EXTREMALS OF THE GENERALIZED EULER-BERNOULLI ENERGY IN REAL SPACE FORMS AND APPLICATIONS

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Varna, June 6^{th} 2008

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1. Introduction

- **In 1691, Jakob Bernoulli posed the problem of the elastic beam. Three years later, he published his own solution.**
- **In 1694, Huygens criticized Jakob for not showing all the solutions.**
- **In 1742, Daniel Bernoulli proposed to minimize the squared radius of curvature in order to determine the shape of an elastic rod subject to pressure at both ends.**

Introduction

- **Following the D. Bernoulli's simple geometric model, an elastic curve is a minimizer of the bending energy:**

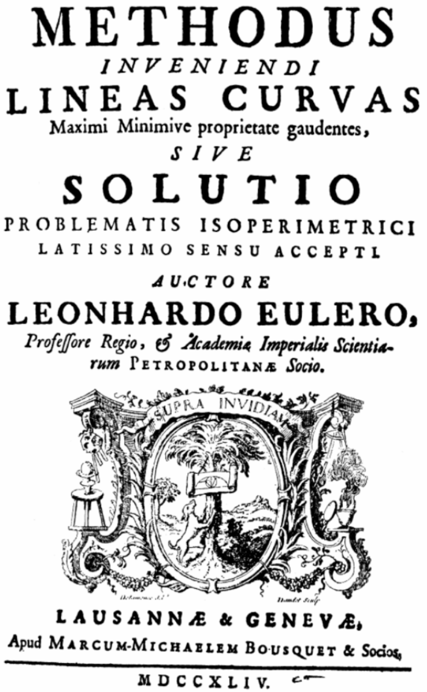
$$\mathcal{F}^2_\lambda(\gamma) = \int_\gamma (\kappa^2 + \lambda) ds, \tag{1.1}$$

κ **being the curvature of** γ**.**

λ **corresponds to a constraint on the length.**

$\lambda = 0$**: free elastica.**

Introduction



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SOLUTIO
PROBLEMATIS ISOPERIMETRICI
LATISSIMO SENSU ACCEPTI
AUCTORE
LEONHARDO EULERO,
Professore Regio, & Academiæ Imperialis Scientiarum PETROPOLITANÆ Socio.

SUPRA INVIDIAM

LAUSANNÆ & GENEVÆ,
Apud MARCUM-MICHAELEM BOUSQUET & Socios.
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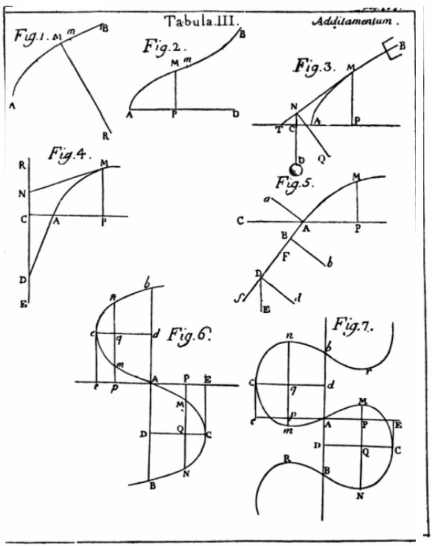
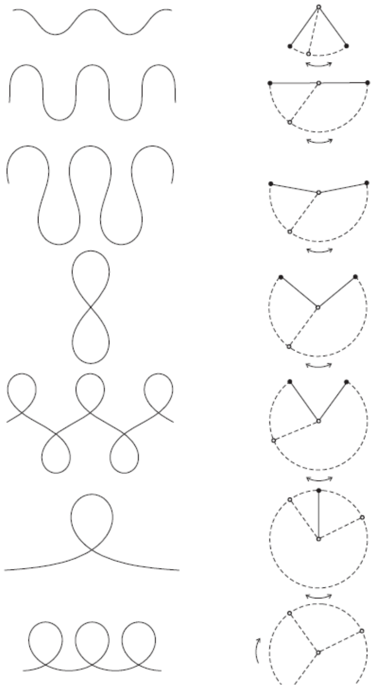




I. Mladenov et all have recently obtained explicit expressions for the plane elastic curves.

- **In 1743, L. Euler determined the plane elastic curves.**
- **J. Radon (1910) and R. Irrgang (1933) analyzed the free elastic curves in $\mathbb{R}^3$.**
- **More recently, in 1982-3 Bryant and Griffiths studied related variational problems in real space forms.**

REMARK

The study of the closed elastic curves is a problem of special geometric significance.

Introduction

- **J. Langer and D. Singer in 1987 and Koiso en 1993, showed by different methods that there exist closed elastic curves of a given length in a compact Riemannian manifold.**
- **J. Langer and D. Singer classified the closed free elastic curves in 2-dimensional space forms (1984); They showed also that there exist a countable family of closed elastic curves in $\mathbb{R}^3$, (1985)**
- **Closed elasticae in $\mathbb{S}^3$ were studied by J. Arroyo, O.J. Garay and J.J. Mencía in**

More generally, we consider the following:

PROBLEM

existence and classification of critical points and minimizers of the generalized Euler-Bernoulli energy functional

$$\mathcal{F}(\gamma) = \int_\gamma \mathbf{P}(\kappa). \qquad (1.2)$$

acting on spaces of curves in a Riemannian mani fold ($P(t)$ is a C^∞ function)

They include:

- geodesics;
- classical elasticae;
- elasticae with constant length;
- elasticae circular at rest;
- closed elasticae enclosing a fixed area;
- etc...

Introduction

Some applications:

- models of relativistic particles (massive or massless);
- models of p-branes;
- models of membranes and vesicles;
- construction of Chen-Willmore submanifolds;
- etc...

2. Order one functionals

We consider two cases:

- $\frac{dP'}{ds} = 0.$ **Order one functionals**
- $\frac{dP'}{ds} \neq 0.$ **Higher order functionals.**
- **Techniques are different.**

2.1. Notation

- $\mathbb{M}^n$, n-dimensional **Riemannian manifold** with metric $<,>$.
- $\mathbb{M}^n(G)$, n-dimensional **real space form** with constant curvature G.
- **Levi-Civita connection** ∇.
- **curvature tensor** $\mathbf{R}$.
- $\mathcal{H} \equiv$ **a certain space of curves**, $\gamma : \mathbb{I} = [0,1] \to \mathbb{M}^n$, satisfying suitable boundary conditions.

Notation

- $\mathcal{H} \equiv$ **will satisfy at least:**

  1. $\gamma \in C^4(\mathbb{I})$**,**

  2. γ **is immersed in** $\mathbb{M}^n$**, that is,** $\frac{\partial \gamma}{\partial t} \neq 0$ **and**

  3. there is a well defined normal vector on γ **(for instance,** $n = 2$ **and** $\mathbb{M}^2$ **is orientable or** $\frac{\partial^2 \gamma}{\partial t^2} \neq 0$**).**

- $\mathbf{\Omega} \equiv$ **space of closed curves**.

Notation

- $\mathbf{V(t)} = \frac{\partial\gamma}{\partial t} = \gamma'(t)$ **is the tangent vector to the curve.**
- $\mathbf{v}(t) =< \mathbf{V}, \mathbf{V} >^{\frac{1}{2}}$ **the speed of** γ.
- **Frenet Frame** $\begin{cases} \mathbf{T}(t) \text{ \textbf{unit tangent} to } \gamma. \\ \mathbf{N}(t) \text{ \textbf{unit normal}}. \\ \mathbf{B}(t) \text{ \textbf{unit binormal}}. \end{cases}$
- $\kappa(t) = \|\nabla_T \mathbf{T}\|$ **the curvature (**κ **denotes the oriented curvature if** γ **is a curve in an oriented surface** $\mathbb{M}^2$**).**

Notation

- $\gamma_w(t) = \gamma(w,t) : (-\varepsilon, \varepsilon) \times \mathbb{I} \to \mathbb{M}^n$ **denotes a variation of** $\gamma(t) = \gamma\ (0,t)$
- $\mathbf{W} = \mathbf{W}(t) = \frac{\partial \gamma}{\partial w}(0,t)$ **variational vector field along the curve** γ
- $s \in [0, L]$ **denotes the arclength parameter of** $\gamma(s)$ **(**L **is the length of** γ**)**

Notation

A vector field W defined on regular curve γ immersed in $\mathbb{M}^3(G)$, is called a Killing field along γ, if for any variation in the direction of W, we have

$$\frac{\partial v}{\partial w} = \frac{\partial \kappa}{\partial w} = \frac{\partial \tau}{\partial w} = 0. \tag{2.3}$$

(Langer-Singer) A Killing field along γ is the restriction of a Killing field defined on $\mathbb{M}^3(G)$.

2.2. A useful tool I: Hopf Cylinders

- **We recall that the Hopf map, $\pi : \mathbb{S}^3(1) \to \mathbb{S}^2(\frac{1}{2})$ is a Riemannian submersion when the base space $\mathbb{S}^2(\frac{1}{2})$ is chosen to have radius $\frac{1}{2}$.**

- **If β is a curve in the two sphere, then $\bar{\beta}$ will denote a horizontal lift of β in the three sphere.**

- **For any curve $\beta(s)$ in $\mathbb{S}^2$, its complete lift**

$$\mathbf{T}_\beta = \pi^{-1}(\beta) = \{e^{it}.\overline{\beta}(s) \,:\, (s,t) \in \mathbb{R}^2\}$$

is called the Hopf Cylinder shaped on β.

A useful tool I: Hopf Cylinders

- **They are flat surfaces with the induced metric from $\mathbb{S}^3$.**
- **A Hopf cylinder $\mathbf{T}_\beta$ is embedded in $\mathbb{S}^3$ if β is a simple curve in $\mathbb{S}^2$.**
- **If β is a closed curve, then the Hopf tube $\mathbf{T}_\beta$ is a flat torus, whose isometry type depends on the length and enclosed area of β.**
- **The whole extrinsic geometry of $\mathbf{T}_\beta$ is governed by the curvature function of β in $\mathbb{S}^2$.**

A useful tool I: Hopf Cylinders

- **The map $\phi = \phi(z,t) : \mathbb{R}^2 \to \mathbf{T}_\beta$, defined by**

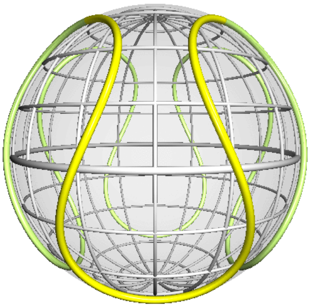
$$\phi(z,t) = e^{iz}\bar{\beta}(t) = \cos z\bar{\beta}(t) + \sin z\eta(t),$$

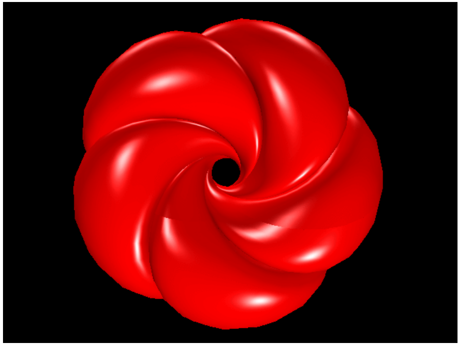
works as a covering map.

