ON GEODESIC MAPPINGS

and

THEIR GENERALIZATIONS

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1. Introduction

Diffeomorphisms and automorphisms of geometrically generalized spaces constitute one of the contemporary actual directions in differential geometry. A large number of works is devoted to

geodesic, quasigeodesic, holomorphically projective, almost geodesic, F-planar

and other mappings, transformations and deformations.

This lecture is dedicated to some results concerning the fundamental equations of these mappings and deformations.

Obviously the existence of a solution of these fundamental equations imply the existence of the mentioned mappings, transformations and deformations.

These fundamental equations were found in several forms. Among these forms there is the particularly important form of a

system of differential equations of Cauchy type.

For their linear forms the question of solvability can be answered by algebraic methods.

2. On systems of differential equations of Cauchy type

Here we introduce the basic notions of the theory of systems of differential equations of Cauchy type. We restrict ourselves to the local theory.

Assume a smooth domain $D \subset R^n$ with coordinates $x = (x^1, x^2, \ldots, x^n)$ and smooth functions $F_i^A(x, y)$, $i = 1, \ldots, n$; $A = 1, \ldots, N$, on $\mathcal{D} \subset D \times R^N$.

The system of differential equations of Cauchy type has the following form

$$\frac{\partial y^A(x)}{\partial x^i} = F_i^A(x, y(x)), \qquad \begin{matrix} A, B = 1, \ldots, N, \\ i = 1, \ldots, n, \end{matrix} \tag{1}$$

where $y(x) = (y^1(x), \ldots, y^N(x))$ are unknown functions.

For initial Cauchy conditions: $y^A(x_o) = \underset{o}{y}^A, \quad A = 1, \ldots, N, \tag{2}$

where $x_o \in D$ and $\underset{o}{y}^A \in R^N$, then the system (1) has at most one solution.

For this reason the general solutions of the system (1) depends on $r \leq N$ real parameters.

The system (1) may be written in terns of covariant derivatives. A fundamental investigation of (1) consists in a check of the integrability conditions, which are essentially algebraic equations for the unknown variables y^A. In the case when they are identifically fulfilled, we have $r = N$.

A homogeneous system of linearly differential equations of Cauchy type has the following for

$$\frac{\partial y^A(x)}{\partial x^i} = f^A_{B\,i}(x)\, y^B(x), \qquad \begin{matrix} A, B = 1, \ldots, N, \\ i = 1, \ldots, n, \end{matrix} \tag{3}$$

where $f^A_{B\,i}(x)$ are functions on D.

The integrability conditions of the homogeneous linear system (3):

$$\frac{\partial^2 y^A(x)}{\partial x^i \partial x^j} = \frac{\partial^2 y^A(x)}{\partial x^j \partial x^i}$$

constitute a system of homogeneous linear algebraic equations with respect to the unknown functions $y^A(x)$.

Differential continuation of their integrability conditions forms also a system of homogeneous linear algebraic equations with respect to the unknown functions $y^A(x)$.

This means that with the aid of linear algebra we may convince ourselves, whether or not the system (3) has solutions and determine on how many parameters $r \leq N$ it depends

Many problems of differential geometry have been successfully solved by homogeneous systems of linearly differential equations of Cauchy type, for example:

- isometric and homothetic transformations of Riemannian spaces,
- affine and projective transformations of Riemannian spaces and spaces with affine connections,
- holomorphically projective transformations of Kählerian spaces.
- affine mappings of Riemannian spaces and spaces with affine connections,

The above results were found in the years 1900 – 1960 and shown in many monographs – L.P. Eisenhart, S. Kobayashi, A.Z. Petrov, K. Yano, ...

Now I want to introduce new results which were obtaind in the last 40 years and are connected which the mentioned systems of Cauchy type. This means that for the mentioed types of geometrical problems regular methods of solution were found.

