Nonlinear gravitational waves and their polarization

Geometry, Integrability and Quantization

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Collaboration

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Some research lines

- *Complete integrability in field theory*

- *General relativity*:

  Reduction of Einstein field equations

  Nonlinear gravitational waves

  Integrable gravitational models and their quantization

Nonlinearity

- The need of taking into full account the *nonlinearity* of Einstein's equations when studying gravitational waves from strong sources is generally recognized.
- Despite the great distance of the sources from Earth (where most of detectors are located) there are situations where the nonlinear effects cannot be neglected.

Christodoulou memory

- When the source is a coalescing binary a *secondary wave* is generated via the non linearity of Einstein's field equations.
- The *memory* seems to be too weak to be detected from the present generation of interferometers (even if ω is in the optimal band for LIGO/VIRGO interferometers)

MAIN DETECTORS

VIRGO

NAUTILUS

Exact gravitational waves

- However, the Christodoulou memory is of the same order as the linear effects related to the same source, thus stressing the relevance of the nonlinearity of the Einstein's equations also from an experimental (LIGO/VIRGO/NAUTILUS) point of view.
- For these reasons exact solutions of the Einstein equations deserve special attention when they are of propagative nature.

Role of exact solutions

Explicit solutions enable to discriminate more easily among *physical* or *pathological* features.

- Where are there singularities?
- What is their character?
- How do test particles and fields behavior in given background space-times?
- What are their global structures?
- Is a solution stable and generic?

Problem

- Classification of gravitational fields (not only Ricci-flat metrics) invariant for a Lie algebra $\mathcal{G}$ of Killing vector fields, such that:

  I. The distribution $\mathcal{D}$, generated by vector fields belonging to $\mathcal{G}$, is 2-dimensional.

An integrable gravitational case

- Ernst, Maison, Harrison, …,Belinsky, Zakharov:

Einstein field equations for a metric of the form

$$g = f(z,t)(dt^2 - dz^2) + h_{11}(z,t)dx^2 + h_{22}(z,t)dy^2 + 2h_{12}(z,t)dxdy$$

reduce essentially to

$$\partial_\xi(\alpha H^{-1}\partial_\eta H) + \partial_\eta(\alpha H^{-1}\partial_\xi H) = 0$$

$$H = ||h_{ab}||; \quad \xi = (t+z)/\sqrt{2}; \quad \eta = (t-z)/\sqrt{2}\,; \quad \alpha = \sqrt{|det H|}.$$

- *G-Inverse Scattering Transform yields solitary wave solutions*.

Geon

The choice of coordinates

- The choice of coordinates also depends on

1. the properties of the distribution, $\mathcal{D}^{\perp}$, orthogonal to $\mathcal{D}$,
2. the rank of the metric restricted to the leaves of $\mathcal{D}$.

Several cases

II. The distribution $\mathcal{D}^{\perp}$ is:

IIa integrable and transversal to $\mathcal{D}$.

IIb semintegrable and transversal to $\mathcal{D}$

IIc non integrable and transversal to $\mathcal{D}$

IId integrable and not transversal to $\mathcal{D}$.

IIe semintegrable and not transversal to $\mathcal{D}$

IIf non integrable and not transversal to $\mathcal{D}$

$(\mathcal{G}_2, r)$-type metrics

- The case, in which the metric *g* restricted to any integral (2-dimensional) submanifold (*Killing leaf*) of the distribution $\mathcal{D}$ is degenerate, splits naturally into two sub-cases according to whether the rank *r* of *g* restricted to Killing leaves is 1 or 0.
- In order to distinguish various cases occurring in the sequel, the notation $(\mathcal{G}_2, r)$ will be used: metrics satisfying the conditions I and IIa will be called of $(\mathcal{G}_2,2)$-type; metrics satisfying conditions I and II d,e or f ($\mathcal{D}$ and $\mathcal{D}^{\perp}$, are *not transversal*) will be called of $(\mathcal{G}_2,1)$-type or of $(\mathcal{G}_2,0)$-type according to the rank of their restriction to Killing leaves.