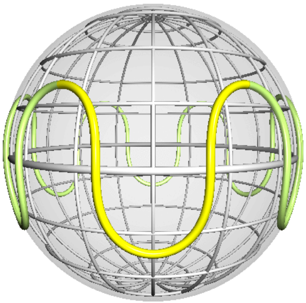
- **$\mathbf{T}_\beta = \pi^{-1}(\beta)$ is isometric to $\mathbb{R}^2/R$, where R is the lattice in $\mathbb{R}^2$ span by $(2A, L)$ and $(2\pi, 0)$.**

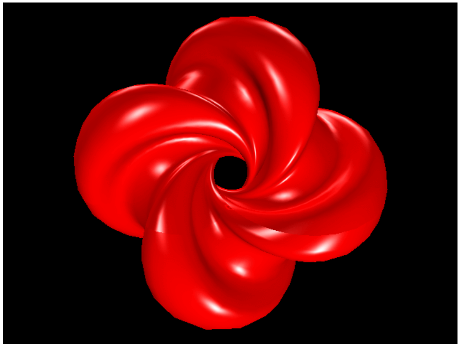
- **Here L denotes the length of β and $A \in (-\pi, \pi)$ the oriented area enclosed by β in the two sphere.**

Examples of Hopf Tori









2.3. A useful tool II: Lancret's curves

- A generalized helix (or Lancret's curve) in $\mathbb{R}^3$ is a curve which makes a constant angle with a fixed straight line (the axis of the general helix).
- Algebraic characterization: the ratio of torsion to curvature is constant (M.A. Lancret, 1802; B. de Saint Venant, 1845.)
- Geometric characterization: A curve in $\mathbb{R}^3$ is a Lancret's one if and only if it is a geodesic of a right cylinder shaped on a plane curve.

Ordinary helices (constant curvature and torsion) are called trivial Lancret's curves.

A useful tool II: Lancret's curves

A curve unit $\gamma(s)$ in $\mathrm{M}^3(\mathrm{G})$ will be called a general helix if there exists a Killing vector field $V(s)$ with constant length along γ (the axis), such that the angle between V and γ' is a non-zero constant along γ.

Obvious examples of general helices are:

- Any curve in $\mathrm{M}^3(\mathrm{G})$ with $\tau \equiv 0$. In this case just take $V = B$ to have an axis.
- Ordinary helices. In this case $V(s) = \cos\theta \cdot T(s) + \sin\theta \cdot B(s)$ with $\cot\theta = \frac{\tau^2 - c}{\tau\kappa}$ works as an axis.

(The Lancret theorem in 3-space forms)
M. Barros proved the following:

- **A curve γ in $\mathbb{H}^3(-1)$ is a general helix if and only if either (1) $\tau \equiv 0$ and γ is a curve in some hyperbolic plane, or (2) γ is an ordinary helix.**

- **A curve γ in $\mathbb{S}^3(1)$ is a general helix if and only if either (1) $\tau \equiv 0$ and γ is a curve in some unit 2-sphere, or (2) there exists a constant b such that**

$$\tau = b\kappa \pm 1.$$

Lancret's curves and Hopf Cylinders

The geometric integration of natural equations is obtained as follows:

- **A curve in $\mathbb{S}^3(1)$ is a general helix if and only if it is a geodesic of a Hopf cylinder.**
- **A curve in $\mathbb{S}^3(1)$ is an ordinary helix if and only if it is a geodesic of a Hopf torus shaped on a circle.**

2.4. Total curvature functional

Closed critical points of the total curvature functional

$$\mathcal{F}(\gamma) = \int_\gamma (\kappa + \lambda) ds \qquad (2.4)$$

in space forms $\begin{cases} \lambda = 0 : \textbf{free model;} \\ \lambda \neq 0 : \textbf{constrained model.} \end{cases}$

The Euler-Lagrange equations are:

$$R(N, T)T = (\tau^2 + \lambda\kappa)N - \tau_s B + \tau\Upsilon, \qquad (2.5)$$

where Υ belongs to the Frenet frame normal bundle

Solutions to the free model: $\lambda = 0$.

1. **The Gaussian curvature vanishes on critical points γ lying on surfaces.**

2. **In a real space form $\mathbb{M}^n(G)$, trajectories actually lie in $\mathbb{M}^3(G)$.**

3. **If γ is a critical point for $\mathcal{F}$ which is fully immersed in $\mathbb{M}^3(G)$, then:**

   - $\tau^2 = G > 0.$.

We only need to consider $\mathbb{S}^3(1)$. Critical points for $\mathcal{F}$ are horizontal lifts via the Hopf map of curves in $\mathbb{S}^2(\frac{1}{2})$).

Closed solution to the free model: $\lambda = 0$.

Let β be an immersed closed curve in $\mathbb{S}^3(1)$, then β is a critical point for $\mathcal{F}$, if and only if, there exists a natural number, say m, such that

β is a horizontal lift, via the Hopf map, of the m-fold cover of an immersed closed curve γ in $\mathbb{S}^2(\frac{1}{2})$, whose enclosed oriented area A is a rational multiple of π

$A = \frac{p}{m}\pi$, where p and m are relative primes.

Total curvature functional: Examples

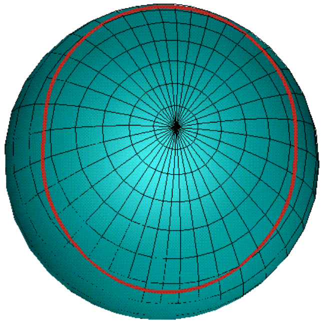
The spherical elliptic lemniscate: In spherical coordinates (ϕ, θ) on $\mathbb{S}^2(\frac{1}{2})$,

$$\gamma : \frac{1}{4}\left(\phi^2 + \sin^2\theta\right)^2 = a^2 \sin^2\theta + b^2\phi^2,$$

with parameters a and b satisfying $b^2 \geq 2a^2$.

This curve is the image under a Lambert projection of an elliptic lemniscate in the plane.

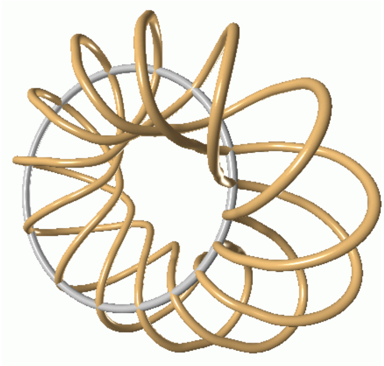
$a^2 = \frac{1}{8}$, $b^2 = 1$ - - →



Total curvature functional: Examples

Since the Lambert projection preserves the area, the area enclosed by γ in $\mathbb{S}^2(\frac{1}{2})$ is $A = \frac{a^2+b^2}{2}\pi$. Now we choose a and b such that $a^2 + b^2$ is a rational number, say $\frac{p}{q}$, with $a^2 + b^2 \leq 1$.

Then, a horizontal lift of the $2q$-fold cover of γ gives a critical point for $\mathcal{F}$ in $\mathbb{S}^3(1)$.



H-lift of the 16^{th}-cover

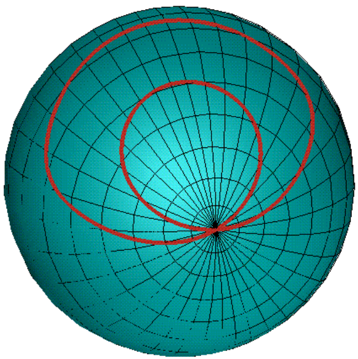
Total curvature functional: Examples

The spherical limaçon or the spherical snail of Pascal. Given real parameters a and h.

$$\gamma : \left(\frac{1}{2}\phi^2 + \frac{1}{2}\sin^2\theta - 2a\phi\right)^2 = h^2(\phi^2 + \sin^2\theta),$$

This is nothing but the image under the Lambert projection of a snail of Pascal.

$a = \frac{1}{4}$, $b = \frac{1}{8}$ $\dashrightarrow$



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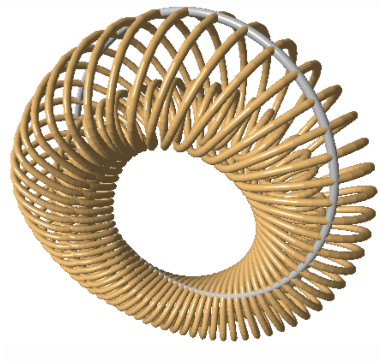
Total curvature functional: Examples

Therefore, γ encloses the area $A = \left(h^2 + \frac{1}{2}a^2\right)\pi$.

Again, for a suitable choice of parameters a and h, we get examples of critical points for $\mathcal{F}$ in $\mathbb{S}^3(1)$ by applying the above proposition.



Horizontal lift of the 64^{th}-cover

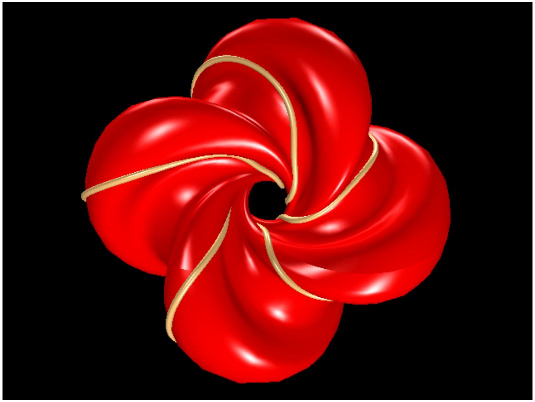
Total curvature functional: constrained model

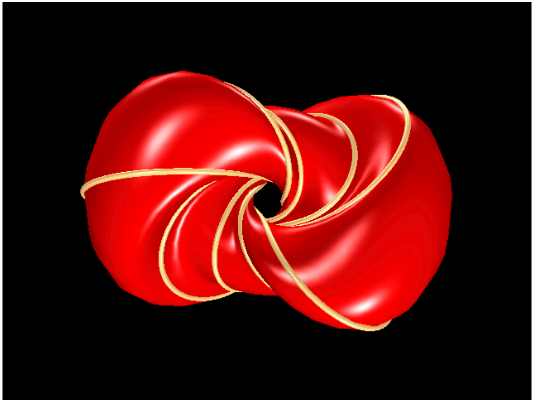
$$\mathcal{F}(\gamma) = \int_\gamma (\kappa + \lambda) ds, \ \lambda \neq 0 \, .$$

- **The whole space of closed trajectories in the constrained model is formed by a rational one-parameter family of closed helices in $\mathbb{S}^3$. Geometrically, they are geodesics of circular Hopf tori which are obtained when the slope is quantized by a rational constraint.**

The solution of our problem is encoded in the geometry of the Hopf Tori.

Examples of closed trajectories





The energy functional is given by

$$\mathcal{F}_{mnp}(\gamma) = \int_\gamma (m + n\kappa + p\tau)ds, \tag{2.6}$$

Second order boundary conditions

Given $q_1, q_2 \in \mathrm{M}^3(\mathbf{c})$ and $\{x_1, y_1\}$, $\{x_1, y_1\}$ orthonormal vectors in $T_{q_1}\mathrm{M}^3(\mathbf{c})$ and $T_{q_2}\mathrm{M}^3(\mathbf{c})$ respectively, define the space of curves

$$\Lambda = \{\gamma : [t_1, t_2] \to \mathbf{M}^3(\mathbf{c})\} \; : \\ \gamma(t_i) = q_i, T(t_i) = x_i, N(t_i) = y_i, \\ 1 \le i \le 2.$$

First order particles models

Then, the critical points of the variational problem $\mathcal{F}_{mnp} : \Lambda \to \mathbb{R}$ **are characterized by the following Euler-Lagrange equations**

$$\begin{aligned} -m\kappa + p\kappa\tau - n\tau^2 + nc &= 0, \\ p\kappa_s - n\tau_s &= 0. \end{aligned}$$

First order particles models

m	n	p	**Solutions in** $\mathbb{R}^3$**,** $c = 0$
$\neq 0$	$= 0$	$= 0$	**Geodesics** $\kappa = 0$
$= 0$	$= 0$	$\neq 0$	**Circles** κ **constant and** $\tau = 0$
$= 0$	$\neq 0$	$= 0$	**Plane curves** $\tau = 0$
$\neq 0$	$\neq 0$	$= 0$	**Ordinary Helices with** $\kappa = \frac{-n\tau^2}{m}$
$\neq 0$	$= 0$	$\neq 0$	**Ordinary Helices with arbitrary** κ **and** $\tau = \frac{m}{p}$
$= 0$	$\neq 0$	$\neq 0$	**Lancret curves with** $\tau = \frac{p}{n}\kappa$
$\neq 0$	$\neq 0$	$\neq 0$	**Ordinary Helices with** $\kappa = \frac{-na^2}{m+ap}$**,** $\tau = \frac{ma}{m+ap}$ **and** $a \in \mathbb{R} - \{-\frac{m}{p}\}$

First order particles models

In the Euclidean space, non-trivial Lancret curves appear just for models with $m = 0$ and $p.n \neq 0$, that is for

$$\mathcal{F}_{mnp}(\gamma) = \int_\gamma (n\kappa + p\tau)ds$$

In this cases the ratio $\frac{p}{n}$ determines the slope of the solutions. In other words, $\frac{p}{n} = \cot\theta$, where θ is the angle that the Lancret curve makes with the axis.