- geodesic mappings of Riemannian spaces (N.S. Sinyukov, 1967),
- geodesic mappings of spaces with affine connections onto Riemannian spaces (V.E. Berezovsky and J. Mikeš, 1989),
- geodesic deformation of Riemannian hypersurfaces in Riemannian spaces (M.L. Gavrilchenko, V.A. Kiosak and J. Mikeš, 2004),
- conformal mappings of Riemannian spaces onto Einstein spaces (M.L. Gavrilchenko, E. Gladysheva and J. Mikeš, 1992),

- holomorphically projective mappings of Kählerian spaces
  (V.V. Domashev and J. Mikeš, 1976),

- holomorphically projective mappings of hyperbolically Kählerian spaces
  (I.N. Kurbatova, 1980),

- holomorphically projective mappings of parabolically Kählerian spaces
  (M. Shiha, 1994),

- F-planar mappings of spaces with affine connections onto Riemannian spaces
  (J. Mikeš, 1994, 1999),

3. Spaces with affine connection, Riemannian and Kählerian spaces

If the contrary is not specified, the present review is given locally in tensor form in the class of real sufficiently smooth functions. The dimension n of the spaces under consideration, as a rule, is greater than 2, and is not mentioned specially. All the spaces are assumed to be connected.

Let us give the basic notions of the theory for space with affine-connected (A_n), Riemannian (V_n), and Kählerian (K_n) spaces.

<u>3.1 Space with affine connection (A_n).</u> In a space A_n with an affine connection without torsion covered by a local coordinate system $x = (x^1, x^2, \dots, x^n)$ together with an object of the affine connection $\Gamma^h_{ij}(x)$ $(h, i, j, \dots = \overline{1, n})$ the Riemannian tensor and Ricci tensor are defined in the following way:

$$R^h_{ijk} \equiv \partial_j \Gamma^h_{ki} + \Gamma^\alpha_{ki}\Gamma^h_{j\alpha} - \partial_k \Gamma^h_{ji} - \Gamma^\alpha_{ji}\Gamma^h_{k\alpha}, \qquad R_{ij} \equiv R^\alpha_{ij\alpha}, \qquad \partial_i \equiv \partial/\partial x^i.$$

An equiaffine space is defined as A_n, with $R_{ij} = R_{ji}$. The spaces where the conditions $R^h_{ijk} = 0$ $(R_{ij} = 0)$ hold are called flat (Ricci-flat, respectively).

The space A_n belongs to the class C^r $(A_n \in C^r)$ if $\Gamma^h_{ij}(x) \in C^r$.

3.2 Riemannian Spaces (V_n).

In the Riemannian space V_n, determined by the symmetric and nondegenerate metric tensor g_{ij}, Christoffel symbols of types I and II are introduced by the formulas

$$\Gamma_{ijk} \equiv \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ik})$$

and

$$\Gamma^h_{ij} \equiv g^{h\alpha}\Gamma_{ij\alpha},$$

where g^{ij} are elements of the inverse matrix to g_{ij}.

The signature of the metrics is assumed, in general, to be arbitrary. Christoffel symbols of type II are the natural connection (the Levi-Civita connection) of Riemannian spaces, with respect to which the metric tensor is covariantly constant, i.e.,

$$g_{ij,k} = 0.$$

Hereafter "," denotes the covariant derivative with respect to the connection of the space V_n (or A_n).

A Riemannian space is equiaffine.

The space V_n belongs to the class C^r ($V_n \in C^r$) if $g_{ij} \in C^r$.

Using g_{ij} and g^{ij}, we introduce in V_n the operation of lowering and rising indices, for example:

$$R_{hijk} \equiv g_{h\alpha} R^{\alpha}_{ijk}; \quad R^{h\ \,k}_{.ij.} \equiv g^{k\alpha} R^{h}_{ij\alpha}; \quad R^{h}_{i} \equiv g^{h\alpha} R_{\alpha i}.$$

Together with the tensors of Riemann, Ricci, and the projective Weyl tensor, the latter is simplified in V_n:

$$W^{h}_{ijk} \equiv R^{h}_{ijk} - \frac{1}{n-1}(\delta^{h}_{k} R_{ij} - \delta^{h}_{j} R_{ik}),$$

where δ^{h}_{i} is the Kronecker symbol, in V_n we introduce into consideration the scalar curvature $R \equiv R_{\alpha\beta} g^{\alpha\beta}$ and the Brinkmann and Weyl tensors of conformal curvature:

$$L_{ij} \equiv \tfrac{1}{n-2}(R_{ij} - \tfrac{R}{2(n-1)} g_{ij})$$

and

$$C_{hijk} \equiv R_{hijk} - g_{hk} L_{ji} + g_{ik} L_{jh} + g_{hj} L_{ki} - g_{ij} L_{kh}.$$

