Fractional Hamilton-Jacobi Equation and WKB Approximation

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Abstract

The Hamilton- Jacobi partial differential equation is generalized to be applicable for systems containing fractional derivatives. The Hamilton- Jacobi function in configuration space is obtained in a similar manner to the usual mechanics.

Wentzel, Kramers, Brillouin (WKB) approximation for fractional systems is investigated. In the fractional case, the wave function is constructed such that the phase factor is the same as the Hamilton's principle function "S". To demonstrate our proposed approach an example is investigated in details

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1-Introduction

The Hamiltonian formulation of non-conservative systems has been developed by Riewe [1, 2]; he used the fractional derivative [3, 4, 5] to construct the Lagrangian and Hamiltonian for non-conservative systems. As a sequel to Riewe's work, Rabei et al. [6] used Laplace transforms of fractional integrals and fractional derivatives to develop a general formula for the potential of any arbitrary forces, conservative or non-conservative. This led directly to the consideration of the dissipative effects in Lagrangian and Hamiltonian formulations. Besides, the canonical quantization of non-conservative systems carried out by Rabei et al. [7].

Other investigations and further developments are given by Agrawal [8] .He presented the fractional variational problems and the resulting equations are found to be similar to those for variation problems containing integral order derivatives. This approach is extended to classical fields with fractional derivatives [9]. Besides, Klimek [10] showed that the fractional Hamiltonian is usually not a constant of motion, even in the case when the Hamiltonian is not an explicit function of time. In addition, as a continuation of Agrawal's work [8], Rabei et al. [11] achieved the passage from the Lagrangian containing fractional derivatives to the Hamiltonian. The Hamilton's equations of motion are obtained in a similar manner to the usual mechanics.

In the present work, the Hamilton – Jacobi partial differential equation (HJPDE) is generalized to be applicable for systems containing fractional derivatives. In addition the system is quantized using the WKB approximation.

2-Hamiltonian Formalism with Fractional Derivative

Several definitions of a fractional derivative have been proposed, these definitions include Riemann–Liouville, Grünwald–Letnikov, Weyl, Caputo, Marchaud, and Riesz fractional derivatives. Here; the problem is formulated in terms of the left and the right Riemann–Liouville fractional derivatives.

The left Riemann–Liouville fractional derivative defined as

$$ {}_aD_x^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau) d\tau, \quad (1) $$

Which is denoted as (LRLFD), and the right Riemann–Liouville fractional derivative reads as

$$ {}_xD_b^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx} \right)^n \int_x^b (\tau - x)^{n-\alpha-1} f(\tau) d\tau, \quad (2) $$

Which, denoted as (RRLFD)? Here α is the order of the derivative such that $n-1 \leq \alpha \leq n$ and Γ represents the Euler gamma function. If α is an integer, these derivatives are defined in the usual sense, i.e.

$$ {}_aD_x^{\alpha} f(x) = \left(\frac{d}{dx} \right)^{\alpha} f(x) \ , \ {}_xD_b^{\alpha} f(x) = \left(-\frac{d}{dx} \right)^{\alpha} f(x) \ , \ \alpha = 1,2,3,.... \ , (3) $$

The fractional operator ${}_aD_x^{\alpha}$ can be written as [13]

$$ {}_aD_x^{\alpha} = \frac{d^n}{dx^n}\, {}_aD_x^{\alpha-n}, (4) $$
,

Where the number of additional differentiations n is equal to [α] +1, where [α] is the whole part of α. The operator ${}_aD_x^{\alpha}$ is a generalization of differential and integral operators and can be introduced as follows:

$$ {}_aD_x^{\alpha} = \begin{cases} \dfrac{d^{\alpha}}{dx^{\alpha}} & \mathrm{Re}(\alpha) > 0 \\ 1 & \mathrm{Re}(\alpha) = 0 \\ \int_a^x (d\tau)^{-\alpha} & \mathrm{Re}(\alpha) < 0 \end{cases} $$

(5)