First order particles models

m	n	p	**Solutions in** $\mathbb{H}^3$, $C=-c^2$
$\neq 0$	$=0$	$=0$	**Geodesics** $\kappa=0$
$=0$	$=0$	$\neq 0$	**Curves with** κ **constant and** $\tau=0$
$=0$	$\neq 0$	$=0$	**Do not exist**
$\neq 0$	$\neq 0$	$=0$	**Ordinary Helices with** $\kappa=\frac{-n(c^2+\tau^2)}{m}$
$\neq 0$	$=0$	$\neq 0$	**Ordinary Helices with arbitrary** κ **and** $\tau=\frac{m}{p}$
$=0$	$\neq 0$	$\neq 0$	**Ordinary Helices with** $\kappa=\frac{-n(c^2+a^2)}{ap}$ **and** $\tau=-\frac{c^2}{a}$ **and** $a\in\mathbb{R}-\{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	**Ordinary Helices with** $\kappa=\frac{-n(c^2+a^2)}{m+ap}$, $\tau=\frac{ma-pc^2}{m+ap}$ **and** $a\in\mathbb{R}-\{-\frac{m}{p}\}$

First order particles models

m	n	p	**Solutions in $\mathbb{S}^3$, $C = c^2$**
$\neq 0$	$= 0$	$= 0$	**Geodesics** $\kappa = 0$
$= 0$	$= 0$	$\neq 0$	**Circles** κ **constant and** $\tau = 0$
$= 0$	$\neq 0$	$= 0$	**Horizontal lifts, via the Hopf map, of curves in $\mathbb{S}^2$**
$\neq 0$	$\neq 0$	$= 0$	**Ordinary Helices with** $\kappa = \frac{n(c^2-\tau^2)}{m}$
$\neq 0$	$= 0$	$\neq 0$	**Ordinary Helices with arbitrary** κ **and** $\tau = \frac{m}{p}$
$= 0$	$\neq 0$	$\neq 0$	**Ordinary Helices with** $\kappa = \frac{n(c^2-a^2)}{ap}$ **and** $\tau = \frac{c^2}{a}$ **and** $a \in \mathbb{R} - \{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	**Ordinary Helices with** $\kappa = \frac{n(c^2-a^2)}{m+ap}$**,** $\tau = \frac{ma+pc^2}{m+ap}$ **and** $a \in \mathbb{R} - \{-\frac{m}{p}\}$
$\neq 0$	$\neq 0$	$\neq 0$	**Lancret curves with** $\tau = \frac{p}{n}\kappa - \frac{m}{p}$ **and** $c = \pm\frac{m}{p}$

First order particles models

The most interesting models on spheres are those where $m.n.p \neq 0$.

$$\mathcal{F}_{mnp}(\gamma) = \int_{\gamma} (m + n\kappa + p\tau) ds, \tag{2.7}$$

Remember: general helices in $\mathbb{S}^3$ are completely determined from both a curve in the $\mathbb{S}^2$ and a slope, that is the angle that the helix makes, in the corresponding Hopf tube, with the axis (i.e. with the fibres).
In this cases the ratio $\frac{m}{p}$ is determined from the radius of the sphere while the ratio $\frac{p}{n}$ gives the slope of the solutions.

Notice that, in particular, the horizontal lifts of curves in the two sphere are general helices of the three sphere with slope $\frac{\pi}{2}$.

Let β_{np} be the geodesic in $M_\beta = \pi^{-1}(\beta)$ with slope θ, $\cot\theta = \frac{p}{n}$. From the third table one sees, for example, the following.

Let γ be a curve in $\mathbb{S}^3(1)$, then it is a critical point of $\mathcal{F}_{nnp}$, $n.p \neq 0$, if and only if either

1. **γ is a helix with curvature $\kappa = \frac{n(1-a^2)}{n+ap}$ and torsion $\tau = \frac{na+p}{n+ap}$ and $a \in \mathbb{R} - \{-\frac{n}{p}\}$, or**
2. **$\gamma \in \{\beta_{\mathbf{np}} : \beta \text{ is a curve in } \mathbb{S}^{\mathbf{2}}(\tfrac{\mathbf{1}}{\mathbf{2}})\}$**

First order particles models

We study the variational problem on the space of closed curves.

- **There are no closed critical points in $\mathbb{R}^3$ and $\mathbb{H}^3$ other than closed "plane" curves.**
- **Spherical case. We will restrict ourselves to the unit sphere.**
- **Closed generalized helices in $\mathbb{S}^3(1)$ can be characterized as follows.**

First order particles models

- **For any curve $\beta(s)$ in $\mathbb{S}^2$, we take $\mathbf{T}_\beta = \pi^{-1}(\beta)$ the Hopf Cylinder shaped on β.**

- **From the isometry type of $\mathbf{T}_\beta$, we have that a geodesic γ of $\mathbf{T}_\beta$ closes up, if and only if, its slope $\omega = \cot\theta$ satisfies**

$$\omega = \frac{1}{L}(2A + q\pi),$$

where q is a rational number.

- **On the other hand, $\gamma \in \Omega$ is a critical point of $\mathcal{F}_{nnp}$ if and only if its slope satisfies $\omega = \frac{p}{n}$.**

Then, we have

Proposition. **Let β be an embedded closed curve in $\mathbb{S}^2(\frac{1}{2})$, with length $L > 0$ and enclosing an oriented area $A \in (-\pi, \pi)$. The geodesic with slope ω in $\mathbf{T}_\beta = \pi^{-1}(\beta)$ is a critical point of the variational problem $\mathcal{F}_{mnp} : \Omega \to \mathbb{R}$ in $\mathbb{S}^3(1)$ if and only if the following relationship holds**

$$\frac{\omega L - 2A}{\pi} \in \mathbb{Q}.$$

We can assume the area A to be positive, changing if necessary the orientation of β.

First order particles models

The only further restriction on (A, L) to define an embedded closed curve in the two sphere is given by the isoperimetric inequality in $\mathbb{S}^2(\frac{1}{2})$:

$$L^2 + 4A^2 - 4\pi A \geq 0.$$

In terms of $(2A, L)$, the above inequality is written as

$$L^2 + (2A - \pi)^2 \geq \pi^2.$$

In the $(2A, L)$-plane, we define the region

$$\Delta = \{(2A, L)\,:\, L^2 + (2A - \pi)^2 \geq \pi^2 \,\text{and}\, 0 \leq A \leq \pi\},$$

First order particles models

For each point $(2A, L) \in \Delta$ there is an embedded closed curve on $\mathbb{S}^2(\frac{1}{2})$ with length L and enclosed area A.

Theorem. **For any couple of parameters, n and p with $n.p \neq 0$, there exists an infinite series of closed general helices that are extremal for the variational problem $\mathcal{F}_{nnp} : \Omega \to \mathbb{R}$ in $\mathbb{S}^3(1)$. This series includes all the geodesics β_{np} in $\mathbf{T}_\beta = \pi^{-1}(\beta)$ with slope $\omega = \frac{p}{n}$ and β determined as above by $(2A, L)$ in the following region**

$$\Delta \cap \left(\cup_{q \in \mathbb{Q}} (\omega L - 2A = q\pi)\right).$$

Particle Models arising from Geometry

- **Lagrangians describing relativistic particles, have a long history in Physics.**
- **The conventional approach considers Lagrangians which depend on higher derivatives of the curve γ that represents the worldline of the particle in the spacetime.**
- **Investigation of these models in the classical variational setting, gives rise to very complicated nonlinear differential equations which are difficult to analyze.**

Some applications: Particle models

- **Recent geometric models are intrinsic. They describe the particles inside the original space-time where the system is evolving.**
- **The motion of the particle is described by an action of the form,**

$$\Theta(\gamma) = \int_\gamma P(\kappa_1, \kappa_2, ..., \kappa_{n-1}),$$

which is a functional of the *Frenet curvatures* of the worldline γ.

Some applications: Particle models

- **For Lagrangians of this form, the Euler-Lagrange equations can be always formulated in terms of the *Frenet curvatures* κ_i.**
- **A basic point here is that in a space-time of constant curvature c, the Frenet frame provides a complete kinematical description of the particle motion: once we know its *Frenet curvatures* κ_i, the trajectory of the particle can be reconstructed up to rigid motions.**

Total curvature functional: Some applications

- **A space-time where the dynamics of particles happens ($\mathbb{M}^n$ Riemannian or Lorentzian);**
- **A regular curve γ with $n-1$ curvature functions, $\kappa_1, \kappa_2, \cdots, \kappa_{n-1}$:**

$$\begin{cases} \textbf{they are invariant under the group of motions} \\ \textbf{sometimes, they uniquely determine the curve} \end{cases}$$

- **An action defined by Lagrangian densities depending on the curvatures**

$$\mathcal{F} : \mathbf{\Omega} \to \mathbb{R}, \quad \mathcal{F}(\alpha) = \int_\alpha P\left(\kappa_1, \kappa_2, \cdots, \kappa_{n-1}\right)(s)\, ds.$$

Total curvature functional: Some applications

- **Y.A. Kuznetsov and M.S. Plyushchay**, ***Nucl. Phys. B***, **253(1-2)(1991) 50–55.**
- **M.S. Plyushchay**, ***Phys. Lett. B***, **389 (1993) 181.**
- **V.V. Nesterenko, A. Feoli and G. Scarpetta** , ***J. Math. Phys.***, **36 (1995) 5552.**
- **A. Nersessian**, ***Phys. Lett. B***, **473 (1996) 1201.**
- **G. Arreaga, R. Capovilla, and J. Guven**, ***Class. Quantun Grav.***, **18 (2001) 5065–5083.**

Particular cases:

- **1. Geodesics.**

$$P(\kappa_1, \kappa_2, \cdots, \kappa_{n-1}) = c, \quad \textbf{constant}.$$

This model describes free fall particles in M^n.

- **2. Massless Bosons, (Plyushchay, 1990). Trajectories are critical points of the total curvature**

$$P(\kappa_1, \kappa_2, \cdots, \kappa_{n-1}) = c\,\kappa_1, \quad \mathcal{F}(\alpha) = c \int_\alpha \kappa(s)\, ds.$$

Particular cases:

- **3. Massive Bosons.**

$$P\left(\kappa_1, \kappa_2, \cdots, \kappa_{n-1}\right) = c\,\kappa_1 + m,$$

$$\mathcal{F}(\alpha) = \int_\alpha \left(c\,\kappa(s) + m\right)\, ds.$$

- **4. Tachyonless models of relativistic particles.**

$$\mathcal{F}_{mnp}(\alpha) = \int_\alpha \left(m + n\,\kappa_1 + p\,\kappa_2\right)\, ds.$$

The order one rigidity model (Plyushchay)

$$\mathcal{F}_m : \mathbf{\Omega} \to \mathbb{R}, \qquad \mathcal{F}_m(\gamma) = \int_\gamma (\kappa(s) + m)\, ds,$$

In Riemannian and Lorentzian Surfaces, trajectories of particles are the solutions of the following equations:

$$m\,\kappa = \varepsilon_2 \mathbf{G}.$$

Trajectories of the free model i.e. massless model $m = 0$ correspond with those curves made up of parabolic points.