3.3 Kählerian Spaces (K_n).

In the present lecture, by a Kählerian space we mean a wide class of spaces defined as follows:

A Riemannian space is called a Kählerian space K_n if, together with the metric tensor $g_{ij}(x)$, an affine structure $F_i^h(x)$ is defined that satisfies the relations

$$F_\alpha^h F_i^\alpha = e\delta_i^h; \; F_{(i}^\alpha g_{j)\alpha} = 0; \; F_{i,j}^h = 0, \text{ where } e = \pm 1, 0.$$

- ∗ If $e = -1$, then K_n is said to be an elliptical Kählerian space K_n^-,
- ∗ if $e = +1$, then K_n is said to be a hyperbolic Kählerian space K_n^+, and
- ∗ if $e = 0$ and Rg$\| F_i^h \| = m \geq 2$, then
  K_n is said to be an m-parabolic Kählerian space $K_n^{o(m)}$.
- ∗ The space $K_n^{o(n/2)}$ is called a parabolic Kählerian space K_n^o.

The spaces K_n^+, K_n^- and K_n^o must be of even dimension.

The spaces K_n^- were first considered by P.A. Shirokov, the spaces K_n^+ were considered by P.K. Rashevskii, and the spaces $K_n^{o(m)}$ were studied by V.V. Vishnevskii. In the investigations mentioned these spaces are referred to as A-spaces. Independently from P.A. Shirokov the spaces K_n^- were studied by E. Kähler. In the international literature these spaces are preferably referred to as Kählerian spaces.

4. Conformal mappings onto Einstein spaces

4.1 Conformal mappings of Riemannian spaces

Considering concrete mappings of spaces, for example, $f\colon V_n \to \bar{V}_n$, both spaces are referred to the general coordinate system x with respect to this mapping. This is a coordinate system where two corresponding points

$$M \in V_n \quad \text{and} \quad f(M) \in \bar{V}_n$$

have equal coordinates $x = (x^1, x^2, \ldots, x^n)$; the corresponding geometric objects in V_n will be marked with a bar.
For example, $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols in $\bar{V}_n$.

The mapping from V_n onto $\bar{V}_n$ is conformal if and only if, in the common coordinate system x with respect to the mapping, the conditions

$$\bar{g}_{ij}(x) = \mathrm{e}^{2\psi(x)} g_{ij}(x),$$

where $\psi(x)$ is a function on V_n, g_{ij} and $\bar{g}_{ij}$ are metric tensors of V_n and $\bar{V}_n$, respectively.

4.2 Conformal mappings onto Einstein spaces

One of the directions of investigation is the study of conformal mappings of V_n onto the Einsteinian spaces begun by H.W. Brinkmann (1925). He found fundamental equation of these problem in the nonlinear differential equations in covariant derivatives of Cauchy type. These results were presented in detail by A.Z. Petrov, and M.L. Gavril'chenko continued these investigations.

J. Mikeš, M.L. Gavril'chenko, and E.A. Gladysheva proved that V_n admit a conformal mapping onto the Einsteinian space $\bar{V}_n$ if and only if in V_n the system of linear differential equations of Cauchy type in covariant derivatives

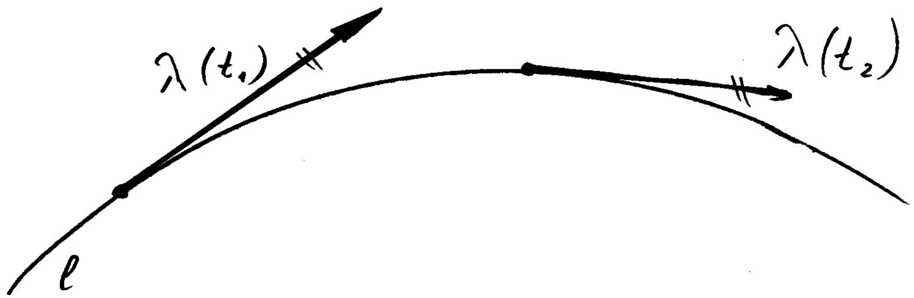
$$\begin{aligned} s_{,i} &= s_i; \\ s_{i,j} &= s\,L_{ij} + u\,g_{ij}; \\ u_{,i} &= s_\alpha L_i^\alpha, \end{aligned}$$

where L_{ij} is the Brinkmann tensor, had a solution for the unknown invariants $s(>0)$, u and the vector s_i. In this case $\bar{g}_{ij} = s^{-2} g_{ij}$.