2-dimensional Lie algebra of isometries

	$\mathcal{D}^{\perp}$, $r=2$	$\mathcal{D}^{\perp}$, $r=1$	$\mathcal{D}^{\perp}$, $r=0$
$\mathcal{G}_2$	**integrable**	**integrable**	*integrable*
$\mathcal{G}_2$	**semi-integrable**	**semi-integrable**	**semi-integrable**
$\mathcal{G}_2$	non-integrable	**non-integrable**	*non-integrable*
$\mathcal{A}_2$	integrable	**integrable**	**integrable**
$\mathcal{A}_2$	**semi-integrable**	semi-integrable	semi-integrable
$\mathcal{A}_2$	non-integrable	*non-integrable*	*non-integrable*

The integrable case. Local aspects

- Complete classification of gravitational fields (not only Ricci-flat metrics) invariant for a Lie algebra $\mathcal{G}$ of Killing vector fields, such that:
  I. The distribution $\mathcal{D}$, generated by vector fields belonging to $\mathcal{G}$, is 2-dimensional.
  II. The distribution $\mathcal{D}^{\perp}$, orthogonal to $\mathcal{D}$, is integrable and transversal* to $\mathcal{D}$.

The integrable case. Global aspects

- Global solutions of the Einstein field equations can also be constructed.

  Two cases: dim $\mathcal{G}$ =2 or dim $\mathcal{G}$ =3. They are qualitatively different but all manifolds satisfying the assumptions I and II are in a sense fibered over ζ-complex curves. $|\!\rightarrow$ *global* solutions of vacuum Einstein equations.
- If dim $\mathcal{G}$=3, condition II follows from I.

History

- Only two 2-d Lie algebras: $\mathcal{A}_2$ and $\mathcal{G}_2$.
- A gravitational field g satisfying I and II, with $\mathcal{G} = \mathcal{A}_2$ or $\mathcal{G}_2$, is said to be $\mathcal{G}$-integrable.

- 1916 $\mathcal{A}_2$-integrable gravitational fields by Weyl.
- 1937 $\mathcal{A}_2$-integrable grav. waves by Einstein-Rosen.
- 1958 $\mathcal{A}_2$-integrable grav. fields by Kompaneyets and Landau.
- 1979 $\mathcal{A}_2$-integrable grav. solitary waves by Belinsky-Zakharov.
- 2000 $\mathcal{G}_2$-integrable gravitational fields by G.S, G.V, A.V.

Notation

Manifolds $\mathcal{M}$ are connected and C^{∞},

Metric: a non-degenerate symmetric (0,2) tensor field,

Kil(g): the Lie algebra of all Killing fields of a metric *g*,

Killing algebra: a sub-algebra of *Kil(g)*

Killing leaves of g: integral sub-manifolds of the distribution generated by vector fields of *Kil(g)*

$\mathcal{A}_2$: a *2d* Abelian Lie algebra,

$\mathcal{G}_2$: a *2d* non-Abelian Lie algebra,

$\mathcal{G}$-*integrable metric*: metric satisfying *I & II,* with $\mathcal{G}=\mathcal{A}_2$ or $\mathcal{G}_2$

Semi-adapted coordinates

- Let g be a metric on the space-time $\mathcal{M}$ (a connected smooth manifold), $\mathcal{G}_2 = Span(X, Y)$ one of its Killing algebra

$$[X, Y] = sY, \qquad s = 0,1$$

- The Frobenius distribution $\mathcal{D}$ (possibly with singularities) generated by X and Y is *2d*.
- In a neighbourhood of a non-singular point of $\mathcal{D}$ a chart (x^μ) (*semi-adapted*) exists such that

$$X = \partial_3 \ , \quad Y = e^{sx^3}\partial_4$$

Invariant gravitational fields

- A gravitational field g admits X and Y as Killing fields *iff* in a *s-a* chart has the form

$$g|_S = g_{ij}dx^i dx^j + 2(l_i + s m_i)\, dx^i dx^3 - 2m_i dx^i dx^4 +$$
$$[s^2\lambda (x^4)^2 - 2s\mu x^4 + \nu]\, dx^3 dx^3 + 2\,[\mu - s\lambda x^4]\, dx^3 dx^4 +$$
$$\lambda\, dx^4 dx^4$$