In higher dimensions, the free total curvature (Plyushchay), model is consistent only in three spheres or in anti-de-Sitter three spaces.

The Dynamics in the three sphere has been previously described.

To completely describe the Dynamics in the anti de Sitter three space $\mathbf{AdS}_3$, one has to determine the family of helices:

$$\{(\kappa,\tau)\in\mathbb{R}^2 \,:\, \tau^2-\varepsilon_2 m\kappa=1\}.$$

Total curvature functional: constrained model

M. Barros, A. Ferrandez, M.A. Javaloyes and P. Lucas, *Class. Quantun Grav.*, **35 489–513 (2005)**

Massive spinning particles in AdS$_3$ described by the Lagrangian $\mathcal{F}_m$, with $m \neq 0$, evolve generating worldlines that are helices in AdS$_3$. The complete solution of the motion equations consists of a one-parameter family of non-congruent helices. The moduli space of solutions may be described by three different (but equivalent) pairs of dependent real moduli.

First order particles models

The previous program can be extended to study models describing relativistic particles where Lagrangian densities depend linearly on both the curvature and the torsion of the trajectories in D = 3 Lorentzian spacetimes with constant curvature:

- Y.A. Kuznetsov and M. S. Plyushchay, ***Nucl. Phys. B***, 389 (1993) 181.
- M. Barros, A. Ferrandez, M.A. Javaloyes and P. Lucas, ***Class. Quantun Grav.***, 35 (2005) 489–513.

First order particles models

$$\mathcal{F}_{mnp}(\gamma) = \int_{\gamma} (m + n\kappa + p\tau)ds,$$

- **The moduli spaces of trajectories are completely and explicitly determined.**
- **Trajectories are Lancret curves including ordinary helices.**
- **The geometric integration of the solutions is obtained using the Lancret program as well as the notions of B-scrolls and Hopf tubes.**
- **The moduli subspaces of closed solitons in anti-de Sitter settings are also obtained.**

3. Higher Order Functionals: Euler-Lagrange Equations

3.1. First variation formula

PROBLEM

existence and classification of critical points and minimizers of the generalized Euler-Bernoulli energy functional

$$\mathcal{F}(\gamma) = \int_\gamma \mathbf{P}(\kappa). \tag{3.8}$$

acting on spaces of curves in a Riemannian mani fold ($P(t)$ is a C^∞ function)

Lemma 1.(J. Langer and D. Singer, 1985)

With the previous notation, we have:

1. $[V, W] = 0.$

2. $[W, T] = gT, \ where < \nabla_T W, T >= -g.$

3. $[[W, T], T] = -T(g)T = -g_s T.$

4. $\frac{\partial v}{\partial w} =< \nabla_T W, T > v = -gv.$

5. $$\frac{\partial \kappa}{\partial w} =< R(W, T)T, \nabla_T T > + < \nabla_T^2 W, N > -2 < \nabla_T W, T > \kappa$$

First variation formula

Moreover, if $M^n(G)$ is a Riemannian manifold of constant sectional curvature G then

$$\frac{\partial \tau}{\partial w} = < \frac{1}{\kappa}\nabla_T^3 W - \frac{\kappa_s}{\kappa^2}\nabla_T^2 W, B > + \quad \left(\frac{G}{\kappa} + \kappa\right)\nabla_T W$$
$$-\frac{\kappa_s}{\kappa^2} < GW, B >$$

where τ is the torsion of the curve

First variation formula

We take $P(t)$ a smooth function and consider the following curvature energy functional

$$\mathcal{F}(\gamma) = \int_\gamma \mathbf{P}(\kappa) = \int_\mathbf{0}^\mathbf{L} \mathbf{P}(\kappa)\mathbf{ds} = \int_\mathbf{0}^\mathbf{1} \mathbf{P}(\kappa) \cdot \mathbf{v} \cdot \mathbf{dt}. \quad (3.9)$$

acting on $\mathcal{H}$. ($P(t)$ is a C^∞ function and $\mathbf{v}(\mathbf{t}) =< \gamma', \gamma' >^{\frac{1}{2}}$).

First variation formula

By using

- **lemma 1,**
- **the first Frenet formula $\nabla_T T = \kappa N$, and**
- **integration by parts,**

we can obtain the first derivative of $\mathcal{F}$.

Notation $\begin{cases} P'(\kappa) = \frac{dP}{d\kappa} \\ \mathcal{K} = P'(\kappa) \cdot N, \\ \mathcal{J} = \nabla_T \mathcal{K} + (2\kappa P'(\kappa) - P(\kappa)) \cdot T, \\ \mathcal{E} = \nabla_T \mathcal{J} + P'(\kappa) \cdot R(N, T)T, \end{cases}$

First variation formula

Proposition 1. (First Variation Formula) **Under the above conditions and notation, the following formula holds:**

$$\frac{d}{dw}\mathcal{F}(\gamma)_{|w=o} = \int_0^L < \mathcal{E}, W > ds + \mathcal{B}[W,\gamma]\,_0^L,$$

where

$$\mathcal{B}[W,\gamma]\,_0^L = [< \mathcal{K}, \nabla_T W > - < \mathcal{J}, W >]\,_0^L.$$

Thus, under suitable boundary conditions, one sees that a critical point of $\mathcal{F}$ will satisfy the following Euler-Lagrange equation

$$\begin{aligned}\mathcal{E} \;=\; & \nabla_T^2 P'(\kappa)\cdot N + \nabla_T\left(2\kappa P'(\kappa) - P(\kappa)\right)\cdot T + \\ & + P'(\kappa)\cdot R(N,T)T = 0.\end{aligned}$$

Proposition 1.(Euler-Lagrange equations in real space forms of constant curvature G, $\mathbb{M}^n(G)$)

$$\left(\kappa^2 - \tau^2 + G\right) \cdot P'(\kappa) + \frac{d^2 P'}{ds^2} = \kappa \cdot P(\kappa), \tag{3.10}$$

$$2 \cdot \frac{dP'}{ds} \cdot \tau + P'(\kappa) \cdot \tau_s = 0, \tag{3.11}$$

$$P'(\kappa) \cdot \eta = 0, \tag{3.12}$$

- η **belongs to the normal bundle to** $span\,\{T, N, B\}$.

Hence, a critical point γ must lie fully in either a 2-dimensional or a 3-dimensional totally geodesic submanifold of $M^n(G)$.

Thus our problem in space forms reduces to:

To determine explicitly the closed critical curves in a 3-dimensional real space form $M^3(G)$:

3.4. Solving the Euler-Lagrange equation

1. **To explicitly integrate** $\mathcal{E} = 0$
   - **Impossible for a general** P**.**
2. **Even if we assume the existence of periodic solutions** κ, τ, **the corresponding periodic curves** γ **in** $\mathbb{M}^3(G)$ **are not necessarily closed**
   - **We need to establish closure conditions for these critical points**
3. **We need to compute the second variation formula to locate minima.**

Solving the Euler-Lagrange equation

1. **For a general P:**

$$\begin{cases} \textbf{compute first integrals of } \mathcal{E} = 0 \\ \textbf{give closure conditions of critical } \gamma. \\ \textbf{compute the second variation formula} \end{cases}$$

2. **For "suitable" choices of P : solve the Euler-Lagrange equations (explicitly or by quadratures) and determine the closed critical points**

Solving the Euler-Lagrange equation

- **to establish closure conditions for critical points γ associated to periodic solutions of the Euler-Lagrange equation**
- **we construct and adapted coordinate system**
- **depends on** $\begin{cases} \textbf{space of Killing fields of } \mathbb{M}^{\mathbf{3}}(\mathbf{G}) \\ \textbf{choice of } \mathbf{P} \end{cases}$

3.5. First integrals of $\mathcal{E} = 0$

Assumption: $\frac{dP}{ds} \neq 0$.

To integrate the E-L equations in this case, we use the following method

- **Find Killing fields along a critical point $\gamma(s)$ expressible in terms of the local invariants of the curve.**
- **Use them along with a sort of Noether's argument to facilitate integration of the Euler-Lagrange equations**

First integrals of $\mathcal{E} = 0$

A vector field W defined on regular curve γ immersed in $\mathbb{M}^3(G)$, is called a Killing field along γ, if for any variation in the direction of W, we have

$$\frac{\partial v}{\partial w} = \frac{\partial \kappa}{\partial w} = \frac{\partial \tau}{\partial w} = 0. \tag{3.13}$$

- **(Langer-Singer) A Killing field along γ is the restriction of a Killing field defined on $M^3(G)$**

  .

First integrals of $\mathcal{E} = 0$

From Lemma 1, we can see that W is a Killing field along γ, if and only if,

$$< \nabla_{\mathbf{T}}\mathbf{W}, \mathbf{T} >= \mathbf{0},$$

$$< \nabla^{\mathbf{2}}_{\mathbf{T}}\mathbf{W}, \mathbf{N} > +\mathbf{G}\cdot < \mathbf{W}, \mathbf{N} >= \mathbf{0},$$

$$< \frac{\mathbf{1}}{\kappa}\nabla^{\mathbf{3}}_{\mathbf{T}}\mathbf{W} - \frac{\kappa_{\mathbf{s}}}{\kappa^{\mathbf{2}}}\nabla^{\mathbf{2}}_{\mathbf{T}}\mathbf{W} + \left(\frac{\mathbf{G}}{\kappa} + \kappa\right)\nabla_{\mathbf{T}}\mathbf{W} - \frac{\kappa_{\mathbf{s}}}{\kappa^{\mathbf{2}}}\mathbf{G}\cdot\mathbf{W}, \mathbf{B} >= \mathbf{0}.$$

First integrals of $\mathcal{E} = 0$

Consider the following vector fields along γ

$$\mathcal{J} = (\kappa P'(\kappa) - P(\kappa))\,\mathbf{T} + \frac{dP'}{d\kappa} \cdot \mathbf{N} + \tau P'(\kappa)\mathbf{B}, \qquad (3.14)$$

$$\mathcal{I} = -P'(\kappa)\mathbf{B}, \qquad (3.15)$$

Proposition 2.
Let $\gamma : \mathbb{I} = [0,1] \rightarrow M^3(G)$ be a critical point of $\mathcal{F}$.
Then the vector fields $\mathcal{J}$ and $\mathcal{I}$ defined in (3.14) and (3.15) respectively, are Killing fields along γ.

First integrals of $\mathcal{E} = 0$

Now if γ happens to be a critical point of $\mathcal{F}$ (under any boundary conditions), then standard arguments imply that $\mathcal{E} = 0$ on γ. The variation formulas continue to hold when L is replaced by any intermediate value $t \in (0, L)$ and, therefore, the first variational formula

$$\frac{d}{dw}\mathcal{F}(\gamma)_{|w=o} = \int_0^t < \mathcal{E}, W > ds \; + \mathcal{B}\left[W, \gamma\right]\,_0^t.$$

reduces to

$$\frac{d}{dw}\mathcal{F}(\gamma)_{|w=o} = \mathcal{B}\left[W, \gamma\right]\,_0^t. \tag{3.16}$$

First integrals of $\mathcal{E} = 0$

Therefore, for any Killing field W on $\mathbb{M}^3(G)$, we have from (3.16)

$$0 = \mathcal{B}\left[W, \gamma\right]_0^t , \tag{3.17}$$

and $\mathcal{B}\left[W, \gamma\right](t)$, is constant along γ. Applying this to $\mathcal{I}, \mathcal{J}$, we have

$$< \mathcal{I}, \mathcal{J} >= c, \tag{3.18}$$

$$< \mathcal{I}, \mathcal{J} > +G < \mathcal{I}, \mathcal{I} >= e, \tag{3.19}$$

on γ, where c is and e are constant.