The starting point of Agrawal is to consider the Lagrangian of the form

$$ L\,(q,\ {}_aD_t^{\alpha} q\ ,\ {}_tD_b^{\beta} q\ ,t) $$

Which, satisfies the following variation

$$ \delta \int \ L\,(q,\ {}_aD_t^{\alpha} q\ ,\ {}_tD_b^{\beta} q\ ,t)\,dt \ = \ 0 $$

Then, the Euler-Lagrange equation for fractional calculus of variations problem is obtained as

$$\frac{\partial L}{\partial q} + {}_tD_b^{\alpha} \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q} + {}_a D_t^{\beta} \frac{\partial L}{\partial\, {}_tD_b^{\beta} q} = 0, (6)$$

The generalized momenta are introduced as [11]

$$p_\alpha = \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q} \quad , \quad p_\beta = \frac{\partial L}{\partial\, {}_tD_b^{\beta} q}, (7)$$

And the Hamiltonian depending on the fractional time derivatives reads as

$$H = p_\alpha\, {}_aD_t^{\alpha} q + p_\beta\, {}_tD_b^{\beta} q - L, (8)$$

Calculating the total differential of this Hamiltonian we obtain

$$dH = p_\alpha d\, {}_aD_t^{\alpha} q + dp_\alpha\, {}_aD_t^{\alpha} q + p_\beta d\, {}_tD_b^{\beta} q + dp_\beta\, {}_tD_b^{\beta} q - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial\, {}_aD_t^{\alpha} q} d\, {}_aD_t^{\alpha} q$$

$$- \frac{\partial L}{\partial\, {}_tD_b^{\beta} q} d\, {}_tD_b^{\beta} q - \frac{\partial L}{\partial t} dt, (9)$$

Substituting the values of the momenta from Eq(7), we get

$$dH = dp_\alpha\, {}_aD_t^{\alpha} q + dp_\beta\, {}_tD_b^{\beta} q - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt, (10)$$

Making use of the Euler-Lagrangian equation (6), We obtain

$$dH = dp_\alpha\, {}_aD_t^{\alpha} q + dp_\beta\, {}_tD_b^{\beta} q - [\, {}_aD_t^{\beta} p_\beta + {}_tD_b^{\varepsilon} p_\beta] dq - \frac{\partial L}{\partial t} dt, (11)$$

This means that the Hamiltonian is a function of the form

$$H = H(q, p_\alpha, p_\beta, t)$$

Thus the total differential of this Hamiltonian reads as

$$dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p_\alpha} dp_\alpha + \frac{\partial H}{\partial p_\beta} dp_\beta + \frac{\partial H}{\partial t} dt, (12)$$

Comparing Eq.(11) and Eq.(12), we get the following Hamilton's Equations of motion

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}\ ; \quad \frac{\partial H}{\partial p_\alpha} = {}_aD_t^\alpha q\ , \quad \frac{\partial H}{\partial p_\beta} = {}_tD_b^\beta q\ ; \quad \frac{\partial H}{\partial q} = {}_tD_b^\alpha p_\alpha + {}_aD_t^\beta p_\beta \quad (13)$$

It is observed that the fractional Hamiltonian is not a constant of motion even though the Lagrangian does not depend on the time explicitly.

3. Hamilton-Jacobi Partial Differential Equation with Fractional Derivatives

In this section, the determination of the Hamilton-Jacobi partial differential equation for systems with fractional derivatives is discussed. According to Rabei et al. [11], the fractional Hamiltonian is written as

$$H(q,p_\alpha,p_\beta,t)=p_\alpha\,{}_aD_t^\alpha q+p_\beta\,{}_tD_b^\beta q-L(q,{}_aD_t^\alpha q,{}_tD_b^\beta q,t),(14)$$

Consider the canonical transformation with a generating function

$$F_2\left({}_aD_t^{\alpha-1}q,{}_tD_b^{\beta-1}q,P_\alpha,P_\beta,t\right)$$

Then, the new Hamiltonian will take the form

$$K(Q,P_\alpha,P_\beta,t)=P_\alpha\,{}_aD_t^\alpha Q+P_\beta\,{}_tD_b^\beta Q-L'(Q,{}_aD_t^\alpha Q,{}_tD_b^\beta Q,t),(15)$$