- with g_{ij}, m_i, l_i, λ, μ, ν functions of (x^1, x^2)
- (Note: $det\, \mathrm{H} = \lambda\nu - \mu^2$)

Invariant anholonomic basis

$e_i = \partial_i\ ,\ e_3 = \partial_3 + s\partial_4\ ,\ e_4 = -\partial_4;$ $[\,e_\mu\,,\,e_\nu] = c^\alpha{}_{\mu\nu} e_\alpha$

- $\theta^i = dx^i,\ \theta^3 = dx^3,\ \theta^4 = sx^4\, dx^3 - dx^4;$ $2d\theta^\alpha = -\,c^\alpha{}_{\mu\nu}\,\theta^\mu \wedge \theta^\nu$

(g_{ij})	(l_i) (m_i)
(l_i) $(m_i)^+$	ν $-\mu$ $-\mu$ λ

Killing leaves

- Condition II allows to construct semi-adapted charts $\{x, y, x^3, x^4\}$ such that the fields $e_1=\partial_x$ and $e_2=\partial_y$, belong to $\mathcal{D}^{\perp}$.
- In such a chart, called from now on *adapted*, the components l_i, m_i vanish.
- Related s-adapted vierbeins will be called *adapted vierbeins*.
- *Killing leaf*: integral submanifold of $\mathcal{D}$.
- *Orthogonal leaf*: integral submanifold of $\mathcal{D}^{\perp}$.
- Since $\mathcal{D}^{\perp}$ is transversal to $\mathcal{D}$, the restriction $g|_S$ of g to any Killing leaf S is non degenerate.
- Then $(S,g|_S)$ is a *2d* Riemannian manifold on which the isometry group G_2 act transitively. Thus, it is homogeneous; in particular, the Gauss curvature $K(S)$ of the Killing leaves is constant.

Gauss curvature of Killing leaves

The Gauss curvature $K(S)$ of the Killing leaves can be easily computed in the chart ($p=x^3/_S$, $q=x^4/_S$), where the metric $g/_S$ has the form

$$g/_S=(s^2\underline{\lambda}q^2-2s\underline{\mu}q+\underline{\nu})dp^2+2(\underline{\mu}-s\underline{\lambda}q)dpdq+\underline{\lambda}dq^2,$$

$$K(S)=-s^2\underline{\lambda}(\underline{\lambda}\,\underline{\nu}-\underline{\mu}^2)^{-1};$$

The function $K(x^1, x^2)=-s^2\lambda(\lambda\nu-\mu^2)^{-1}$ describes the behavior of Gauss curvature in passing from one Killing leaf to another.

The Ricci tensor field

Even in the adapted invariant vielbein its components are too complicated and we will not write them.

Reduced Einstein equations for $s \neq 0$.

- $\partial_\xi(\alpha H^{-1}\partial_\eta H) + \partial_\eta(\alpha H^{-1}\partial_\xi H) = 2s\mathrm{l}\, f g(Y,Y)/\alpha$
- ………………….
- ………………….
- ………………….