Now, plug (3.15) and (3.14) into (3.18) and (3.19) to obtain

Proposition 2.
(First Integrals of the Euler-Lagrange equations in space forms)
With the above notation,

$$e = \tau \cdot \left(P'(\kappa)\right)^2 , \tag{3.20}$$

$$d = \left(P''(\kappa)\right)^2 \cdot \kappa_s^2 + \left(\kappa \cdot P'(\kappa) - P(\kappa)\right)^2 + +G \cdot \left(P'(\kappa)\right)^2 + \frac{e^2}{\left(P'(\kappa)\right)^2} \tag{3.21}$$

$\kappa(s)$, $\tau(s)$ periodic solutions of Euler-Lagrange equations; $\gamma(s)$ the corresponding curve in $\mathbb{M}^3(G)$; $\mathcal{J}$, $\mathcal{I}$ the associated Killing fields and their extensions to $\mathbb{M}^3(G)$

Proposition 3.

The Killing fields $\mathcal{J}, \mathcal{I}$ commute : $[\mathcal{J}, \mathcal{I}] = \mathbf{0}$.

We use this to find a coordinate system where:

$$\left\{ \begin{array}{l} \text{the coordinates of } \gamma \\ \text{closure conditions} \end{array} \right\} \text{ in terms of } \left\{ \begin{array}{c} P \\ \kappa \end{array} \right\}$$

3.7. Closure conditions in $\mathbb{S}^3(1)$.

Choose cylindrical coordinates in the 3-sphere

$$x(\theta, \varphi, \psi) = ...$$

$$... = (\cos\theta\cos\psi, \sin\theta\cos\psi, \cos\varphi\sin\psi, \sin\varphi\sin\psi),$$

$\theta, \varphi \in (0, 2\pi), \psi \in (0, \frac{\pi}{2})$

$$\gamma(s) = x(\theta(s), \varphi(s), \psi(s)). \tag{3.22}$$

By using (1) the above proposition; (2) the expressions for $\mathcal{J}$, $\mathcal{I}$: (3.14),(3.15); and (3) the first integrals of $\mathcal{E} = 0$:(3.18), (3.19), one can obtain

Closure conditions in $\mathbb{S}^3(1)$.

$$\begin{aligned} \theta'(s) &= \frac{b(\kappa P'(\kappa) - P(\kappa))}{b^2 - (P'(\kappa))^2}, \\ \varphi'(s) &= \frac{a(\kappa P'(\kappa) - P(\kappa))}{a^2 - (P'(\kappa))^2}, \\ \cos 2\psi &= 2\frac{\left(P'\left(\kappa\right)\right)^2 - b^2}{a^2 - b^2} - 1. \end{aligned} \tag{3.23}$$

So, from the above equations we have that the curvature κ, and the energy function P, basically determine the cylindrical coordinates $\theta(s), \varphi(s), \psi(s)$ of a critical point $\gamma(s)$

Moreover, closure conditions for critical point $\gamma(s)$ can be formulated in this system.

Proposition 4.

A critical point of periodic curvature γ will close up, if and only if, the angular progressions

$$\Lambda_\theta(\gamma) = \int_o^\rho \frac{b(\kappa P'(\kappa) - P(\kappa))}{b^2 - (P'(\kappa))^2},$$

$$\Lambda_\varphi(\gamma) = \int_o^\rho \frac{a(\kappa P'(\kappa) - P(\kappa))}{a^2 - (P'(\kappa))^2}.$$

are rational multiples of 2π.

Similarly

$$\left\{\begin{array}{c} \textbf{adapted cylindrical coordinates} \\ \underbrace{\textbf{more difficult process}} \\ \downarrow \\ \left\{\begin{array}{c} r(s),\ \ z(s),\ \ \varphi(s) \\ \textbf{closure conditions} \end{array}\right\} \textbf{expressed} \left\{\begin{array}{c} \kappa(s) \\ P(\kappa) \end{array}\right\} \end{array}\right.$$

Closure conditions in $\mathbb{R}^3$.

A critical point of periodic curvature γ, will close up in R^3, if and only if,

$$0 = \int_o^\rho (\kappa P'(\kappa) - P(\kappa))ds \ ,$$

and the angular progression

$$\Lambda_\varphi(\gamma) = \int_o^\rho \frac{e\sqrt{d}(\kappa P'(\kappa) - P(\kappa))}{e^2 - d(P'(\kappa))^2} ds$$

is a rational multiple of 2π.

Closure conditions in $\mathbb{H}^3$.

2-dimensional cases are obtained by taking $b = 0$ and $e = 0$ in the above formulas.

- **Proceeding in a similar way we can give closure conditions in $\mathbb{H}^2$.**
- **We are working out the closure conditions in $\mathbb{H}^3$.**

3.9. Particular cases

We shall discuss the above results for suitable choices of P . By ”suitable” we mean:

- $\mathcal{E} = 0$ **can be explicitly solved (at least, they can be solved by quadratures)**
- $P(\kappa)$ **has** $\begin{cases} \textbf{mathematical significance,} \\ \textbf{physical significance.} \end{cases}$

Particular cases

Examples of suitable choices where the method works

$$\begin{cases} \mathbf{P}(\kappa) = \kappa^{\mathbf{r}} \begin{cases} \text{hyperelastic curves} \\ \text{Chen-Willmore submanifolds} \\ \text{string theory} \end{cases} \\[2em] \mathbf{P}(\kappa) = (\kappa + \lambda)^{\mathbf{2}} \begin{cases} \text{elasticae circular at rest} \\ \text{membranes, vesicles} \end{cases} \\[2em] \mathbf{P}(\kappa) = (\kappa + \lambda)^{\frac{1}{2}} \begin{cases} \text{total curvature} \\ \text{relativistic particle models} \end{cases} \end{cases}$$

Particular cases: Closed solutions

$$
P(\kappa) = \kappa^r \begin{cases}
r = 1 \begin{cases} \textbf{total curvature functional} \\ \underline{\hspace{3cm}}\textbf{:}\ \mathbb{M}^n(c), n = 2, 3. \end{cases} \\
r = 2 \begin{cases} \textbf{classical elasticae functional} \\ \textbf{Euler-Rado-Langer-Singer and} \\ \underline{\hspace{3cm}}\textbf{:}\ \mathbb{M}^n(c), n = 2, 3.\ \textbf{except}\ \mathbb{H}^3. \end{cases} \\
r > 2 \begin{cases} \textbf{generalized elasticae functional} \\ \textbf{non-existence in}\ \mathbb{R}^2, \mathbb{S}^2, \mathbb{R}^3. \\ \mathbb{H}^2 : \textbf{solved for}\ r = 3; \textbf{exist. other.} \\ \mathbb{S}^3 : \textbf{solved for constant}\ \kappa; \textbf{e. o.} \\ \mathbb{H}^3 : \textbf{unknown so far.} \end{cases}
\end{cases}
$$

4. Classical elasticae in $\mathbb{S}^3(1)$

Critical points of the elastic energy functional

$$\mathcal{F}(\gamma) = \int_\gamma \kappa^2 \tag{4.24}$$

acting on closed curves of the 3-sphere.

- **Constant curvature**
- **Non-constant curvature**

4.1. Elasticae in $\mathbb{S}^3(1)$: Constant curvature

The set of constant curvature closed critical curves of $\mathcal{F}(\gamma) = \int_\gamma \kappa^2 ds$ in $\mathbb{S}^3(G)$ (and therefore, also with constant non-zero torsion: helices) is completely determined and forms a rational 1-parameter family $\{\gamma_q / q \in \mathbb{Q}^+ - \{\frac{1}{2}\}\}$.

- **The main point of the proof is that: Helices in $\mathbb{S}^3(G)$ can be considered as geodesics of Hopf tori.**

Elasticae in $\mathbb{S}^3(1)$: Constant curvature

Given a helix of known curvature and torsion (κ, τ), it may be seen as the geodesic of slope $g = \frac{1-\tau}{\kappa}$ contained in the Hopf torus $\mathbf{T}_\alpha$ shaped on the circle α of curvature $\rho = \frac{\kappa^2+\tau^2-1}{\rho}$ and enclosing an oriented area A of the sphere $\mathbb{S}^2(\frac{1}{2})$.

$\mathbf{T}_\alpha$ is determined by the lattice

$$\Gamma = span\{(0, 2\pi), (L, 2A)\},$$

where L is the length of α.

Elasticae in $\mathbb{S}^3(1)$: Constant curvature

A helix will be close, iff exists a rational number $q \neq 0$, such that

$$g = q\sqrt{\rho^2 + 4} - \frac{\rho}{2} \tag{4.25}$$

- **Given $\rho \in \mathbb{R}$, $q \in \mathbb{Q}$ we determine g by (4.25)**
- **The curvature and torsion (κ, τ) of the closed helix are obtained from $g = \frac{1-\tau}{\kappa}$, $\rho = \frac{\kappa^2+\tau^2-1}{\rho}$.**
- **In order to be a critical point, it must satisfy the Euler-Lagrange equation.**

Elasticae in $\mathbb{S}^3(1)$: Constant curvature

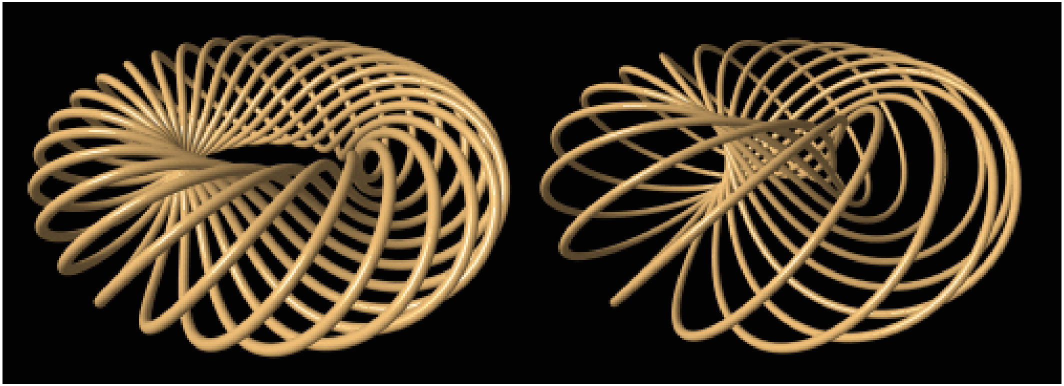
- **Hence the point is to find a real number ρ and a rational number q satisfying**

$$\mathcal{E}(\kappa(\rho, \mathbf{q}), \tau(\rho, \mathbf{q})) = \mathbf{0}.$$

- **We can show that, for any rational number $q \neq 0$, there exists a unique positive solution.**

Elasticae in $\mathbb{S}^3(1)$: Constant curvature

The following Figure shows the stereographic projection of the closed elastic helices corresponding to $q = 1$ and $q = \frac{1}{32}$.



Closed elastic helices γ_1 and $\gamma_{\frac{1}{32}}$

4.2. Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

To determine the closed critical points, our method required

1. **to explicitly obtain the periodic solutions κ, τ, of the Euler-Lagrange equations (first integrals);**
2. **to compute the ingredients in the closure conditions;**
3. **to check that closure conditions are satisfied.**

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

First step

Assume now that κ is a non-constant function. By applying previous results, we get that the first integrals of the Euler-Lagrange equations are

$$\begin{aligned} 16\kappa^2\kappa_s^2(s) &= 4d\kappa^2 - 16G\kappa^4 - 4\kappa^6 - e^2, \\ \tau(s) &= \left(\frac{e}{4\kappa^2(s)}\right), \end{aligned}$$

where d and e are constants of integration.