5 Fundamental equations of Theory of Geodesic Mappings

5.1 Geodesic curves.

In Riemannian spaces V_n and spaces A_n with affine connection straight lines generalise to geodesic curves, which are characterised by the property that there is a parallel tangent vector field along them.



This is expressed by the equation

$$\nabla_{\lambda(s)}\lambda(s) = 0 \quad \text{or} \quad \nabla_{\lambda(t)}\lambda(t) = \varrho(t)\lambda(t),$$

that ∇ is the covariant derivative of the tangent vector $\lambda(t)$ along a geodesic is equal to zero or parallel to itself.

5.2 Geodesic mappings.

The diffeomorphism f from the space of an affine connection A_n onto the space of an affine connection $\bar{A}_n$ is called a geodesic mapping if f maps any geodesic line of A_n into a geodesic line of $\bar{A}_n$.

As an example we may take projective mappings, which map straight lines in Euclidean space to straight lines.

The GM problem was first posed by E. Beltrami (1865). Significant contributions to the investigation of the general laws of this theory were made by T. Levi-Civita, T. Thomas, H. Weyl, A.S. Solodovnikov, G.I. Kruchkovich, and N.S. Sinyukov; see also the books of L.P. Eisenhart, A.Z. Petrov, A.P. Norden, G. Vranceanu, and others.

The second direction of GM-theory is the integration of basic GM-equations.

- U. Dini found metrics of geodesically corresponding surfaces.
- The problem of finding metrics for the Riemannian spaces V_n and $\overline{V}_n$ was formulated by T. Levi-Civita, and he solved it for the case of proper Riemannian spaces.
- By the method used by T. Levi-Civita, this problem was solved, also, in the case in which one of the spaces is proper Riemannian and the other is pseudo-Riemannian.
- For pseudo-Riemannian spaces this problem was solved by
  - P.A. Shirokov for V_2,
  - A.Z. Petrov for V_3,
  - V.I. Golikov for Lorentzian spaces V_4,
  - G.I. Kruchkovich for Lorentzian spaces V_n, and
  - A.V. Aminova for arbitrary V_n.

A detailed description is given in the review by A.V. Aminova.

5.3 Levi-Civita equations.

The mapping from A_n onto $\bar{A}_n$ is geodesic if and only if, in the common coordinate system x with respect to the mapping, the conditions

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i \psi_j + \delta^h_j \psi_i \tag{4}$$

hold, where $\psi_i(x)$ is a covector.

If $\psi_i \not\equiv 0$, then a geodesic mapping is called nontrivial (NGM); otherwise it is said to be trivial or affine. Under GM the following conditions hold:

$$\bar{R}^h_{ijk} = R^h_{ijk} + \delta^h_i \psi_{[jk]} + \delta^h_k \psi_{ij} - \delta^h_j \psi_{ik}; \quad \bar{R}_{ij} = R_{ij} + (n-1)\psi_{ij} + \psi_{[ij]}, \tag{5}$$

where $\psi_{ij} \equiv \psi_{i,j} - \psi_i \psi_j$. The Weyl tensor of the projective curvature W^h_{ijk} is an invariant object of the geodesic mapping, i.e. $\bar{W}^h_{ijk} = W^h_{ijk}$.

If the spaces A_n and $\bar{A}_n$ are equiaffine, the covector ψ_i is gradient-like.

If $\bar{A}_n$ is the Riemannian space $\bar{V}_n$ with the metric tensor $\bar{g}_{ij}$, condition (4) is equivalent to

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}. \tag{6}$$

Conditions (4) and (6) are called the **Levi-Civita equations**.