The old canonical coordinates q, p_α, p_β, satisfy the fractional Hamilton's principle that can be put in the form

$$\delta\int_{t_1}^{t_2}\left(p_\alpha\,{}_aD_t^\alpha q+p_\beta\,{}_tD_b^\beta q-H\right)dt=0,(16)$$

At the same time the new canonical coordinates Q, P_α, P_β, of course satisfy a similar principle.

$$\delta\int_{t_1}^{t_2}\left(P_\alpha\,{}_aD_t^\alpha Q+P_\beta\,{}_tD_b^\beta Q-K\right)dt=0,(17)$$

The simultaneous validity of Eq. (16) and Eq. (17) does not mean of course that the integrands in both expressions are equal. Since the general form of the Hamilton's principle has zero variation at the end points, both statements will be satisfied if the integrands connected by a relation of the form [12]

$$p_{\alpha}\, {}_aD_t^{\alpha} q + p_{\beta}\, {}_tD_b^{\beta} q - H = P_{\alpha}\, {}_aD_t^{\alpha} Q + P_{\beta}\, {}_tD_b^{\beta} Q - K + \frac{dF}{dt}, \quad (18)$$

The function F can be given as:

$$F = F_2\left({}_aD_t^{\alpha-1} q, {}_tD_b^{\beta-1} q, P_{\alpha}, P_{\beta}, t\right) - P_{\alpha}\, {}_aD_t^{\alpha-1} Q - P_{\beta}\, {}_tD_b^{\beta-1} Q, \quad (19) \; .$$

The function $\boldsymbol{F_2}$ called Hamilton's principal function S for a contact transformation.

$$F_2 = S\left({}_aD_t^{\alpha-1} q, {}_tD_b^{\beta-1} q, P_{\alpha}, P_{\beta}, t\right), \quad (20)$$

Thus,

$$\frac{dF}{dt} = \frac{dS}{dt} - \frac{dP_{\alpha}}{dt}\, {}_aD_t^{\alpha-1} Q - P_{\alpha} \frac{d}{dt}\, {}_aD_t^{\alpha-1} Q - \frac{dP_{\beta}}{dt}\, {}_tD_b^{\beta-1} Q - P_{\beta} \frac{d}{dt}\, {}_tD_b^{\beta-1} Q$$

By using definitions of fractional calculus given in Eq. (4) then we have

$$\frac{dF}{dt} = \frac{dS}{dt} - \frac{dP_{\alpha}}{dt}\, {}_aD_t^{\alpha-1} Q - P_{\alpha}\, {}_aD_t^{\alpha} Q - \frac{dP_{\beta}}{dt}\, {}_tD_b^{\beta-1} Q - P_{\beta}\, {}_tD_b^{\beta} Q, \quad (21)$$

Substituting the values of the $\frac{dF}{dt}$ from Eq. (21) into the Eq. (18) we have

$$p_{\alpha}\, {}_aD_t^{\alpha} q + p_{\beta}\, {}_tD_b^{\beta} q - H = -K + \frac{dS}{dt} - \frac{dP_{\alpha}}{dt}\, {}_aD_t^{\alpha-1} Q - \frac{dP_{\beta}}{dt}\, {}_tD_b^{\beta-1} Q \quad (22)$$

But

$$\frac{dS}{dt} = \frac{\partial S}{\partial\, {}_aD_t^{\alpha-1} q} \frac{d}{dt}\, {}_aD_t^{\alpha-1} q + \frac{\partial S}{\partial\, {}_tD_b^{\beta-1} q} \frac{d}{dt}\, {}_tD_b^{\beta-1} q + \frac{\partial S}{\partial P_{\alpha}} \frac{dP_{\alpha}}{dt} + \frac{\partial S}{\partial P_{\beta}} \frac{dP_{\beta}}{dt} + \frac{\partial S}{\partial t}$$