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

The family of periodic solutions of the Euler-Lagrange equations can be parameterized in

$$\mathcal{D} = \{(\beta, \alpha)\ ;\ \alpha > \beta > 0\}, \quad \begin{cases} e^2 = 4\,(4G + \alpha + \beta)\,\alpha\beta \\ d = (\alpha + \beta)\,(4G + \alpha + \beta) - \alpha\beta \end{cases}$$

and is given by

$$\kappa^2_{\beta,\alpha}(s) = \alpha - (\alpha - \beta)\,\mathrm{sn}^2\left(\frac{\sqrt{\alpha - \alpha_o}}{2}s - \mathbf{K}(p)\,, p\right)$$

with K (p) **denoting the complete elliptic integral of the first kind of modulus** $p = \sqrt{\frac{\alpha-\beta}{\alpha-\alpha_o}}$**.**

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

Therefore, we have proved

There exists a 2-parameter family of curves in $\mathbb{S}^3(G)$, $\mathcal{R}_{\beta,\alpha} = \{\gamma_{\beta,\alpha};\ \alpha > \beta > 0\}$, whose curvature and torsion functions $\kappa_{\beta,\alpha}$ and $\tau_{\beta,\alpha}$, as given previously are periodic solutions of the Euler-Lagrange equations corresponding to the elastic energy functional $\mathcal{F}$.

- **Members of $\mathcal{R}_{\beta,\alpha}$ are candidates to be closed critical points of $\mathcal{F}$.**
- **Without loss of generality we assume $G = 1$.**

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

Second step: Closure conditions

Take $\gamma_{\beta,\alpha} \in \mathcal{R}_{\beta,\alpha}$ and let ρ the period of its curvature $\kappa_{\beta,\alpha}(s)$. Then $\gamma_{\beta,\alpha}$ is a closed critical point of $\mathcal{F}$, if and only if,

$$\Lambda\theta\left(\gamma_{\beta,\alpha}\right) = -\frac{b}{4}\int_0^{\rho(\beta,\alpha)}\left(\frac{\kappa^2}{\kappa^2-\frac{b^2}{4}}\right)ds,$$

$$\Lambda\varphi\left(\gamma_{\beta,\alpha}\right) = -\frac{a}{4}\int_0^{\rho(\beta,\alpha)}\left(\frac{\kappa^2}{\kappa^2-\frac{a^2}{4}}\right)ds$$

are rationally related to 2π (to simplify the notation, we are using κ instead of $\kappa_{\beta,\alpha}$ in the above formulas).

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

In the above expression a and b were given by

$$a^2 = \frac{d + \sqrt{d^2 - 4e^2}}{2}, \; b^2 = \frac{d - \sqrt{d^2 - 4e^2}}{2}$$

We define new parameters (w, r) by

$$w = \frac{b^2}{4} \text{ and } r = \frac{a^2}{4}.$$

Then we can show after a long computation that

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

$$
\begin{aligned}
\Lambda\theta\left(\gamma_{\beta,\alpha}\right) &= -2\left(\frac{w}{\alpha-\alpha_o}\right)^{\frac{1}{2}}\left(\mathbf{K}\left(p\right)+\ldots\right. \\
&\qquad \left.\ldots+\frac{w}{\alpha-w}\mathbf{\Pi}\left(\frac{\pi}{2},\frac{\alpha-\beta}{\alpha-w},\sqrt{\frac{\alpha-\beta}{\alpha-\alpha_o}}\right)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\Lambda\varphi\left(\gamma_{\beta,\alpha}\right) &= -2\left(\frac{r}{\alpha-\alpha_o}\right)^{\frac{1}{2}}\left(\mathbf{K}\left(p\right)+\ldots\right. \\
&\qquad \left.\ldots+\frac{r}{\alpha-r}\mathbf{\Pi}\left(\frac{\pi}{2},\frac{\alpha-\beta}{\alpha-r},\sqrt{\frac{\alpha-\beta}{\alpha-\alpha_o}}\right)\right),
\end{aligned}
$$

with

- $\mathbf{\Pi}\left(\frac{\pi}{2},\upsilon,p\right)$ (respectively, $\mathbf{K}\left(p\right)$) is the complete elliptic integral of third kind (respectively, of the first kind) of modulus $p=\sqrt{\frac{\alpha-\beta}{\alpha-\alpha_o}}$.

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

Third step

For any $(\beta, \alpha) \in \mathcal{D}$, let $\kappa_{\beta,\alpha}(s)$ be the corresponding non-constant periodic solutions of the Euler-Lagrange equations, it determines a curve $\gamma_{\beta,\alpha}$ in $\mathbb{S}^3(1)$ belonging to $\mathcal{R}_{\beta,\alpha}$.

Then we define the map

$$\Lambda : \mathcal{D} \to \mathbb{R}^2$$

$$\Lambda(\beta, \alpha) = \left(\frac{\Lambda\varphi(\gamma_{\beta,\alpha})}{2\pi}, \frac{\Lambda\theta(\gamma_{\beta,\alpha})}{2\pi} \right).$$

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

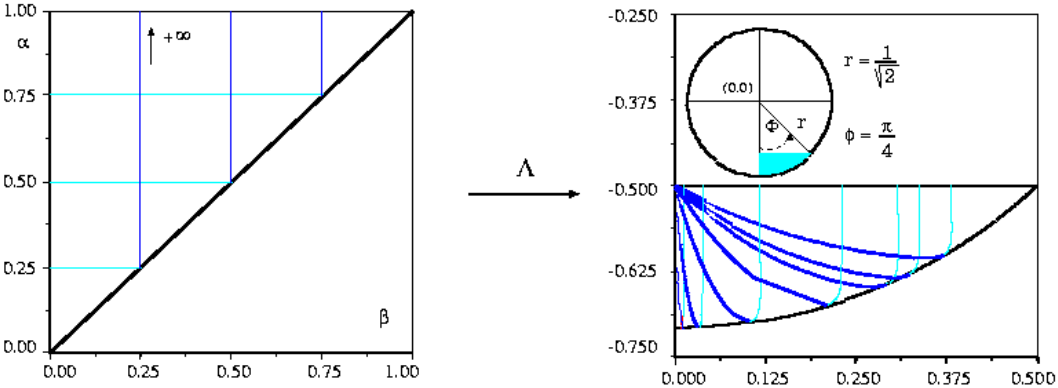
To determine the closed critical curves of $\mathcal{R}_{\beta,\alpha}$, we must check the closure conditions given previously. Hence we must:

- **compute $\Lambda\ (\mathcal{D})$ as accurately as possible (it is quite complicate generally), and**
- **show that $\Lambda\,(\mathcal{D}) \cap \mathbb{Q}^2 \neq \varnothing$.**

Elasticae in $\mathbb{S}^3(1)$: Constant curvature

In our case, we can prove that

$$\Lambda\left(\mathcal{D}\right)=\left\{(x,y)\,;\;x^2+y^2<\frac{1}{2},x>0\text{ and }y<-\frac{1}{2}\right\}.$$



Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

Hence, closed non-constant curvature elastic curves in $\mathbb{S}^3(1)$ are indexed in $\Lambda(\mathcal{D}) \cap \mathbb{Q}^2$ (multiple covers of a closed elastica correspond to the same point of the region).

Points in the upper boundary of this region, represent closed elastic curves that lie in $\mathbb{S}^2(1)$ (geodesics correspond to the "vertex" $(\frac{1}{2}, \frac{-1}{2})$).

Points in the lower boundary, $\Lambda(\mathcal{L}_2) \cap \mathbb{Q}$, correspond to closed elastic helices fully immersed in $\mathbb{S}^3(1)$.

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature

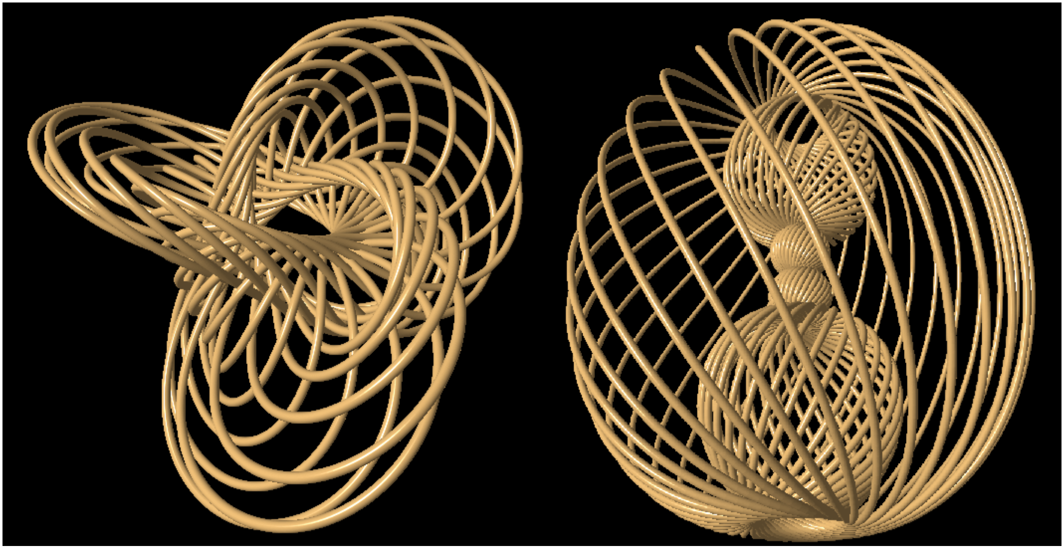
For any choice of natural parameters $n, m, l \in \mathbb{N}$ satisfying

$$(n,m,l)=1,\ 0 < m < \frac{n}{2} < l < \frac{n}{\sqrt{2}},\ m^2 + l^2 < \frac{n^2}{2},$$

there exists a closed elastica $\gamma_{n,m,l}$ which is totally determined and fully immersed in $\mathbb{S}^3(1)$, that closes up after n periods of its curvature, m trips around the "equator" of x_φ and l trips around the "equator" of x_θ.

Every closed elastica in $\mathbb{S}^3(1)$ can be obtained in this way.

Elasticae in $\mathbb{S}^3(1)$: Non-constant curvature



Stereographic projections of the closed elasticae $\gamma_{75,22,47}$ and $\gamma_{150,30,97}$

5. Elastic curves circular at rest

We consider the problem of the existence and classification of elastic curves which are circular at rest, that is critical points of

$$\mathcal{F}^{\lambda}(\gamma) = \int_{\gamma} (\kappa - \lambda)^2 ds \,. \tag{5.26}$$

in a surface of constant curvature $\mathbb{M}^2(c)$.

- **intrinsic interest.**
- **they provide solutions to the membranes problem.**

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Closed critical points satisfy the Euler-Lagrange equation:

$$2\kappa_{ss} + \kappa^3 + (2 - \lambda^2)\kappa + 2\lambda = 0\,,$$

whose first integral is:

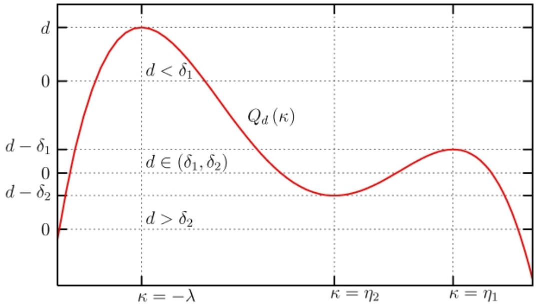
$$4\,\kappa_s^2 = d - (\kappa + \lambda)^2\left((\kappa - \lambda)^2 + 4\right)\;;\;\; d > 0.$$

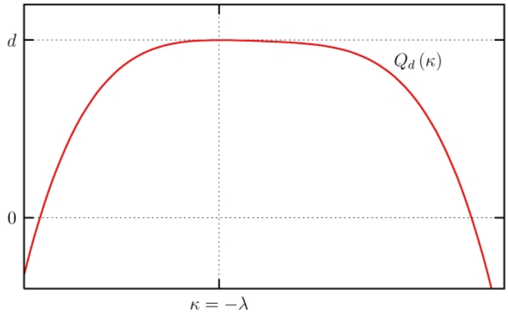
Denote by

$$Q_d(x) = d - (x + \lambda)^2\left((x - \lambda)^2 + 4\right)\,,$$

Depending on the values of λ and d this polynomial may have two or four simple roots.