Solutions of Einstein equations for $g(Y,Y) \neq 0$

- If the Killing field Y is not of *light type*, then in the adapted coordinates (x, y, p, q) one has

$$g = f(dx^2 \pm dy^2) + \beta^2[(s^2kq^2 - 2slq + m)dp^2 + 2(l - skq)dpdq + kdq^2]$$

$f = -\Delta_\pm \beta^2/(2s^2k)$ and $\beta(x,y)$ a solution of the ***tortoise equation***

$$\beta + A\ ln|\beta - A| = u(x,y)$$

with $\Delta_\pm u = 0$ s. t. $|grad_\pm u| \neq 0$; $\Delta_\pm = \partial^2_{xx} \pm \partial^2_{yy}$; k, l, m consts, $km - l^2 = \pm 1$, $k \neq 0$, $(\Delta_\pm \beta^2 \neq 0 \Leftrightarrow |grad_\pm u| \neq 0;\ \Delta_\pm \beta^2 = \beta(grad_\pm \beta)^2/(\beta - A))$

Canonical form of metrics for $g(Y,Y) \neq 0$

- It is not clear whether if the given local metrics are pairwise different or not. The gauge freedom can be eliminated as follows:

The general integral of the equation $\Delta_{\pm} u = 0$ satisfying the condition $|\mathrm{grad}_{\pm} u| \neq 0$ defines, in both cases ±, two non constant functions

+ ↵ u and its conjugate harmonic v, for $\Delta_{+} u = 0$

- ↵ $u = F(x+y) + G(x-y)$, $v = F - G$, for $\Delta_{-} u = 0$

- By using (u,v) as new coordinates on the orthogonal leaves
- $g = [e^{[(u-\beta)/A]}/(2s^2k\beta)](du^2 \pm dv^2) + \beta^2[(s^2 k q^2 - 2slq + m)dp^2 + 2(l - skq)\, dp dq + k dq^2]$

Normal form of metrics for $g(Y,Y) \neq 0$

- On the Killing leaves it is also possible to introduce coordinates (θ,ϕ) diagonalizing the metric $g|_S$ to the form

$$g|_S = \beta^2[d\theta^2+\Theta(\theta)d\phi^2],$$

where $\Theta(\theta)$ is equal either to $sinh^2\theta$ or $-cosh^2\theta$, depending on the signature of the metric. In the *normal coordinates,* $(r=2s^2k\beta, \tau=v, \theta, \phi)$, the metric takes the *normal form* (with $\varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1$)

$$g = \varepsilon_1([1-A/r]d\tau^2 \pm [1-A/r]^{-1}dr^2)+\varepsilon_2 r^2[d\theta^2+\Theta(\theta)d\phi^2]$$

The geometric reason for this form is that when Y is not isotropic, a third Killing field exists, say Z, which together with X and Y constitute a basis of the $so(2,1)$ Lie algebra.

so(3) - invariant Ricci-flat metrics

- The above results lead to expect that vacuum gravitational fields, with Killing algebras isomorphic to *so(3)* with *2d* leaves, can be essentially described as it was done in the case of *so(2,1)* .
- Also in this case the solutions depend on the tortoise equation and this gives new insight to the physical meaning of the so called *Regge-Wheeler tortoise coordinate*.

$$g=f(dx^2 \pm dy^2)+r^2(x,y)[d\theta^2+\sin^2\theta\, d\phi^2]$$

$$f=\Delta_{\pm} r^2\,, \qquad r+A\,\ln|r-A| = x$$

Ricci-flat metrics for $g(Y,Y) = 0$

- $\partial_\xi(\alpha H^{-1}\partial_\eta H) + \partial_\eta(\alpha H^{-1}\partial_\xi H) = 2s^2 l f g(Y,Y)/\alpha$

General solution, in *a. c.* (x, y, p, q),

$$g = 2f(dx^2 \pm dy^2) + \mu[(w(x,y) - 2sq)dp^2 + 2\, dp\, dq],$$

$$\mu = D\Phi + B;\ D, B \text{ in } \mathrm{R},\ \Delta\Phi = 0,\ f = (\mathrm{grad}\,\Phi)^2/\sqrt{|\mu|}$$

$$\mu\Delta w + \partial_x(\mu)\partial_x w + \partial_y(\mu)\partial_y w = 0.$$

- *Lorentzian* (+); *Kleinian* (−)
- *The Gauss curvature of Killing leaves vanishes.*
- *Two superposition laws*
- Special solutions: $w = \mu'$, $w = \ln|\mu|$, μ' is h. conj. with μ.