Again using definitions of fractional calculus given in Eq. (4) we have the following form

$$\frac{dS}{dt}=\frac{\partial S}{\partial\, {}_aD_t^{\alpha-1}q}\,{}_aD_t^{\alpha}q+\frac{\partial S}{\partial\, {}_tD_b^{\beta-1}q}\,{}_tD_b^{\beta}q+\frac{\partial S}{\partial P_\alpha}\frac{dP_\alpha}{dt}+\frac{\partial S}{\partial P_\beta}\frac{dP_\beta}{dt}+\frac{\partial S}{\partial t} \qquad (23)$$

Substituting the values of the $\frac{dS}{dt}$ from Eq. (23) into the Eq. (22) we get

$$p_\alpha=\frac{\partial S}{\partial\, {}_aD_t^{\alpha-1}q} \quad , \quad p_\beta=\frac{\partial S}{\partial\, {}_tD_b^{\beta-1}q} \qquad (24)$$

$${}_aD_t^{\alpha-1}Q=\frac{\partial S}{\partial P_\alpha} \quad , \quad {}_tD_b^{\beta-1}Q=\frac{\partial S}{\partial P_\beta} \qquad (25)$$

$$H+\frac{\partial S}{\partial t}=K \qquad (26)$$

We can automatically ensure that the new variables are constant in time by requiring that the transformed Hamiltonian K shall be identically zero, In other words, Q, P_α, P_β are constants. We see by putting K = 0 that this generating function must satisfy the partial differential equation.

$$H+\frac{\partial S}{\partial t}=0$$

(27)

This equation is called the Hamilton –Jacobi equation. Let us assume that

$$P_\alpha=E_1 \quad , \quad P_\beta=E_2$$

Where E_1, E_2 are constants, then the action function (20), can be expressed as

$$S = S\left({}_aD_t^{\alpha-1}q, {}_tD_b^{\beta-1}q, E_1, E_2, t\right), (28)$$

Further insight into the physical significance of Hamilton's principal function S is furnished by an examination of the total time derivative, which can be computed from the formula

$$\frac{dS}{dt} = \frac{\partial S}{\partial\, {}_aD_t^{\alpha-1}q}\, {}_aD_t^{\alpha}q + \frac{\partial S}{\partial\, {}_tD_b^{\beta-1}q}\, {}_tD_b^{\beta}q + \frac{\partial S}{\partial t} \qquad (29)$$

By using Eq. (24) we have

$$\frac{dS}{dt} = p_\alpha\, {}_aD_t^{\alpha}q + p_\beta\, {}_tD_b^{\beta-1}q - H$$

And using Eq. (9) we have

$$\frac{dS}{dt} = L \quad .$$

Thus

$$S = \int_{t_1}^{t_2} L dt \quad ,$$

If we restrict our considerations to the time -independent Hamiltonians, then the Hamilton-Jacobi function can be written in the form

$$S = W_1\left({}_aD_t^{\alpha-1}q, E_1\right) + W_2\left({}_tD_b^{\beta-1}q, E_2\right) + f\left(E_1, E_2, t\right), (30)$$

Where, W called Hamilton's characteristic function and the function, f takes the following form:

$$f\left(E_1, E_2, t\right) = -Et$$

Making use of equations (24) and (25) we obtain:

$$p_{\alpha} = \frac{\partial W_1}{\partial \; {}_aD_t^{\alpha-1} q} \quad , \quad p_{\beta} = \frac{\partial W_2}{\partial \; {}_tD_b^{\beta-1} q}, (31)$$

$${}_aD_t^{\alpha-1} Q = \frac{\partial S}{\partial E_1} = \lambda_1 \quad , \quad {}_tD_b^{\beta-1} Q = \frac{\partial S}{\partial E_2} = \lambda_2 . \qquad (32)$$

Here λ_1 , λ_2 are constants

4. Illustrative Example

To demonstrate the application of our formalism, let us discuss the following model:

Consider the Lagrangian given by Agrawal [8]

$$L = \frac{1}{2}\left({}_0D_t^{\alpha} q\right)^2$$

The (HJPDE) for this Lagrangian is calculated as

$$\frac{1}{2}\left(p_{\alpha}\right)^2 + \frac{\partial S}{\partial t} = 0$$

Using Eq. (31) we obtain

$$\frac{1}{2}\left(\frac{\partial W_1}{\partial_0 D_t^{\alpha-1} q}\right)^2 - E = 0$$