λ-elastic curves in $\mathbb{S}^2(1)$





4-roots: Two solutions. The first one is given by

$$\kappa_{\mathbf{d}}^{\lambda}(\mathbf{s}) = \frac{\alpha_{\mathbf{2}}(\alpha_{\mathbf{4}}-\alpha_{\mathbf{1}}) - \alpha_{\mathbf{4}}(\alpha_{\mathbf{2}}-\alpha_{\mathbf{1}})\,\mathbf{cn}^{\mathbf{2}}(\mathbf{rs},\mathbf{M})}{(\alpha_{\mathbf{4}}-\alpha_{\mathbf{1}}) - (\alpha_{\mathbf{2}}-\alpha_{\mathbf{1}})\,\mathbf{cn}^{\mathbf{2}}(\mathbf{rs},\mathbf{M})},$$

where

$$r = \frac{\sqrt{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}{4}, \; M = \sqrt{\frac{(\alpha_4-\alpha_3)(\alpha_2-\alpha_1)}{(\alpha_4-\alpha_2)(\alpha_3-\alpha_1)}}$$

and $cn(rs, M)$ **is the Jacobi Elliptic cosine.**
The second solution $\widetilde{\kappa}_d^{\lambda}(s)$ **is obtained by interchanging** $1 \leftrightarrow 3$, $2 \leftrightarrow 4$

2-roots: One solution.

$$\kappa_d^\lambda(s) = \frac{(p+q)(q\alpha_2 + p\,\alpha_1) - 2p\,q\,(\alpha_2-\alpha_1)\,cn\,(r s, M)}{(p+q)^2 - (p-q)^2 cn^2\,(r s, M)} + \\ + \frac{(p-q)(q\alpha_2 - p\alpha_1) cn^2\,(r s, M)}{(p+q)^2 - (p-q)^2 cn^2\,(r s, M)},$$

with

$$p^2 = (\alpha_2+\alpha_1)^2 + 2\alpha_2^2 - 2\lambda^2 + 4 \quad ,$$
$$q^2 = (\alpha_2+\alpha_1)^2 + 2\alpha_1^2 - 2\lambda^2 + 4 \quad ,$$
$$M = \frac{1}{2}\sqrt{\frac{(\alpha_2-\alpha_1)^2 - (p-q)^2}{pq}} \quad , \qquad r = \frac{\sqrt{p\,q}}{2}.$$

λ-elastic curves in $\mathbb{S}^2(1)$

- **Let $\kappa(s)$ be a solution to the E-L equation with period ρ.**
- **Take $\gamma(s)$ the associate curve in $\mathbb{S}^2(1)$.**
- **We show that there exist geographic coordinates,**

$$x(\theta, \phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

such that $\gamma(s) = x(\theta(s), \phi(s))$ **and**

$$\theta_{\mathbf{s}}(\mathbf{s}) = \frac{\kappa^{\mathbf{2}} - \lambda^{\mathbf{2}}}{\mathbf{b}\left(\mathbf{d} - \mathbf{4}\,(\kappa+\lambda)^{\mathbf{2}}\right)}, \quad \mathbf{b}^{\mathbf{2}}(\mathbf{d} - \mathbf{4}(\kappa+\lambda)^{\mathbf{2}}) = \sin^{\mathbf{2}}\phi.$$

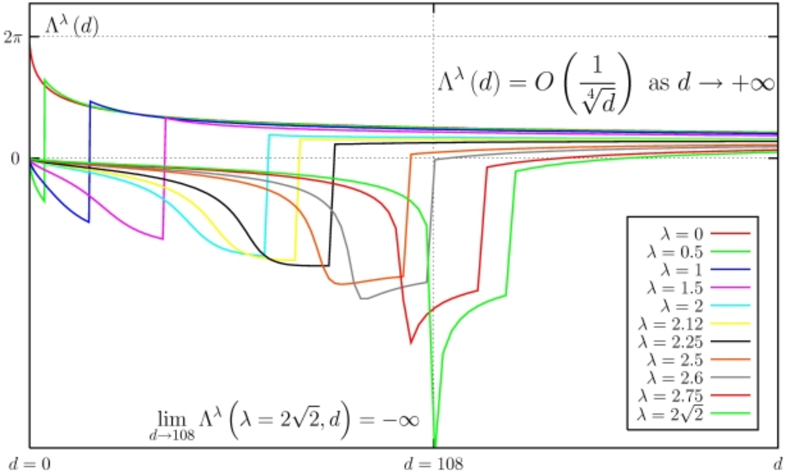
Closedness condition:

Let $\gamma(s)$ **be a curve in** $\mathbb{S}^2(1)$ **corresponding to a periodic solution of the E-L equation** $\kappa(s)$ **with period** ρ**. Then** $\gamma(s)$ **is a closed** $\lambda-$**elastic curve, if and only if, its progression angle in one period of its curvature,**

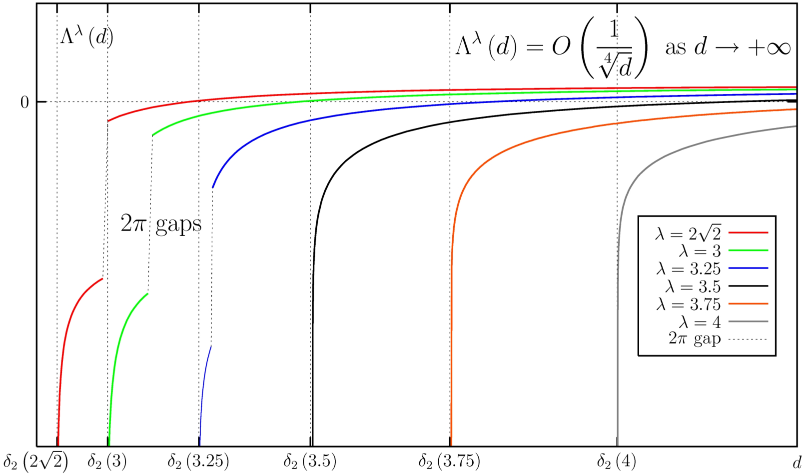
$$\Lambda^\lambda(d) = \sqrt{d}\int_0^\rho \frac{(\kappa^2 - \lambda^2)}{d - 4\left(\kappa + \lambda\right)^2} ds \, ,$$

is a rational multiple of π**.**

Variation of $\Lambda^\lambda(d)$ for $\lambda^2 < 8$.



Variation of $\Lambda^\lambda(d)$ for $\lambda^2 > 8, d > 16\lambda^2$



Let Λ_1 be defined as

$$\begin{aligned}\Lambda_1 = & -4\lambda\frac{K(M)}{r} \\ & +8\lambda^2\int_{\varsigma}^{\lambda}\frac{d\kappa}{(\kappa+3\lambda)\sqrt{(\lambda-\kappa)(\kappa-\varsigma)\left((\kappa-u)^2+v^2\right)}},\end{aligned} \tag{5.27}$$

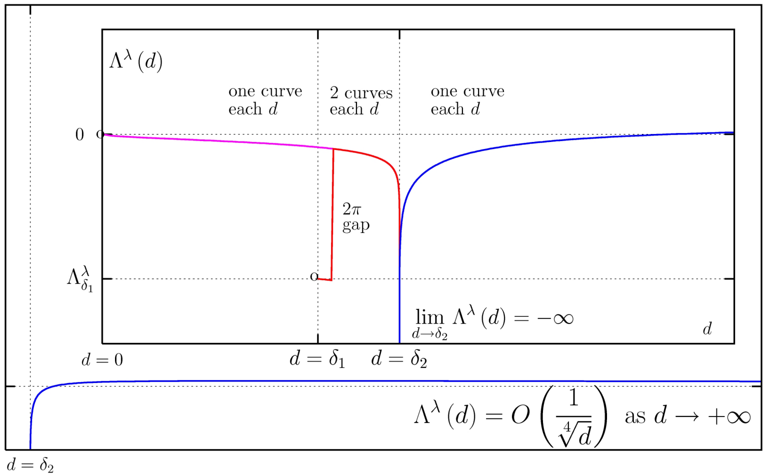
where M and r were given previously, $K(M)$ denotes the complete elliptic integral of the first kind, and ς is the only negative root of $\beta^3+\lambda\beta^2+\beta(\lambda^2-4)-\lambda(\lambda^2-12)=0$.

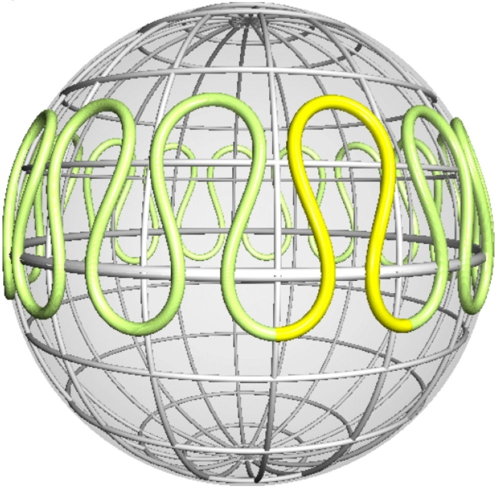
λ-elastic curves in $\mathbb{S}^2(1)$

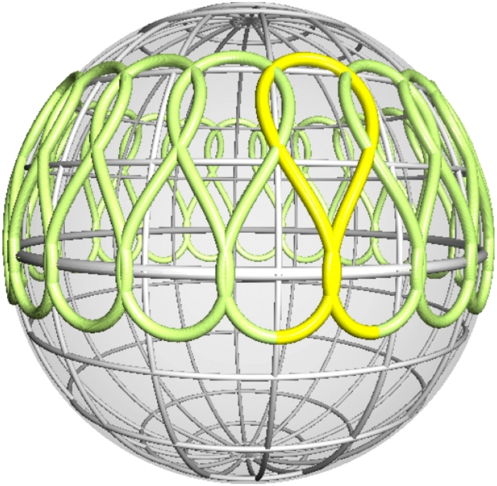
- **If $0 \leq \lambda < 2\sqrt{2}$, then for every pair of integer numbers $m, n \in \mathbb{Z}$ satisfying $\left|\frac{\Lambda_1}{2\pi} - \frac{m}{n}\right| < \frac{1}{2}$, there exists a closed λ-elastic curve $\gamma_{\mathbf{mn}}(\mathbf{s})$ in $\mathbb{S}^2(1)$.**
- **If $\lambda \geq 2\sqrt{2}$, then for every pair of integer numbers $m, n \in \mathbb{Z}$ satisfying $\frac{m}{n} < 0$, there exists a closed λ-elastic curve $\gamma_{\mathbf{mn}}(\mathbf{s})$ in $\mathbb{S}^2(1)$.**

- **In any of the above cases,** $\gamma_{\mathbf{mn}}(\mathbf{s})$ **closes up after** n **periods of its curvature and** m **trips around the equator.**
- **For any** $\lambda \geq 2\sqrt{2}$ **there exists a closed "figure eight"** λ**-elastic curve in** $\mathbb{S}^2(1)$**.**

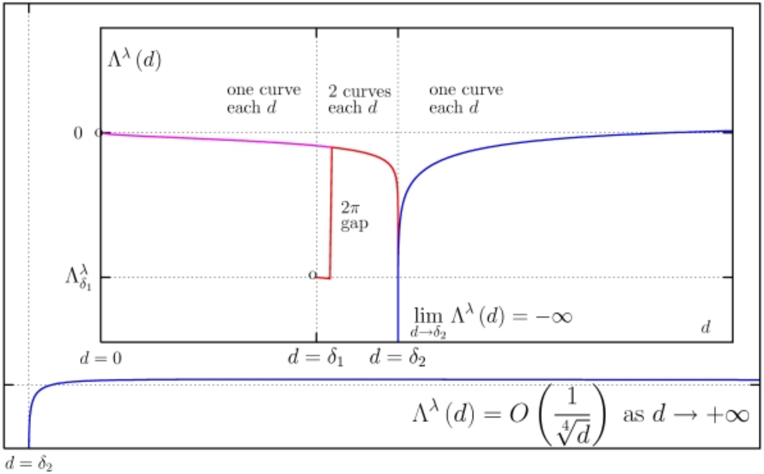
Variation of $\Lambda^\lambda(d)$ for $\lambda = 4$





