partial x}+dy\otimes\frac{\partial}{\partial y}-\varepsilon\, udz\otimes\frac{\partial}{\partial x}-udz\otimes\frac{\partial}{\partial y}-\varepsilon\, pd\xi\otimes\frac{\partial}{\partial x}-pdz\otimes\frac{\partial}{\partial y},$$

$$\tilde{P}_V = dx\otimes\frac{\partial}{\partial x}+dy\otimes\frac{\partial}{\partial y}+pdz\otimes\frac{\partial}{\partial x}+\varepsilon pdz\otimes\frac{\partial}{\partial y}-ud\xi\otimes\frac{\partial}{\partial x}-\varepsilon ud\xi\otimes\frac{\partial}{\partial y},$$

The corresponding matrices look like:

$$P_V = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\varepsilon u & -u & 1 & 0 \\ -\varepsilon p & -p & 0 & 1 \end{Vmatrix}, \quad P_H = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \varepsilon u & u & 0 & 0 \\ \varepsilon p & p & 0 & 0 \end{Vmatrix},$$

$$(P_V)^* = \begin{Vmatrix} 0 & 0 & -\varepsilon u & -\varepsilon p \\ 0 & 0 & -u & -p \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad (P_H)^* = \begin{Vmatrix} 1 & 0 & \varepsilon u & \varepsilon p \\ 0 & 1 & u & p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix},$$

$$\tilde{P}_V = \begin{Vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & \varepsilon p & 1 & 0 \\ -u & -\varepsilon u & 0 & 1 \end{Vmatrix}, \quad \tilde{P}_H = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -p & -\varepsilon p & 0 & 0 \\ u & \varepsilon u & 0 & 0 \end{Vmatrix},$$

$$(\tilde{P}_V)^* = \begin{Vmatrix} 0 & 0 & p & -u \\ 0 & 0 & \varepsilon p & -\varepsilon u \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad (\tilde{P}_H)^* = \begin{Vmatrix} 1 & 0 & -p & u \\ 0 & 1 & -\varepsilon p & \varepsilon u \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix},$$

The projections of the coordinate bases are:

$$\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \xi},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right).P_V=\left(-\varepsilon u\frac{\partial}{\partial x}-\varepsilon p\frac{\partial}{\partial y},-u\frac{\partial}{\partial x}-p\frac{\partial}{\partial y},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right);$$

$$\left(\frac{\partial}{\partial z},\frac{\partial}{\partial \xi},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right).P_H=\left(\frac{\partial}{\partial z}+\varepsilon u\frac{\partial}{\partial x}+\varepsilon p\frac{\partial}{\partial y},\frac{\partial}{\partial \xi}+u\frac{\partial}{\partial x}+p\frac{\partial}{\partial y},0,0\right.$$

$$(dz,d\xi,dx,dy).(P_V)^*=(0,0,-\varepsilon u dz-ud\xi+dx,-\varepsilon p dz-pd\xi+dy$$

$$(dz,d\xi,dx,dy).(P_H)^*=(dz,d\xi,\varepsilon u dz+ud\xi,\varepsilon p dz+pd\xi))$$

Consider now the 2-forms:

$$G=(P_V)^*dx\wedge(P_H)^*dx+(P_V)^*dy\wedge(P_H)^*dy=$$
$$\varepsilon u\,dx\wedge dz+\varepsilon p\,dy\wedge dz+u\,dx\wedge d\xi+p\,dy\wedge d\xi$$

$$\tilde G=(\tilde P_V)^*dx\wedge(\tilde P_H)^*dx+(\tilde P_V)^*dy\wedge(\tilde P_H)^*dy=$$
$$-p\,dx\wedge dz+u\,dy\wedge dz-\varepsilon p\,dx\wedge d\xi+\varepsilon u\,dy\wedge d\xi$$

It follows: $G=A\wedge\zeta,\ \tilde G=A^*\wedge\zeta$ and $\tilde G=*G$, where $*$ is the Hodge star operator defined by η. Clearly, the two 2-forms $(G,*G)$ represent the two nonintegrable Pfaff systems (A,ζ) and (A^*,ζ).