Solving this equation we have

$$W_1 = \sqrt{2E}\; {}_0D_t^{\alpha-1} q$$

Thus

$$p_{\alpha} = \sqrt{2E}$$

Making use of Eq. (30) we obtain the function S as:

$$S = \sqrt{2E}\; {}_0D_t^{\alpha-1} q - Et$$

Eq. (32) leads to

$${}_0D_t^{\alpha-1} Q = \frac{\partial S}{\partial E} = \frac{1}{\sqrt{2E}}\; {}_0D_t^{\alpha-1} q - t = \lambda_1$$

Thus

$$ {}_0D_t^{\alpha-1}q = \sqrt{2E}\left(t+\lambda_1\right) $$

Or

$$ {}_0D_t^{\alpha}q = \sqrt{2E} = p_\alpha $$

This is the same result obtained by Rabei et al. [11], which is equivalent to Agrawal formalism [8].

5- Fractional WKB Approximation

The outstanding result regarding the meaning of the state function ψ and its relationship to Hamilton's principle function S enables us to write the exponential solution of Schrödinger equation [13].

$$ \psi(q,t) = \exp\left(\frac{i\,S(q,t)}{\hbar}\right) \tag{33} $$

The phase of state function obeys the same mathematical equation, as does Hamilton's principle function S. The physical significance of S in classical mechanics is that it represents the generator of trajectories [12] for fractional systems; the fractional Hamilton's principle function is become the phase of the state function ψ. One can write the solution of Schrödinger equation under the postulated constrains by the WKB approximation and using the fractional Hamilton's principle function eq (12). Thus we propose the fractional state function as:

$$ \psi\left({}_aD_t^{\alpha-1}q,\ {}_tD_b^{\beta-1}q,t\right) = \exp\left(\frac{i}{\hbar}S\left({}_aD_t^{\alpha-1}q,\ {}_tD_b^{\beta-1}q,t\right)\right) \tag{34} $$

From the quantization using WKB approximation [16,17,18,19] a general solution of Schrödinger equation is obtained using the expansion for S and then using the transformation to the N-dimensional system as:

$$ \psi = \left[\prod_{i=1}^{N}\psi_{io}(q_i)\right]\exp\left(\frac{i\,S\left(q_i,t\right)}{\hbar}\right) \tag{35} $$

Where

$$\psi_{io}(q_i) = \frac{1}{\sqrt{p(q_i)}} \tag{36}$$

In our case, S behaves like a 2-dimensional problem with two distinct momenta. Thus,

$$q_1 \equiv {}_aD_t^{\alpha-1} q \qquad P_1 \equiv \hat{P}_\alpha \tag{37}$$

$$q_2 \equiv {}_tD_b^{\beta-1} q \qquad P_2 \equiv \hat{P}_\beta \tag{38}$$

And the momenta are defined as operators. Therefore, we can propose the wave function ψ of the fractional system in the following form

$$\psi({}_aD_t^{\alpha-1}q,\ {}_tD_b^{\beta-1}q,t) = \frac{1}{\sqrt{P_\alpha P_\beta}} \exp\left(\frac{i}{\hbar} S\left({}_aD_t^{\alpha-1}q,\ {}_tD_b^{\beta-1}q, E_1, E_2, t\right)\right) \tag{39}$$

And the momenta operators in the form

$$\hat{P}_\alpha = \frac{\hbar}{i}\left(\frac{\partial}{\partial\ {}_aD_t^{\alpha-1}q}\right), \quad \hat{P}_\beta = \frac{\hbar}{i}\left(\frac{\partial}{\partial\ {}_tD_b^{\beta-1}q}\right) \tag{40}$$

We conclude that (39) is the solution of Schrödinger equation for any given fractional systems. If α and β both are equal to unity, then we will return to the usual classical solution of Schrödinger equation, also we can notice how the probability is inversely proportional to the momentum

$$|\psi|^2 \cong \frac{1}{p(q)}.$$

Example

Let us consider the following fractional Lagrangian,

$$L = \frac{1}{2}\left({}_0D_t^{\alpha} q\right)^2 + \frac{1}{2}\left({}_tD_1^{\beta} q\right)^2 \tag{41}$$