5.4 Sinyukov's equations. $(V_n \to \bar{V}_n)$

N.S. Sinyukov proved that Riemannian space V_n admits GM onto Riemannian space $\bar{V}_n$ if and only if in V_n the linear homogeneous differential equations in covariant derivatives of Cauchy type

$$\begin{aligned}
&\text{(a)} \quad a_{ij,k} = \lambda_i\, g_{jk} + \lambda_j\, g_{ik};\\
&\text{(b)} \quad n\,\lambda_{i,j} = \mu g_{ij} + a_{i\alpha} R^{\alpha}_{j} - a_{\alpha\beta} R^{\alpha\ \beta}_{.ij.};\\
&\text{(c)} \quad (n-1)\,\mu_{,i} = 2(n+1)\lambda_\alpha R^{\alpha}_{i} + a_{\alpha\beta}\left(2R^{\alpha\ \beta}_{.i,.} - R^{\alpha\beta}_{..,i}\right)
\end{aligned} \qquad (7)$$

have a solution respectively to the unknown symmetric nondegenerated tensor a_{ij}, the gradient vector λ_i, and the function μ.

Notice that $\mu \equiv \lambda_{\alpha,\beta} g^{\alpha\beta}$. The solutions of Eqs. (6) and (7) are related by the following equalities:

$$a_{ij} = e^{2\psi} \overline{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}; \;\; \lambda_i = -e^{2\psi} \overline{g}^{\alpha\beta} g_{\alpha i} \psi_\beta.$$

The function ψ generates the vector ψ_i $(\equiv \partial_i \psi)$.

Formula $(7\mathrm{a})$ gives the necessary and sufficient condition for the existence of GM: $V_n \to \bar{V}_n$. This mapping is nontrivial if and only if $\lambda_i \not\equiv 0$.

5.5 Mikeš-Berezovski's equations. $(A_n \to \bar{V}_n)$

J. Mikeš and V. E. Berezovski showed that the affine-connection space A_n admits GM onto Riemannian space $\bar{V}_n$ with the metric tensor $\overline{g}_{ij}$ if and only if the complete set of differential equations of Cauchy type in covariant derivatives

$$
\begin{aligned}
\overline{g}_{ij,k} &= 2\psi_k \overline{g}_{ij} + \psi_{(i}\overline{g}_{j)k};\\
n\psi_{i,j} &= n\psi_i\psi_j + \mu\overline{g}_{ij} - R_{ij} - \overline{g}_{i\alpha}\overline{g}^{\beta\gamma}R^{\alpha}_{\beta\gamma j} - \tfrac{2}{n+1}R^{\alpha}_{\alpha ij};\\
(n-1)\mu_{,i} &= 2(n-1)\psi_\alpha\overline{g}^{\beta\gamma}R^{\alpha}_{\beta\gamma i} + \psi_\alpha\overline{g}^{\alpha\beta}\left(5R_{\beta i} + \tfrac{6}{n+1}R^{\gamma}_{\gamma\beta i} - R_{i\beta}\right)\\
&\quad + \overline{g}^{\alpha\beta}\left(R^{\gamma}_{\alpha\beta i,\gamma} - R_{\alpha i,\beta} - \tfrac{2}{n+1}R^{\gamma}_{\gamma\alpha i,\beta}\right)
\end{aligned}
$$

has a solution in A_n respectively to the unknown symmetric nondegenerated tensor $\overline{g}_{ij}$, the covector ψ_i, and the function μ.

This system is nonlinear.

5.6 Mikeš-Berezovski's equations. (Equiaffine $A_n \to \bar{V}_n$)

The equiaffine space A_n admits GM onto $\overline{V}_n$ if and only if the complete set of linear differential equations of Cauchy type in the covariant derivatives in A_n

$$a^{ij}_{,k} = \delta^i_k \lambda^j + \delta^j_k \lambda^i; \qquad n\lambda^i_{,j} = \mu\delta^i_j + a^{i\alpha} R_{\alpha j} - a^{\alpha\beta} R^i_{\alpha\beta j};$$
$$(n-1)\mu_{,i} = 2(n+1)\lambda^\alpha R_{\alpha i} + a^{\alpha\beta}(2R_{\alpha i,\beta} - R_{\alpha\beta,i})$$

has a solution respectively to the unknown symmetric nondegenerated tensor a^{ij}, the vector λ^i, and the function μ.

The solutions of this system and (6) are related by the equality

$$a^{ij} = e^{2\psi}\bar{g}^{ij}; \quad \lambda^i = -e^{2\psi}\bar{g}^{i\alpha}\psi_\alpha.$$

In this case, the set of equations obtained is linear and its solution is reduced to the investigation of the integrability conditions and their differential continuations, which are a set of algebraic (homogeneous with respect to the unknown tensors a^{ij}, λ^i, and μ) equations with coefficients from A_n. Thus, in principle, we can solve the following problem if the given equiaffine space A_n admits geodesic mappings onto the Riemannian space $\bar{V}_n$ and if the choice of this mapping is arbitrary.

