CONFORMALLY-PROJECTIVE HARMONIC DIFFEOMORPHISMS OF EQUIDISTANT SPACES

HINTERLEITNER Irena

Department of Mathematics, FSI, Technical University of Brno, Czech Republic
Hinterleitner.Irena@seznam.cz

V A R N A 2 0 0 7

1. Introduction

The theory of conformal, geodesic and harmonic mappings is an important part of the differential geometry of Riemannian and pseudo-Riemannian spaces.

S.E. Stepanov and I.G. Shandra [8] studied compositions of conformal and geodesic (projective) diffeomorphisms in the case when these mappings are harmonic. We call such mappings conformally-projective harmonic.

Our consideration is given in tensor form, locally, in the class of real sufficiently smooth functions. The dimension n of the spaces under consideration is greater than 2. All the spaces are assumed to be connected. Let us give the basic notions of the theory of Riemannian spaces V_n, using the notations by L.P. Eisenhart, A.Z. Petrov, and others.

2. Conformal, geodesic and harmonic mappings

In the Riemannian space V_n referred to a local coordinate system $x = (x^1, x^2, \ldots, x^n)$, determined by the symmetric and nondegenerate metric tensor $g_{ij}(x)$, Christoffel symbols of types I and II are introduced by the formulas

$$\Gamma_{ijk} \equiv \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \quad \text{and} \quad \Gamma^h_{ij} \equiv g^{h\alpha}\Gamma_{ij\alpha},$$

where g^{ij} are elements of the inverse matrix to g_{ij}.

The signature of the metrics is assumed, in general, to be arbitrary. Christoffel symbols of type II are the natural connection (the Levi-Civita connection) of Riemannian spaces, with respect to which the metric tensor is covariantly constant, i.e. $g_{ij,k} = 0$.

Hereafter "," denotes the covariant derivative with respect to the connection of the space V_n.

2.1 Conformal mappings

Considering concrete mappings of spaces, for example, $f\colon V_n \to \bar{V}_n$, both spaces are referred to the common coordinate system x with respect to this mapping. This is a coordinate system where two corresponding points $M \in V_n$ and $f(M) \in \bar{V}_n$ have equal coordinates $x = (x^1, x^2, \dots, x^n)$; the corresponding geometric objects in V_n will be marked with a bar.
For example, $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols in $\bar{V}_n$.

The mapping from V_n onto $\bar{V}_n$ is conformal if and only if, in the common coordinate system x with respect to the mapping, the conditions

$$\bar{g}_{ij}(x) = \mathrm{e}^{2\sigma(x)} g_{ij}(x) \tag{1}$$

are satisfied, where $\sigma(x)$ is a function on V_n.

Under conformal mapping the following conditions hold:

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \sigma_j + \delta^h_j \sigma_i - \sigma^h g_{ij}, \tag{2}$$

where $\sigma_i = \partial_i \sigma(x)$, $\sigma^h = \sigma_\alpha g^{\alpha h}$, δ^h_i is the Kronecker delta.

2.2 Geodesic mappings

The diffeomorphism f: $V_n \to \bar{V}_n$ is called a geodesic mapping if f maps any geodesic line of V_n into a geodesic line of $\bar{V}_n$.

The mapping from V_n onto $\bar{V}_n$ is geodesic if and only if, in the common coordinate system x with respect to the mapping, the conditions

$$\bar{\Gamma}_{ij}^{h}(x) = \Gamma_{ij}^{h}(x) + \delta_i^h \psi_j + \delta_j^h \psi_i \tag{3}$$

hold, where $\psi_i(x)$ is a gradient vector.

If $\psi_i \not\equiv 0$, then a geodesic mapping is called nontrivial; otherwise it is said to be trivial or affine.

2.3 Harmonic mappings

A harmonic diffeomorphism is a map that preserves Laplace´s equation.
The mapping from V_n onto $\bar{V}_n$ is harmonic if and only if, in the common coordinate system x with respect to the mapping, the following conditions hold

$$\left(\bar{\Gamma}_{ij}^{h}(x) - \Gamma_{ij}^{h}(x)\right) g^{ij} = 0. \tag{4}$$

3. Conformally-projective harmonic mapping

The compositions of conformal and geodesic (projective) mappings in the case when these mappings are harmonic are called conformally-projective harmonic.