The corresponding curvatures are:

$$\mathcal{R}=\varepsilon(u_\xi-\varepsilon u_z)dz\wedge d\xi\otimes\frac{\partial}{\partial x}+\varepsilon(p_\xi-\varepsilon p_z)dz\wedge d\xi\otimes\frac{\partial}{\partial y}$$
$$\tilde{\mathcal{R}}=-(p_\xi-\varepsilon p_z)dz\wedge d\xi\otimes\frac{\partial}{\partial x}+(u_\xi-\varepsilon u_z)dz\wedge d\xi\otimes\frac{\partial}{\partial y}$$

We obtain

$$\mathcal{R}\left(P_H\frac{\partial}{\partial z},P_H\frac{\partial}{\partial \xi}\right)=[\bar A,\bar\zeta]$$
$$\tilde{\mathcal{R}}\left(\tilde P_H\frac{\partial}{\partial z},\tilde P_H\frac{\partial}{\partial \xi}\right)=[\varepsilon\bar A^*,\bar\zeta]$$

Again Physics + Mathematics

The two 2-forms obtained $(G,*G)$ suggest to test them as basic constituents of Classical electrodynamics, i.e. if they satisfy Maxwell equations. However, it turns out that $\mathbf{d}G\neq 0$ and $\mathbf{d}*G\neq 0$ in general. As for the energy-momentum part of Maxwell theory, determined by the corresponding energy-momentum tensor

$$T_\mu{}^\nu=\frac{1}{2}\big[G_{\mu\sigma}G^{\nu\sigma}+(*G)_{\mu\sigma}(*G^{\nu\sigma}\big],\quad \text{and } T_{44}=u^2+p^2=\phi^2,$$

we obtain the following relations:

$$\nabla_\nu T^\nu_\mu=\frac{1}{2}\Big[G^{\alpha\beta}(\mathbf{d}G)_{\alpha\beta\mu}+(*G)^{\alpha\beta}(\mathbf{d}*G)_{\alpha\beta\mu}\Big]\ .$$

$$G^{\alpha\beta}(\mathbf{d}G)_{\alpha\beta\mu}dx^\mu=(*G)^{\alpha\beta}(\mathbf{d}*G)_{\alpha\beta\mu}dx^\mu=$$

$$\frac{1}{2}L_{\bar\zeta}(u^2+p^2).\zeta=\frac{1}{2}L_{\bar\zeta}\phi^2.\zeta$$

On the other hand

$$(*G)^{\alpha\beta}(\mathbf{d}G)_{\alpha\beta\mu}dx^\mu=-G^{\alpha\beta}(\mathbf{d}*G)_{\alpha\beta\mu}dx^\mu=$$

$$\big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\big]\,\zeta=\mathbf{R}.\zeta.$$

Also, we find

$$\left\langle A,\tilde{\mathcal{R}}\left(\tilde P_H\frac{\partial}{\partial z},\tilde P_H\frac{\partial}{\partial\xi}\right)\right\rangle=-\left\langle\varepsilon A^*,\mathcal{R}\left(P_H\frac{\partial}{\partial z},P_H\frac{\partial}{\partial\xi}\right)\right\rangle=-\mathbf{R}.$$

So, if $L_{\bar\zeta}\phi=0$ we can say that our two 2-forms $G=A\wedge\zeta$ and $*G=A^*\wedge\zeta$, having zero invariants, are nonlinear solutions to the nonlinear equations

$$G^{\alpha\beta}(\mathbf{d}G)_{\alpha\beta\mu}=0,\quad (*G)^{\alpha\beta}(\mathbf{d}*G)_{\alpha\beta\mu}=0,$$

$$G^{\alpha\beta}(\mathbf{d}*G)_{\alpha\beta\mu}+(*G)^{\alpha\beta}(\mathbf{d}G)_{\alpha\beta\mu}=0.$$

From physical point of view these three equations say that the two subsystems of our PhLO, mathematically represented by the G and $*G$

keep the energy-momentum they carry, and are in permanent energy-momentum exchange with each other in equal quantities, i.e. in permanent dynamical equilibrium. The mathematical quantity that guarantees the dynamical nature of this equilibrium is the nonzero curvature $\mathbf{R}$ or $\mathcal{R}$. The permanent nature of this dynamical equilibrium suggests to look for corresponding parameter(s), which should represent relation(s) between/among the state at a given moment of PhLO and its intrinsical capability to overcome the destroying tendencies of the existing nonintegrabilities by means of appropriate propagation properties.