5.7 Infinitesimal Geodesic Deformations.

Let $V_n \subset V_m\,(n < m)$. The relation $\widetilde{y}^\alpha = y^\alpha(x) + \varepsilon\,\xi^\alpha(x)$, where $(x^1,\, x^2,\, \dots,\, x^n)$ and $(y^1,\, y^2,\, \dots,\, y^m)$ are local coordinates in V_n and V_m, ξ^α is a vector field on V_m, determined at the points of V_n, and ε is a small parameter, defines $\widetilde{V}_n$, which is an infinitesimal deformation of V_n.

The infinitesimal deformations of V_n (of first order) are called (M.L. Gavrilčenko) geodesic if, under these deformations, all geodesic lines of V_n are preserved with accuracy up to ε^2. The deformation is geodesic if and only if

$$\delta g_{ij,k} = 2\psi_k g_{ij} + \psi_i g_{jk} + \psi_j g_{ik}, \qquad (8)$$

where δg_{ij} is the first variation of the metric tensor of V_n:

$$\widetilde{g}_{ij}(x) = g_{ij}(x) + \varepsilon\,\delta g_{ij}(x).$$

5.8 Infinitesimal Geodesic Deformations of hypersurfaces of Riemannian spaces.

If $m = n + 1$ and $\eta^\alpha(x)$ a nonisotropic normal vector of V_n, then the set of vectors $\{y^\alpha_{,i};\ \eta^\alpha\}$ forms a basis in V_m.

Let us assume $\xi^\alpha(x) = \lambda^i(x)y^\alpha_{,i} + \mu(x)\eta^\alpha$, where $\lambda^i(x)$ and $\mu(x)$ are a vector field and a function on V_n respectively; then

$$\delta g_{ij} \equiv \lambda_{(i,j)} - 2\mu\Omega_{ij}$$

and Eqs. (8) are reduced to the following system:

$$\lambda_{i,jk} = \lambda_\alpha R^\alpha_{kji} + g_{i(j}\psi_{k)} + (\mu\Omega_{i(j})_{,k)} - (\mu\Omega_{ij})_{,k}, \qquad (9)$$

where Ω_{ij} is the second fundamental tensor of V_n.

M.L. Gavrilchenko, V.A. Kiosak and J. Mikeš proved that, if $Rg \left\|\Omega_{ij}\right\| \geq 3$, then the system (9) may be reduced to a linear complete system of differential Cauchy-type equations in covariant derivatives with respect to components of some second valency tensor, three vectors, and three functions. Hence, the general solution of the Eqs. (9) depends on r $(\leq n(n+3)+3)$ parameters.

6. Holomorphically Projective Mappings of Kählerian Spaces

6.1 Introduction

The holomorphically projective mappings (HPM) of Kählerian spaces K_n are natural generalizations of geodesic mappings. In the HPM theory we can isolate problems similar to those considered in the GM theory. Moreover, it turns out that many results that are valid for GM can be extended, almost completely, to the case of HPM. Note that HPM were considered, as a rule, under the condition of preservation of the structure. It turned out that in the case of HPM the structure is necessarily preserved.

The works by Tashiro, Ishihara, Otsuki and Tashiro, Domashev and Mikeš as well are devoted to general questions concerning the theory of holomorphically projective mappings of the classic Kählerian spaces K_n^-, the works by Prvanović, Kurbatova, Mikeš are devoted to the theory of hyperbolic Kählerian spaces K_n^+, and the works by Vishnevsky, Shiha and Mikeš are devoted to parabolically Kählerian spaces K_n^0 and $K_n^{o(m)}$.

6.2 Definitions and the basic equations

An analytically planar curve of the Kählerian space K_n is a curve defined by the equations $x^h = x^h(t)$ whose tangent vector $\lambda^h = dx^h/dt$, being translated, remains in the area element formed by the tangent vector λ^h and its conjugate $\bar{\lambda}^h \equiv \lambda^\alpha F^h_\alpha$, i.e., the conditions

$$\nabla_t \lambda^h \equiv \frac{d\lambda^h}{dt} + \Gamma^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_1(t)\lambda^h + \varrho_2(t)\bar{\lambda}^h,$$

where ϱ_1, ϱ_2 are functions of the argument t, are fulfielled.