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Physical properties

- Sources
- Asymptotic flatness
- Wave-like character
- Energy
- Polarization

Which signature?

- Metrics may be *Lorentzian* or *Kleinian*.

Ricci flat manifolds of Kleinian signature appear in the *no boundary proposal* of Hartle and Hawking in which the idea is suggested that the signature of the space-time metric may have changed in the early universe.

Dust and cosmic strings sources

- The simplest source for previous metrics is *dust* with density ρ and velocity U^μ and, then, with $T_{\mu\nu} = \rho U_\mu U_\nu$.
- When U^μ is a *light-like* vector field, $T_{\mu\nu}$ can describe the energy and momentum of *electromagnetic waves* ($F_{\mu\nu}$=0, $\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} = 0$). Such metrics could describe the emission of gravitational waves from *γ-ray bursts.*
- Being the *time coordinate* in the Killing leaves, the dust cannot move orthogonally to them and it will be chosen to move parallel to the *light-like* Killing field Y , i.e., with *velocity* $U^\mu = \delta^{\mu 4}$.

Non vacuum Einstein metrics when $g(Y, Y) = 0$

- General solution, in *a. c.* (x, y, p, q),

$$g = 2f(dx^2 + dy^2) + \mu[(w(x,y) - 2q)dp^2 + 2\,dp\,dq],$$

$\mu = D\Phi + B$; D and B in $\mathbb{R}$, $\Delta\Phi = 0$, $f = (\mathrm{grad}\,\Phi)^2/\sqrt{|\mu|}$

$$\mu\Delta w + \mathrm{grad}\,\mu \cdot \mathrm{grad}\,w = 2\mu^2 f\rho$$

$$\Delta w + \partial_x(\ln|\mu|)\partial_x w + \partial_y(\ln|\mu|)\partial_y w = 2\mu f\rho.$$

- *The Gauss curvature of Killing leaves vanishes.*
- *Superposition law*

Physical (?) coordinates

- The coordinates (p,q) on Killing leaves have a transparent geometric interpretation.

- In the "diagonalizing" coordinates (x,y,z,t)

$$2z=(2q-w)\,e^{-p}+e^{p}; \qquad 2t=(2q-w)\,e^{-p}-e^{p}$$

$$g = 2f\,(dx^2 + dy^2) + \mu[dz^2 - dt^2 + dw \cdot d(\ln|z-t|)] \qquad \text{(for } z>t\text{)}$$

- In the new coordinates (x, y, u, v)

$$p = \ln|u| \; ; \; q = uv$$

$$g = 2f(x, y)[dx^2 + dy^2] + \mu[2du\,dv + w u^{-2}\, du^2]$$

- Both coordinates (x,y,z,t) and (x,y,u,v) are harmonic

Asymptotic flatness, wave-like character and spin

- For $f = 1/2$ and $\mu = 1$, previous metrics are locally diffeomorphic to a subclass of vacuum Peres solutions corresponding to a special choice of th harmonic function parameterising them (Bonnor, Aichelburg, Sexl).
- In the new (harmonic) coordinates (x, y, u, v)

$$p = \ln|u| \; ; \; q = uv$$

$$g = dx^2 + dy^2 + 2dudv + wu^{-2}\, du^2$$

Flatness

- In order to have everywhere regular spatially asymptotically flat solutions, f and w must be constant functions and the fluid density ρ must vanish fast enough.
- However, if we admit δ-like singularities in the x-y plane, spatially asymptotically flat vacuum solutions with $f \neq const$ and $w \neq const$ can exist. In this limiting case in which $\rho(x,y) \rightarrow \delta(x,y)$, the energy-momentum tensor becomes the one usually employed to describe the gravitational effects of topological defects known as *cosmic strings*.
- This kind of extended objects are predicted in some *particles physics cosmological models with phase transitions*. Moreover, *cosmic strings* could have an important role in the description of two very interesting astrophysical phenomena: the GRBs and *ultra high energy (E ~* 10^{11}GeV) *cosmic rays.*