We note the relations:

$$\left\langle A, P_H\frac{\partial}{\partial\xi}\right\rangle = \left\langle A^*, \tilde{P}_H\frac{\partial}{\partial z}\right\rangle = -\left\langle A, P_V\frac{\partial}{\partial\xi}\right\rangle = \varepsilon\left\langle A, P_H\frac{\partial}{\partial z}\right\rangle =$$

$$-\varepsilon\left\langle A, P_V\frac{\partial}{\partial z}\right\rangle = \varepsilon\left\langle A^*, \tilde{P}_H\frac{\partial}{\partial\xi}\right\rangle = -\left\langle A^*, \tilde{P}_V\frac{\partial}{\partial}\right\rangle =$$

$$-\varepsilon\left\langle A^*, \tilde{P}_V\frac{\partial}{\partial\xi}\right\rangle = u^2+p^2 = \phi^2 = -\eta(A,A) = -\eta(A^*,A^*) \equiv S^2.$$

On the other hand

$$\left\langle (P_V)^*(dx)\wedge(P_V)^*(dy), \mathcal{R}\left(P_H\frac{\partial}{\partial z}, P_H\frac{\partial}{\partial\xi}\right)\wedge\tilde{\mathcal{R}}\left(\tilde{P}_H\frac{\partial}{\partial z}, \tilde{P}_H\frac{\partial}{\partial\xi}\right)\right\rangle$$

$$= \left\langle (\tilde{P}_V)^*(dx)\wedge(\tilde{P}_V)^*(dy), \mathcal{R}\left(P_H\frac{\partial}{\partial z}, P_H\frac{\partial}{\partial\xi}\right)\wedge\tilde{\mathcal{R}}\left(\tilde{P}_H\frac{\partial}{\partial z}, \tilde{P}_H\frac{\partial}{\partial\xi}\right)\right.$$

$$\varepsilon\left[(u_\xi-\varepsilon u_z)^2+(p_\xi-\varepsilon p_z)^2\right] = \varepsilon\left(\mathcal{R}\right)^2 \equiv \varepsilon Z^2.$$

Hence, the relation

$$\frac{S^2}{Z^2} = \frac{u^2+p^2}{\left[(u_\xi-\varepsilon u_z)^2+(p_\xi-\varepsilon p_z)^2\right]} = \frac{\phi^2}{\phi^2(\psi_\xi-\varepsilon\psi_z)^2} = \frac{1}{(L_{\bar{\zeta}}\psi)^2} \equiv (l_o)^2$$

defines the quantity $\kappa l_o, \kappa=\pm 1$ as an appropriate such parameter.

4. Translational-rotational consistency

In order to introduce mathematically the translational-rotational consistency we recall the relations

$$\bar{A}\wedge\bar{A}^* = \varepsilon\phi^2\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} \neq 0;\ \ [\bar{A},\zeta]\wedge[\bar{A}^*,\zeta] = \varepsilon\phi^2(L_{\bar{\zeta}}\psi)^2\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} \neq 0\,.$$

Thus we have two frames $(\bar{A},\bar{A}^*,\partial_z,\partial_\xi)$ and $([\bar{A},\bar{\zeta}],[\bar{A}^*,\bar{\zeta}],\partial_z,\partial_\xi)$. The internal energy-momentum redistribution during propagation is strongly connected with the existence of linear map transforming the first frame into the second one since both are defined by the dynamical nature of our PhLO. Taking into account that only the first two vectors of these two frames change during propagation we write down this relation in the form

$$([\bar{A},\zeta],[\bar{A}^*,\zeta]) = (\bar{A},\bar{A}^*)\begin{Vmatrix}\alpha & \beta\\ \gamma & \delta\end{Vmatrix}.$$

Solving this system with respect to the real numbers $(\alpha,\beta,\gamma,\delta)$ we obtain

$$\begin{Vmatrix}\alpha & \beta\\ \gamma & \delta\end{Vmatrix} = \frac{1}{\phi^2}\begin{Vmatrix}-\frac{1}{2}L_{\bar{\zeta}}\phi^2 & \varepsilon\mathbf{R}\\ -\varepsilon\mathbf{R} & -\frac{1}{2}L_{\bar{\zeta}}\phi^2\end{Vmatrix} = -\frac{1}{2}\frac{L_{\bar{\zeta}}\phi^2}{\phi^2}\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix} + \varepsilon L_{\bar{\zeta}}\psi\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}.$$

Assuming the conservation law $L_{\bar{\zeta}}\phi^2=0$, we obtain that the rotational component of propagation is governed by the matrix $\varepsilon L_{\bar{\zeta}}\psi\, J$, where J denotes the canonical complex structure in $\mathbb{R}^2$, and since $\phi^2\, L_{\bar{\zeta}}\psi = \mathbf{R}$ we conclude that the rotational component of propagation is available if and only if the Frobenius curvature is NOT zero: $\mathbf{R}\neq 0$. We may also say that a consistent translational-rotational dynamical structure is available if the amplitude $\phi^2 = u^2+p^2$ is a running wave along $\bar{\zeta}$ and the phase $\psi = \mathrm{arctg}\frac{p}{u}$ is NOT a running wave along $\bar{\zeta}$.