The diffeomorphism of K_n onto $\bar{K}_n$ is a holomorphically projective mapping (HPM), if it transform all analytically planar curves of K_n into analytically planar curves of $\bar{K}_n$.

Under the HPM, the structure of the spaces K_n and $\bar{K}_n$ is preserved, i.e., in the coordinate system x, general with respect to the mapping the condition $\bar{F}^h_i(x) = F^h_i(x)$ are satisfied.

The holomorphically projective mappings were introduced by Otsuki and Tashiro for K^-_n, by Prvanović for K^+_n, and by Vishnevsky for $K^{o(m)}_n$ under the a priori assumption that the structure was preserved.

6.3 Fundamental equations of Theory HPM

The necessary and sufficient conditions for the holomorphically projective mappings of K_n and $\bar{K}_n$ are ulfillment of the following conditions in the general (with respect to the mapping) coordinate system (K_n^- by Tashiro, K_n^+ by Prvanović, $K_n^{o(m)}$ by Shiha and Mikeš):

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \bar{\varphi}_i\delta^h_j + \bar{\varphi}_j\delta^h_i + \varphi_i F^h_j + \varphi_j F^h_i, \tag{10}$$

where φ_i is a vector, and the vector $\bar{\varphi}_i \equiv \varphi_\alpha F^\alpha_i$ is necessarily a gradient.

When $\varphi_i \not\equiv 0$, we say that the HPM is nontrivial(NHPM).

The Riemannian and Ricci tensors $K_n^\pm$ and $\bar{K}_n^\pm$ are connected by the conditions

$$\begin{aligned}\bar{R}^h_{ijk} &= R^h_{ijk} + \delta^h_k\psi_{ij} - \delta^h_j\psi_{ik} - e\delta^h_{\bar{k}}\psi_{i\bar{j}} + e\delta^h_{\bar{j}}\psi_{i\bar{k}} - 2e\delta^h_{\bar{i}}\psi_{\bar{j}k};\\ \bar{R}_{ij} &= R_{ij} + (n+2)\psi_{ij},\end{aligned}$$

where $\psi_{ij} \equiv \bar{\varphi}_{i,j} - \bar{\varphi}_i\bar{\varphi}_j + \varphi_i\varphi_j$, with $\psi_{ij} + e\psi_{\bar{i}\bar{j}} = 0$.

Relation (10) are equivalent to the equations:

$$\bar{g}_{ij,k} = 2\bar{\varphi}_k\bar{g}_{ij} + \bar{\varphi}_i\bar{g}_{jk} + \bar{\varphi}_j\bar{g}_{ik} + \varphi_i\bar{F}_{jk} + \varphi_j\bar{F}_{ik}, \tag{11}$$

where $\bar{F}_{ij} \equiv \bar{g}_{i\alpha}F^\alpha_j$.

6.3 Fundamental equations of Theory HPM in new linear form

J. Mikeš has found out for K_n^- and Kurbatova for K_n^+ that the Kählerian space $K_n^\pm$ admits of a nontrivial holomorphically projective mapping if and only if the system of equations

$$\begin{aligned} &(a) \quad a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} - e\lambda_{\bar{\imath}} g_{\bar{\jmath}k} - e\lambda_{\bar{\jmath}} g_{\bar{\imath}k};\\ &(b) \quad n\lambda_{i,j} = \mu g_{ij} + a_{i\alpha} R^\alpha_j - a_{\alpha\beta} R^{\alpha}_{\cdot\, ij\cdot}{}^{\beta};\\ &(c) \quad \mu_{,i} = 2\lambda_\alpha R^\alpha_i \end{aligned} \qquad (12)$$

has a nontrivial solution in it for the unknown tensors

$$a_{ij} \ \ (= a_{ji} = -e a_{\bar{\imath}\bar{\jmath}};\ |a_{ij}| \neq 0), \quad \lambda_i \ (\neq 0) \ \text{ and } \ \mu.$$

The solutions of (11) and (12) are connected by the relations

$$a_{ij} = e^{2\psi} \bar{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \qquad \lambda_i = -e^{2\psi} \bar{g}^{\alpha\beta} g_{\alpha i} \bar{\varphi}_\beta,$$

where ψ is an invariant generated by the gradient $\bar{\varphi}_i = \psi_{,i}$.