The fractional Hamilton-Jacobi equation for this fractional Lagrangian can be calculated as:

$$\frac{1}{2}\left(P_\alpha\right)^2 + \frac{1}{2}\left(P_\beta\right)^2 + \frac{\partial S}{\partial t} = 0. \tag{42}$$

where

$$P_\alpha = \frac{\partial L}{\partial {}_0D_t^{\alpha} q} \quad ; \quad P_\beta = \frac{\partial L}{\partial {}_tD_1^{\beta} q}$$

Making use of equation (24), the fractional Hamilton-Jacobi equation (42) becomes:

$$\frac{1}{2}\left(\frac{\partial W_1}{\partial \; {}_0D_t^{\alpha-1} q}\right)^2 + \frac{1}{2}\left(\frac{\partial W_2}{\partial \; {}_tD_1^{\beta-1} q}\right)^2 + \frac{\partial S}{\partial t} = 0 \tag{43}$$

Taking into account

$$H = -\frac{\partial S}{\partial t} \tag{44}$$

If we apply (44) on a wave function it gives:

$$\frac{\partial S}{\partial t} = -E \equiv -(E_1 + E_2) \tag{45}$$

By using the fact that E is the total energy of the system and taking into account (45) we obtain

$$\left[\frac{1}{2}\left(\frac{\partial W_1}{\partial\, {}_0D_t^{\alpha-1}q}\right)^2 - E_1\right] + \left[\frac{1}{2}\left(\frac{\partial W_2}{\partial\, {}_tD_1^{\beta-1}q}\right)^2 - E_2\right] = 0 \qquad (46)$$

Thus, both sides of (46) should be zero, and we obtain

$$W_1 = \sqrt{2E_1}\; {}_0D_t^{\alpha-1}q, \qquad\qquad W_2 = \sqrt{2E_2}\; {}_tD_1^{\beta-1}q \qquad (47)$$

By using (30) we obtain

$$\psi({}_0D_t^{\alpha-1}q, {}_tD_1^{\beta-1}q, t) = \frac{1}{\sqrt{P_\alpha P_\beta}} \exp\left(\frac{i}{\hbar}\left(\sqrt{2E_1}\; {}_0D_t^{\alpha-1}q + \sqrt{2E_2}\; {}_tD_1^{\beta-1}q - Et\right)\right)$$

(48)

Which, represents the wave function of the following Hamiltonian:

$$H = \frac{1}{2}(P_\alpha)^2 + \frac{1}{2}(P_\beta)^2 \qquad (49)$$

Let us deal now with the momenta as operators of the form (40), and applying these operators on the wave function, one obtain the following momenta eigen values

$$\hat{P}_\alpha \psi = \sqrt{2E_1}\,\psi \qquad\qquad \hat{P}_\beta \psi = \sqrt{2E_2}\,\psi \qquad (50)$$

Then,

$$\left|\hat{P}_\alpha\right| = \sqrt{2E_1} \qquad\qquad \left|\hat{P}_\beta\right| = \sqrt{2E_2} \qquad (51)$$

It’s the same as the classical solution. Also, when applying the energy operator it gives the energy eigenvalues:

$$H\psi = \frac{1}{2}\left(\hat{P}_\alpha\right)^2 \psi + \frac{1}{2}\left(\hat{P}_\beta\right)^2 \psi \qquad = (E_1 + E_2)\psi \qquad (52)$$

6. Conclusions

In This work we have studied the Hamilton-Jacobi partial differential equation for systems containing fractional derivatives. A general theory to solve the Hamilton-Jacobi partial differential equation is proposed for systems containing fractional derivatives under the condition that this equation is separable. The Hamilton-Jacobi function is determined in the same manner as for usual systems. Finding this function enables us to get the solutions of the equations of motion.