As we noted before the local conservation law $L_{\bar{\zeta}}\phi^2=0$, being equivalent to $L_{\bar{\zeta}}\phi=0$, gives one dynamical linear first order equation. This equation pays due respect to the assumption that our spatially finite PhLO, together with its energy density, propagates translationally with the constant velocity c. We need one more equation in order to specify the phase function ψ. If we pay corresponding respect also to the rotational aspect of the PhLO nature it is desirable this equation *to introduce and guarantee the conservative and constant character of this aspect of PhLO nature*. Since rotation is available only if $L_{\bar{\zeta}}\psi \neq 0$, the simplest such assumption respecting the constant character of the rotational component of propagation seems to be $L_{\bar{\zeta}}\psi = const$, i.e. $l_o = const$. Thus, the equation $L_{\bar{\zeta}}\phi = 0$ and the frame rotation $(\bar{A}, \bar{A}^*, \partial_z, \partial_\xi) \rightarrow ([\bar{A}, \bar{\zeta}], [\bar{A}^*, \bar{\zeta}], \partial_z, \partial_\xi)$, i.e. $[\bar{A}, \bar{\zeta}] = -\varepsilon \bar{A}^* L_{\bar{\zeta}}\psi$ and $[\bar{A}^*, \bar{\zeta}] = \varepsilon \bar{A}\, L_{\bar{\zeta}}\psi$, give the following equations for the two functions (u, p):

$$u_\xi - \varepsilon u_z = -\frac{\kappa}{l_o} p, \quad p_\xi - \varepsilon p_z = \frac{\kappa}{l_o} u \ .$$

If we now introduce the complex valued function $\Psi = u\, I + p\, J$, where I is the identity map in $\mathbb{R}^2$, the above two equations are equivalent to

$$L_{\bar{\zeta}}\Psi = \frac{\kappa}{l_o} J(\Psi) \ ,$$

which clearly confirms once again the translational-rotational consistency in the form that *no translation is possible without rotation, and no rotation is possible without translation*, where the rotation is represented by the complex structure J. Since the operator J rotates to angle $\alpha = \pi/2$, the parameter l_o determines the corresponding translational advancement, and $\kappa = \pm 1$ takes care of the left/right orientation of the rotation. Clearly, a full rotation (i.e. 2π-rotation) will require a $4l_o$-translation, so, the natural time-period is $T = 4l_o/c = 1/\nu$, and $4l_o$ is naturally interpreted as the PhLO size along the spatial direction of translational propagation.

In order to find an integral characteristic of the PhLO rotational nature in *action units* we correspondingly modify, (i.e. multiply by

$\kappa l_o/c$) and consider any of the two equal Frobenius 4-forms:

$$\frac{\kappa l_o}{c}\,\mathbf{d}A\wedge A\wedge\zeta=\frac{\kappa l_o}{c}\,\mathbf{d}A^*\wedge A^*\wedge\zeta=\frac{\kappa l_o}{c}\,\varepsilon\mathbf{R}\omega_o\ .$$

Integrating this 4-form over the 4-volume $\mathbb{R}^3\times 4l_o$ we obtain the quantity $\mathcal{H}=\varepsilon\kappa ET=\pm\,ET$, where E is the integral energy of the PhLO, which clearly is the analog of the Planck formula $E=h\nu$, i.e. $h=ET$.

As an illustration we show a picture and a moving picture of a class of solutions to the above equations.

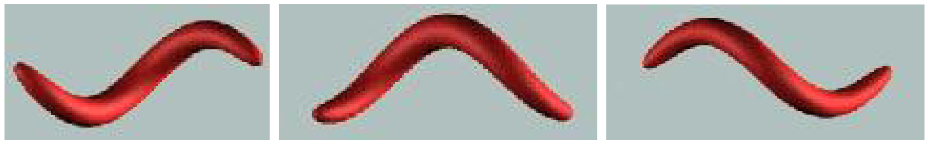

Figure 1: Theoretical example with $\kappa = -1$. The Poynting vector is directed left-to-right.

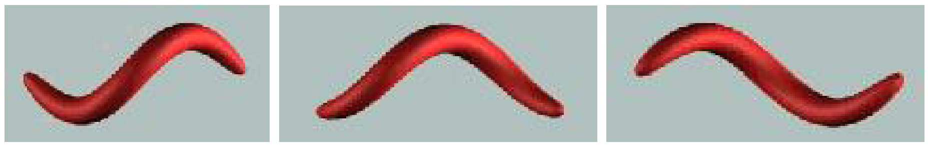

Figure 2: Theoretical example with $\kappa = 1$. The Poynting vector is directed left-to-right.