A diffeomorphism from an n-dimensional Riemannian space V_n onto a Riemannian space $\bar{V}_n$ is a conformally-projective harmonic mapping if and only if in the common coordinate system x the following conditions hold

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \varphi_i\delta^h_j + \varphi_j\delta^h_i - \frac{2}{n}\varphi^h g_{ij}, \tag{5}$$

where g_{ij} are components of the metric tensor on V_n, Γ resp., $(\bar{\Gamma})$ are the Christoffel symbols of V_n resp., $(\bar{V}_n)$, φ_i is a covector, $\varphi^h = g^{h\alpha}\varphi_\alpha$, $\|g^{ij}\| = \|g_{ij}\|^{-1}$.

Theorem 1. A necessary and sufficient condition for $f\colon V_n \to \bar{V}_n$ to be conformally-projective harmonic is

$$\bar{g}_{ij,k} = 2\varphi_k\bar{g}_{ij} + \varphi_i\bar{g}_{jk} + \varphi_j\bar{g}_{ik} - \frac{2}{n}(\bar{\varphi}_i g_{jk} + \bar{\varphi}_j g_{ik}),$$

where $\bar{g}_{ij}$ are components of the metric tensor of $\bar{V}_n$, $\bar{\varphi}_i = \varphi^\alpha\bar{g}_{\alpha i}$.

For $n > 2$ the following theorem holds:

Theorem 2. Let V_n be a Riemannian space. Then V_n admits a conformally-projective harmonic mapping onto a Riemannian space $\bar{V}_n$ if and only if the system of differential equations of Cauchy type:

$$
\begin{aligned}
\bar{g}_{ij,k} &= 2\varphi_k \bar{g}_{ij} + \varphi_i \bar{g}_{jk} + \varphi_j \bar{g}_{ik} - \frac{2}{n}(\bar{\varphi}_i g_{jk} + \bar{\varphi}_j g_{ik}),\\
\varphi_{i,j} &= \alpha\,(\bar{g}_{ij} - \overset{1}{T}(\bar{g})\, g_{ij}) + \overset{2}{T}_{ij}(\bar{g}, \varphi),\\
\alpha_{,i} &= \overset{3}{T}_{i}(\bar{g}, \varphi, \alpha)
\end{aligned}
$$

has a solution in V_n for the unknown tensors $\bar{g}_{ij}(x)$ $(\bar{g}_{ij} = \bar{g}_{ji}, \|\bar{g}_{ij}\| \neq 0)$, the covector $\varphi_i(x)$ and the function $\alpha(x)$.

Here $\overset{s}{T}$ $(s = 1, 2, 3)$ are tensors which are expressed as functions of the shown arguments, also of the objects defined in V_n, i.e. the metric tensor g.

The above system is closed with respect to the unknown tensors $\bar{g}_{ij}(x)$, $\varphi_i(x)$, α.

We know from the theory of differential equations that the initial value problem with initial conditions

$$\bar{g}_{ij}(x_o) = \overset{o}{\bar{g}}_{ij}; \quad \varphi_i(x_o) = \overset{o}{\varphi}_i; \quad \alpha(x_o) = \overset{o}{\alpha},$$

has at most one solution. As the tensor $\bar{g}_{ij}$ is symmetric, the general solution of this system depends on $r \leq \frac{1}{2}(n+1)(n+2)$ real parameters. From this follows

Theorem 3. Let V_n be a Riemannian space. The set of all Riemannian spaces $\bar{V}_n$ for which V_n admits a conformal-projective harmonic mapping onto V_n, depends on at most $r \leq \frac{1}{2}(n+1)(n+2)$ real parameters.

4. Equdistant spaces

A vector field ξ^h is called *concircular*, if $\xi^h_{,i} = \varrho\delta^h_i$, where ϱ is a function.

A Riemannian space V_n with concircular vector field is called *equidistant*.