Wave-character of the field

- The non vanishing independent components of the Riemann tensor are:

$$R_{iuju} = -\partial^2_{ij}h; \quad h=w/u^2 , \quad i, j = x, y$$

- The wave character and the polarization may be analysed in many ways. For example, we could use the Zel'manov criterion to show that these are gravitational waves and the Landau-Lifshitz pd-tensor to find the propagation direction.
- However, the algebraic Pirani's criterion determines the wave character and the propagation direction both at once. Moreover, in the vacuum, the two methods agree. To use this criterion the Weyl scalars must be evaluated according to the Petrov-Penrose classification

The Landau-Lifchitz pseudo-tensor

It has been seen that it yields the correct definition of energy for relevant cases. In facts, the energy flux radiated at infinity for an asymptotically flat space–time, evaluated with the Landau-Lifshitz pseudotensor, has been seen to agree with the Bondi flux that is with the energy flux evaluated in the exact theory.

$$
\begin{aligned}
\tau^{\rho\kappa} &= \frac{1}{16\pi k}\Big\{\left(2\Gamma^{\nu}_{\lambda\mu}\Gamma^{\sigma}_{\nu\sigma} - \Gamma^{\nu}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\nu}_{\lambda\nu}\Gamma^{\sigma}_{\mu\sigma}\right)\left(g^{\rho\lambda}g^{\kappa\mu} - g^{\rho\kappa}g^{\lambda\mu}\right) \\
&+ g^{\rho\lambda}g^{\mu\nu}\left(\Gamma^{\kappa}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\kappa}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\kappa}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\kappa}_{\lambda\mu}\Gamma^{\sigma}_{\nu\sigma}\right) \\
&+ g^{\kappa\lambda}g^{\mu\nu}\left(\Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\rho}_{\nu\sigma}\Gamma^{\sigma}_{\lambda\mu} - \Gamma^{\rho}_{\lambda\mu}\Gamma^{\sigma}_{\nu\sigma}\right) \\
&+ g^{\mu\lambda}g^{\sigma\nu}\left(\Gamma^{\rho}_{\lambda\nu}\Gamma^{\kappa}_{\mu\sigma} - \Gamma^{\rho}_{\lambda\mu}\Gamma^{\kappa}_{\nu\sigma}\right)\Big\}.
\end{aligned}
$$

The Petrov-Penrose classification

- The P-P classification, needs a tetrad basis with 2 real null vector fields (*vf*) and 2 real spacelike (or 2 complex null) *vf.* According to Pirani's criterion, a metric of type Petrov N is a g.w. propagating along one of the two real null vector fields.
- New coordinates adapted to the Petrov-Penrose classification

$$x \to x,\quad y \to y,\quad u \to u,\quad v \to v + \phi(x, y, u)$$

- $$g = dx^2 + dy^2 + 2du dv + 2(\phi_{,x}dx + \phi_{,y}dy)du; \qquad \phi_{,u} = h$$
- Since ∂_u and ∂_v are null real *vf* and ∂_x and ∂_y spacelike real *vf*, in the above coordinates Pirani's criterion can be applied.
- The only nonvanishing components of Riemann tensor are

$$R_{iuju} = -\partial^2_{ij}\,\partial_u\phi, \qquad i, j = x, y,$$

so that these gravitational fields belong to Petrov type N.

Our waves

- As it has been shown, solutions we are considering, represent gravitational waves moving at the velocity of light, that is, in the *would be quantized theory*, particles with zero rest mass.
- Thus, if a classification in terms of Poincaré group invariants could be performed, these waves would belong to the class of unitary (infinite-dimensional) representations of the Poincaré group characterized by $P^2=0$, $W^2=0$.

The polarization

- The definition and the meaning of spin or polarization for a theory, such as general relativity which is non-linear, deserve a careful analysis. It is well known that the concept of particle together with its degrees of freedom like the *spin* may be only introduced for linear theories (for example for the Yang-Mills theories, which are non linear, we need to perform a perturbative expansion around the linearized theory).