Conditions (12a) are necessary and sufficient for the existence of NHPM $K_n^\pm$. For K_n^- they were obtained by Domashev and J. Mikeš.

Equations (12) form a linear homogeneous system of the Cauchy system relative to the components of the unknown tensors a_{ij}, λ_i and μ. Consequently, the general solution of this system depens on $r \le (n/2+1)^2$ parameters.

The solution of Eqs. (12) in $K_n^{\pm}$ reduces to the study of the integrability conditions for (12) and their differential prolongations, which, in turn, constitute a system of homogeneous linear algebraic equations for the unknows a_{ij}, λ_i and μ.

Thus, we can find out whether the given space $K_n^{\pm}$ admits of NHPM, and if it does, then with what arbitrariness.

It has been found for $K_n^{o(m)}$ that (11) are equivalent to the conditions

$$a_{ij,k} = \bar{\tau}_i g_{jk} + \bar{\tau}_j g_{ik} + \tau_i F_{jk} + \tau_j F_{ik},$$

where $\bar{\tau}_i \equiv \tau_\alpha F_i^\alpha$, $F_{ij} = g_{i\alpha} F_j^\alpha$.

M. Shiha showed that these equations could be reduced to a system of the Cauchy type for $r \le (n+2)(n+1)/2 - m(n-m+1)$ parameters.

7. F-planar mappings

7.1 F-planar curves

Let usconsider the space A_n, of affine connection without torsion reffered to the coordinate system x in which, along with the affine connection $\Gamma^h_{ij}(x)$, the affine structure $F^h_i(x)$ is defined.

The curve ℓ: $x^h = x^h(t)$ is said to be F-planar if, being translated along it, the tangent vector $\lambda^h \equiv dx^h/dt$ lies in the surface area formed by the tangent λ^h and its conjugate $\lambda^\alpha F^h_\alpha$, i.e.,

$$\nabla_t \lambda^h = \varrho_1 \lambda^h + \varrho_2 \lambda^\alpha F^h_\alpha,$$

where ϱ_1, ϱ_2 are functions of the parameter t.

F-planar curves generalize, in natural way, geodesic, analytically planar, and quasi-geodesic curves by sence A.Z. Petrov.

7.2 F-planar mappings

The diffeomorphism $A_n \to \bar{A}_n$ is said to be an F-planar mapping if, under this mapping, any F-planar curve A_n passes into the $\bar{F}$-planar curve $\bar{A}_n$.

Theorem 1 *The mapping of A_n onto $\bar{A}_n$ is F-planar if and only if the conditions*

$$\begin{aligned} &\text{(a)} \quad \bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_i \psi_j + \delta^h_j \psi_i + F^h_i \varphi_j + F^h_j \varphi_i, \\ &\text{(b)} \quad \bar{F}^h_i = \alpha F^h_i + \beta \delta^h_i, \end{aligned} \tag{14}$$

where $\psi_i(x)$, $\varphi_i(x)$ are vectors and $\alpha(x)$, $\beta(x)$ are invariants, are satisfied in the coordinate system x which is general with respect to the mapping.

Conditions (14b) mean that F-planar mappings preserve the structure F^h_i.

F-planar mappings generalize geodesic, quasigeodesic, holomorphically projective, planar, and almost geodesic of the type of π_2, subprojective mappings.

If A_n admits an F-planar mappings onto the Riemannian space $\bar{V}_n$, then Eqs. (14a) are equivalent, to the equations:

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik} + \varphi_k (F_{ij} + F_{ji}) + \varphi_i F_{jk} + \varphi_j F_{ik}, \tag{15}$$

where $F_{ij} \equiv \bar{g}_{i\alpha} F^\alpha_j$.

We often encounter equations of this kind in the statement of other problems in works by V.S. Sobchuk and by S.V. Stepanov.

Under the condition that $\mathrm{Rank}\|F^h_i - \varrho\delta^h_i\| > 5$ or $F_{(ij)} = 0$. Eqs. (15) reduce to a system of Cauchy type whose general solution depends on $r \leq n(n+5)/2+3$ parameters.

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