In equidistant spaces V_n, where the concircular vector fields are nonisotropic, there exists a system of coordinates x, where the metric is of the form

$$ds^2 = \frac{1}{f(x^1)} {dx^1}^2 + f(x^1)\, d\tilde{s}^2, \tag{6}$$

where $f \in C^1$ $(f \neq 0)$ is a function,
$d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \dots, x^n) dx^a dx^b$ $(a, b = 2, \dots, n)$ is the metric form of certain Riemannian spaces $\tilde{V}_{n-1}$.

An equidistant space V_n with metric:

$$ds^2 = \frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2, \tag{6}$$

referred to coordinates x admits geodesic mappings onto the Riemannian space $\bar{V}_n$, whose metric form is

$$d\bar{s}^2 = \frac{p}{f\,(1+qf)^2} dx^{1^2} + \frac{p\,f}{1+qf} d\tilde{s}^2, \tag{7}$$

where p, q are some constants such that $1 + qf \neq 0$, $p \neq 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is trivial, and x are common coordinates for V_n and $\bar{V}_n$.

The function $\psi(x)$, which defines a geodesic mapping (see (3)), has the following form:

$$\psi(x) = -\tfrac{1}{2} \ln|1 + q\,f|\,. \tag{8}$$

5. Conformal-projective mappings and equidistant spaces

Analysing formulas (1)-(5), (6) and (8) we can convince ourselves that the following theorem holds:

Theorem 2. An equidistant Riemannian space V_n with the metric

$$ds^2 = (1 + q\, f(x^1))^{\frac{2}{n-2}} \left(\frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2 \right), \tag{9}$$

where $f \in C^1$ $(f \neq 0)$ is a function,
$d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\, dx^a dx^b$ $(a, b = 2, \ldots, n)$ is the metric of some $(n-1)$-dimensional Riemannian space $\tilde{V}_{n-1}$,
is mapped conformally-projectively harmonically on the Riemannian space $\bar{V}_n$ with the metric (7).

Remarks. The Riemannian space V_n with metric

$$ds^2 = (1 + q\, f(x^1))^{\frac{2}{n-2}} \left(\frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2 \right), \tag{9}$$

is conformally mapped onto a Riemannian space with metric

$$ds^2 = \frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2, \tag{6}$$

which is geodesically mapped onto a Riemannian space $\bar{V}_n$ with metric

$$d\bar{s}^2 = \frac{p}{f\,(1 + qf)^2} dx^{1^2} + \frac{p\, f}{1 + qf} d\tilde{s}^2. \tag{7}$$

By comparison of the metric

$$ds^2 = \frac{1}{f(x^1)} {dx^1}^2 + f(x^1)\, d\tilde{s}^2, \tag{6}$$

and

$$ds^2 = (1 + q\, f(x^1))^{\frac{2}{n-2}} \left(\frac{1}{f(x^1)} {dx^1}^2 + f(x^1)\, d\tilde{s}^2 \right), \tag{9}$$

we can convince ourselves that for a suitable choice of the paremeter q the signature of the metric is conserved or can be changed.

There are metrics of the form

$$ds^2 = \frac{1}{f(x^1)} {dx^1}^2 + f(x^1)\, d\tilde{s}^2, \tag{6}$$

which map conformally-projectively harmonically on Einstein spaces.

By a detailed analysis we can convince ourselves of the existence of compact Riemannian spaces, for which global non trivial conformally-projective harmonic mappings exist.

6. Equidistant spaces on geodesic coordinate system and Friedmann metrics

We can make sure that the metrics

$$ds^2 = \frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2, \tag{6}$$

$$d\bar{s}^2 = \frac{p}{f\,(1+q f)^2} dx^{1^2} + \frac{p\, f}{1+q f} d\tilde{s}^2. \tag{7}$$

and

$$ds^2 = (1 + q\, f(x^1))^{\frac{2}{n-2}} \left(\frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2 \right), \tag{9}$$

can be written in the form:

$$ds^2 = e dx^{1^2} + f(x^1)\, d\tilde{s}^2, \tag{10}$$

where $e = \pm 1$, $f \in C^1$ $(f \neq 0)$ is a function, $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n) dx^a dx^b$ $(a, b = 2, \ldots, n)$ is the metric of a certain Riemannian space $\tilde{V}_{n-1}$.