The Pauli-Ljubanski vector

- In linear theories, when Poincaré invariant, the particles are classified in terms of the eigenvalues of two Casimir operators of the Poincaré group, P^2 and W^2 where P_μ are the translation generators and $W_\mu = (1/2)\varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$ is the *Pauli-Ljubanski polarization vector* with $M^{\rho\sigma}$ Lorentz generators. Then, the total angular momentum $J=L+S$ is defined in terms of the generators $M_{\rho\sigma}$ as $J^i=(1/2)\varepsilon^{0ijk}M_{jk}$.

The spin in the linearized theory

- Recall that, in order for such a classification to be meaningful P^2 and W^2 have to be invariants of the theory. This is not the case for general relativity, unless we restrict to a subset of transformations selected for example by some physical criterion or by experimental constraints. For the solutions of the linearized vacuum Einstein equations the choice of the harmonic gauge does the job. There, the residual gauge freedom corresponds to the sole Lorentz transformations. As in the linearized theory, of the whole diffeomorphisms group just the Lorentz transformations preserve the harmonic gauge. That is, we are allowed to speak about polarization if we stay in the harmonic gauge.

Linearized Einstein equations

- $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ with $|h_{\mu\nu}|<<1$, $|\partial_\alpha h_{\mu\nu}|<<1$
- $\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = 0$, $\eta^{\alpha\mu}\partial_\alpha(h_{\mu\nu} - \eta_{\mu\nu}h)=0$ (in vacuum)
- $\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu} = -16\pi G(T_{\mu\nu} +\tau_{\mu\nu})$, $\eta^{\alpha\mu}\partial_\alpha(h_{\mu\nu} - \eta_{\mu\nu}h)=0$
- $h= \eta^{\rho\sigma} h_{\rho\sigma}$, $R^{(1)}{}_{\mu\nu}=\eta^{\alpha\beta}\partial_\alpha\partial_\beta h_{\mu\nu}$

The energy momentum tensor of gravity in the linearized theory

- Any function $h_{\mu\nu}$ of $r=k_\mu x^\mu$, with $k_\mu k^\mu=0$, is solution of the wave equation and the energy and the momentum of the wave are given by

  $\tau^0{}_0 = (u_{22}-u_{11})^2 + (u_{12})^2\,, \quad \tau^3{}_0 = \tau^0{}_0; \quad u_{\mu\nu} = dh_{\mu\nu}/dr$

  $k_\mu=(1,0,0-1)$
- If $\mathcal{R}$ is the generator of a rotation in the x-y plane

  $\mathcal{R}(u_{22}-u_{11})=-4u_{12}$

  $\mathcal{R}u_{12}=u_{22}-u_{11}$

  so that

  $\mathcal{R}^2(u_{22}-u_{11})=-4(u_{22}-u_{11})\,; \quad \mathcal{R}^2u_{12}=-4u_{12}$
- Thus, the eigenvalues of $i\mathcal{R}$ are ± 2

Spin

- Once the propagation direction has been determined, to compute the polarization we only need to look at the transformation properties of physical components of the metric under a rotation in the *x-y* plane orthogonal to the propagation direction.
- A good opportunity!

  The exact gravitational wave

  $$g = dx^2 + dy^2 + dz^2 - dt^2 + dw \cdot d(\ln|z-t|)$$

  $$= \eta + h$$

  is also solution of the linearized Einstein equations

The energy-momentum density

- The Landau-Lifchitz pseudotensor and the Bel superenergy tensor single out the same degrees of freedom:

$$\tau^0{}_0 = (\partial_x h_{tx})^2 + (\partial_x h_{ty})^2 \; ; \qquad \tau^3{}_0 = \tau^0{}_0$$

- This shows that the physical components of these waves have only one index in the *x-y* plane orthogonal to the propagation direction ∂_u.

Spin-1?

Is the Light *heavy* or *light* ?

Spin-1

↓

Attraction or Repulsion !

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