In order to test our formalism, and to get a somewhat deeper understanding, we have examined an example of system with fractional derivatives. The result found to be in exact agreement with Lagrangian formulation given by Agrawal [8] and with Hamiltonian formulation given by Rabei et al. [11].

We use the generating function "S" of the Hamilton-Jaccobi equation in its fractional form to be the phase factor of the wave function describing some potentials valid for the assumptions suggested by the WKB approximation

The proof of our results arises from the new proposed concepts of the momentum and energy operators, that they give the same eigenvalues producing the ordinary results achieved by the classical approach.

Giving the same eigenvalues that means this form of fractional operator also eigen, valid, and useful in effecting on a state functions.

7- References

[1]F. Riewe, Non Conservative Lagrangian and Hamiltonian mechanics. Physical Review E. 53:1890-1899, (1996).

[2] F. Riewe, Mechanics with Fractional Derivatives. Physical Review E.55: 3581-3592, (1997).

[3] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, (2000).

[4] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional Integrals and Derivatives: theory and applications, Gordon and Breach, Amsterdam, (1993).

[5] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego (1999); Fractional Calculus and Applied Analysis 5: 367, (2002).

[6] Eqab M. Rabei, T. Al-halholy, A. Rousan, Potentials of Arbitrary Forces with Fractional Derivatives, International Journal of Modern Physics A, 19: 3083-3092, (2004).

[7] Eqab M. Rabei, Abdul-Wali Ajlouni, Humman B. Ghassib, Quantization of Non-Conservative Systems Using Fractional Calculus, WSEAS Transactions on Mathematics, 5: 853-863, (2006); Quantization of Brownian Motion, International Journal of Theoretical physics, 45: 1619- 1629, (2006).

[8]Om P. Agrawal, Formulation of Euler–Lagrange equations for fractional variational problems, J. Math. Anal. Appl.272: 368-379, (2002).

[9] Dumitru Baleanu, Sami I. Muslih, Lagrangian formulation of classical fields within Riemann-Liouville fractional derivatives, Phys. Scripta (in press).

[10] M. Klimek, Lagrangian and Hamiltonian fractional sequential mechanics; Czech J. Phys., 52: 1247-1253, (2002); Fractional sequential mechanics – models with symmetric fractional derivative, Czech J. Phys. 51: 1348-1354, (2001).

[11] Eqab M. Rabei, Khaled I. Nawafleh, Raed S. Hijjawi , Sami I. Muslih, Dumitru Baleanu, The Hamilton Formalism With Fractional Derivatives, J. Math. Anal. Appl. 327:891-897, (2007).

[12] H. Goldstein, Classical Mechanics, Addison-Wesley Publishing Company, (1980).

[13] Igor M. Sokolov,Joseph Klafter, Alexander Blumen, Fractional Kinetics, Physics Today(2002) American Institute of physics,S-0031-9228-0211-030-1.

[14] B.N.N. Achar, J.W. Hanneken, T. Enck, T.Clarke, Dynamics of the fractional oscillator, Physica A, 297: 361-367, (2001).

[15] David J Griffiths, Introduction to quantum mechanics, prentice hall, New Jersey (1995).

[16] Eqab M. Rabei, Eyad H. Hasan, and Humam B. Ghassib, Hamilton-Jaccobi Treatment of Constrained Systems with Second-Order Lagrangians, International Journal of Theoretical Physics, Vol. **43**, No. 4, April (2004), 1073-1096

[17] Eqab M.Rabei, Khaled I.Nawafleh, Y.S Abdelrahman, H.Y.R Omari. Hamilton-Jaccobi treatment of Lagrangians with linear velocities Modern physics letters A.**18**:(2003), 1591-1596
[18] Eqab M.Rabei, Eyad H. Hasan, Humam B.Ghassib, S Muslih, Quantization of Second-Order constrained Lagrangians systems using the WKB approximation, International Journal of theoretical Physics, Vol. **43,** No. 4, 1073-1096.
[19] Eqab M. Rabei, K.I.Nawafleh and H.B.Ghassib, Quantization of Constrained Systems Using the WKB Approximation, Phys. Rev. A **66**, (2002), 24101.