Generally this function f is not the function, which figures in

$$ds^2 = \frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2, \tag{6}$$

$$d\bar{s}^2 = \frac{p}{f\,(1+qf)^2} dx^{1^2} + \frac{p\,f}{1+qf} d\tilde{s}^2 \tag{7}$$

and

$$ds^2 = (1 + q\, f(x^1))^{\frac{2}{n-2}} \left(\frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2 \right). \tag{9}$$

It is known that this coordinate system x is geodesic.

The *Friedmann metric* is a metric

$$ds^2 = e dx^{1^2} + f(x^1)\, d\tilde{s}^2, \tag{10}$$

with $\tilde{V}_{n-1}$ being a space with constant curvature and with a concrete special function $f(x^1)$.

An equidistant space V_n with metric

$$ds^2 = e dx^{1^2} + f(x^1)\, d\tilde{s}^2, \qquad (10)$$

referred to coordinates x admits geodesic mappings onto the Riemannian space $\bar{V}_n$, whose metric form is

$$d\bar{s}^2 = \frac{ep}{(1+qf)^2} dx^{1^2} + \frac{p\,f}{1+qf} d\tilde{s}^2, \qquad (11)$$

where p, q are some constants such that $1+qf \neq 0$, $p \neq 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is affine.
The function $\psi(x)$ which defines a geodesic mappig has also the form

$$\psi(x) = -\tfrac{1}{2} \ln|1+q\,f|\,.$$

Theorem 3. An equidistant Riemannian space V_n with the metric

$$ds^2 = (1+q\,f(x^1))^{\frac{2}{n-2}} \left(e\,dx^{1^2} + f(x^1)\, d\tilde{s}^2 \right),$$

where $f \in C^1$ $(f \neq 0)$ is a function, $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\, dx^a dx^b$ $(a, b = 2, \ldots, n)$ is the metric of some $(n-1)$-dimensional Riemannian space $\tilde{V}_{n-1}$, is mapped by the identity map conformally-projectively harmonically on the Riemannian space $\bar{V}_n$ with the metric (11).

7. Petrov's conjecture on geodesic mappings of Einstein spaces

A.Z. Petrov extended methods of studying geodesic mappings of four-dimensional Lorentzian-Einstein spaces to Einstein spaces of higher dimensions $n > 4$, and conjectured that

the Lorentzian-Einstein spaces $\mathcal{E}_n$ $(n > 4)$ *which do not have constant curvature, do not admit nontrivial geodesic mappings onto Lorentzian-Einstein spaces* (see [6], pp. 355, 461).

Let us construct a counterexample to A.Z. Petrov's conjecture (see [5] and [4]).

Let $\mathcal{E}_n$ $(n > 4)$ be an equidistant Einstein space of nonconstant curvature with Brinkmann metric

$$ds^2 = \frac{1}{f(x^1)} dx^{1^2} + f(x^1)\, d\tilde{s}^2, \tag{6}$$

satisfying condition

$$f = K x^{1^2} + 2a\, x^1 + b.$$

It is known that the space $\mathcal{E}_n$ with a coordinate system (6) admits a geodesic mapping onto the Einstein space $\bar{\mathcal{E}}_n$ with metric

$$d\bar{s}^2 = \frac{p}{f\,(1+qf)^2} dx^{1^2} + \frac{p\, f}{1+qf} d\tilde{s}^2. \tag{7}$$

If $qf' \neq 0$, the mapping is nontrivial. The coordinates x are common to this mapping. The signatures of the metrics of $\mathcal{E}_n$ and $\bar{\mathcal{E}}_n$ are different if $1 + qf < 0$, otherwise they coincide.

One can easily see that, under an appropriate choice of the constant q, it is possible to construct an example of a nontrivial geodesic mapping between Einstein spaces with Minkowski signature which have nonconstant curvatures and whose dimensions are greather than four.

This provides a counterexample to the reduced Petrov conjecture.

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Thank you for your attention