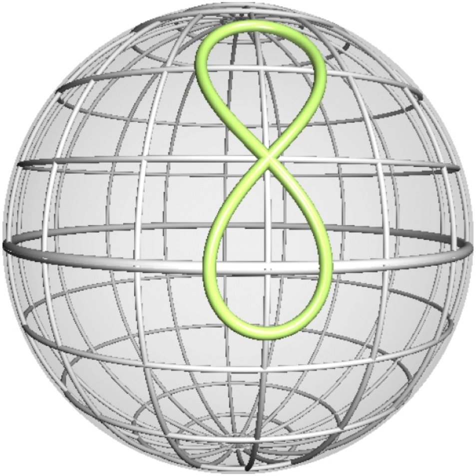


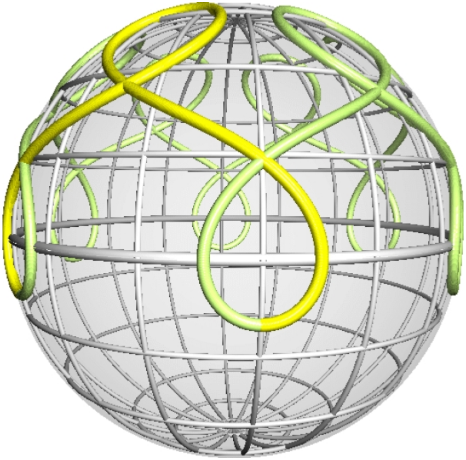
Variation of $\Lambda^\lambda(d)$ for $\lambda = 4$



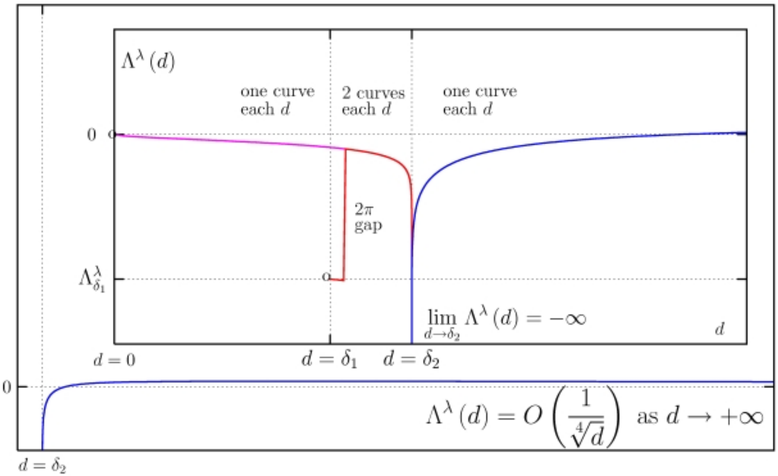
λ-elastic curves in $\mathbb{S}^2(1)$

$\lambda = 4$



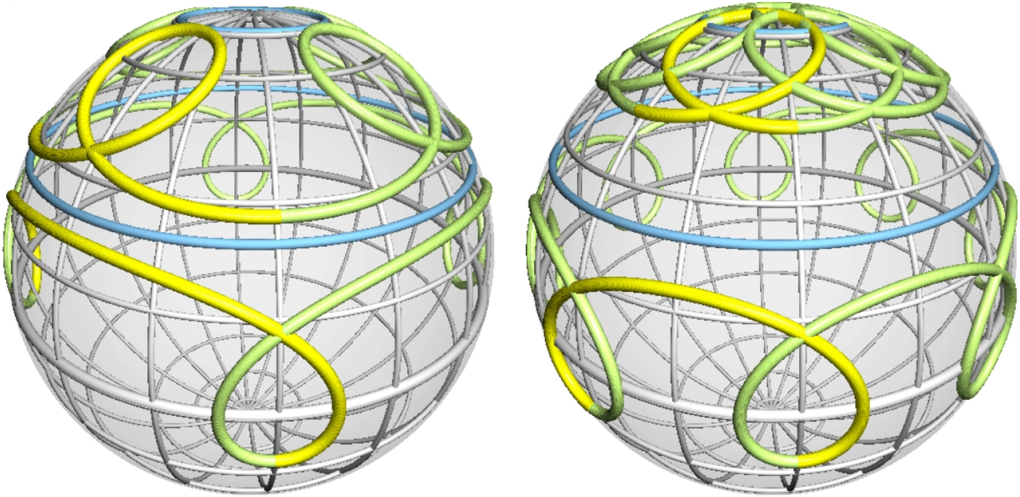


Variation of $\Lambda^\lambda(d)$ for $\lambda = 4$

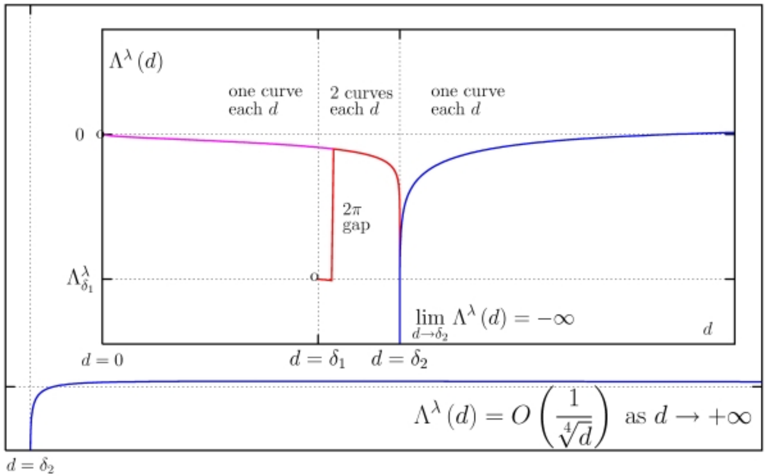


λ-elastic curves in $\mathbb{S}^2(1)$

$\lambda = 4$

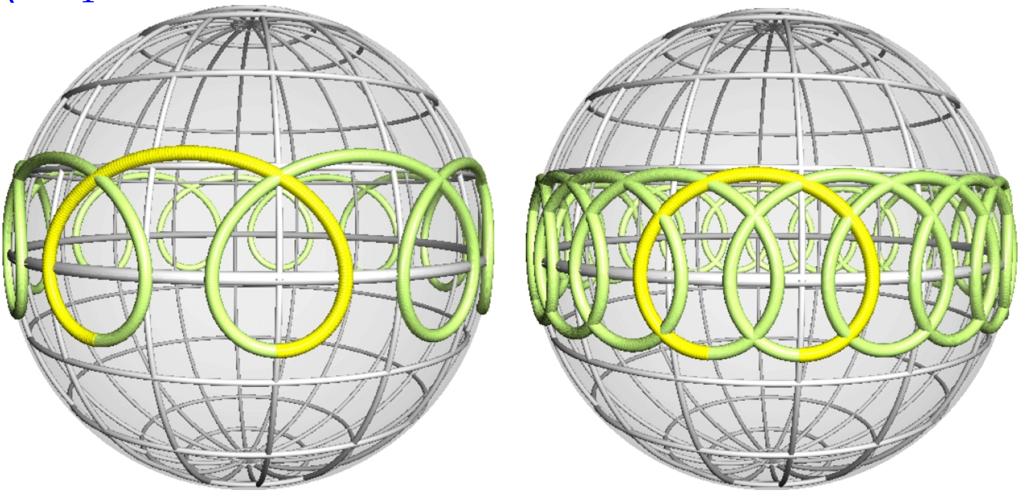


Variation of $\Lambda^\lambda(d)$ for $\lambda = 4$



λ-elastic curves in $\mathbb{S}^2(1)$

$\lambda = 4$



Minima of the energy

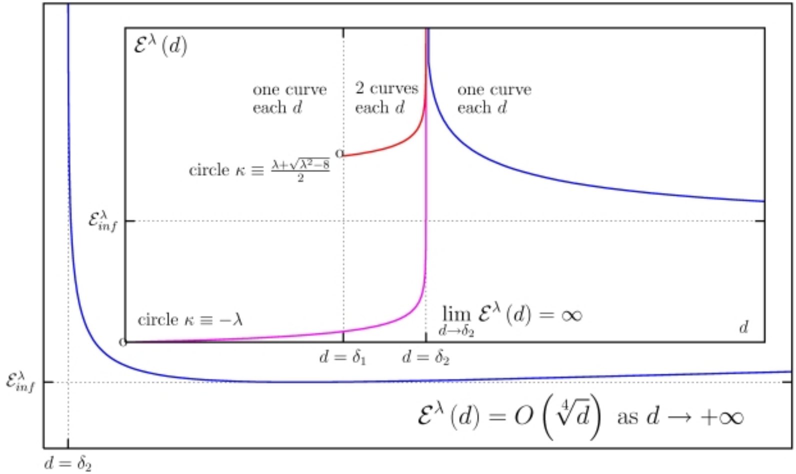
- **numerical searching for minima**
- **derivation of working hypothesis**
- **formal proofs and conclusions**

Minima of the energy: $\lambda^2 \geq 8$

There are three circles which are critical points

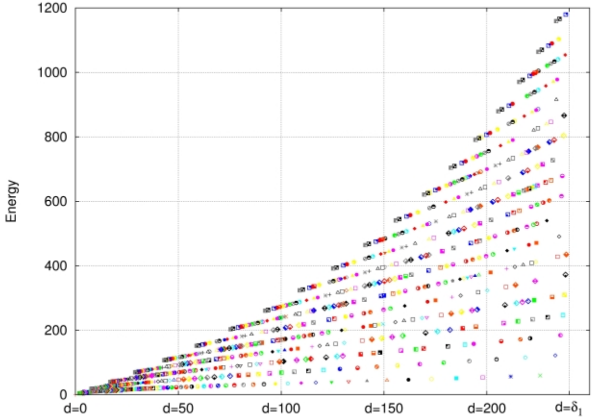
- C_{η_o} **with curvature** $\kappa = -\lambda$. **Obviously they are global minima.**
- C_{η_1} **with curvature** $\eta_1 = \frac{\lambda+\sqrt{\lambda^2-8}}{2}$.
- C_{η_2}, **with curvature** $\eta_2 = \frac{\lambda-\sqrt{\lambda^2-8}}{2}$

Minima of the energy: $\lambda = 4$.



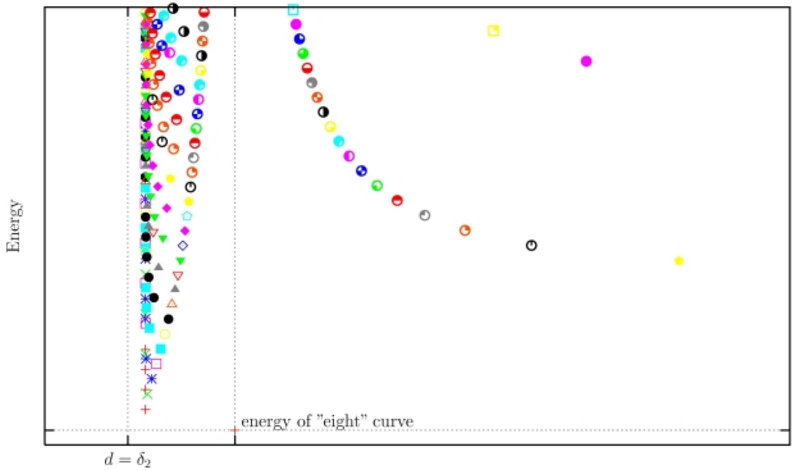
(a) Variation of $\mathcal{E}^\lambda(d)$, energy of γ in one period of its κ.

Minima of the energy: $\lambda = 4$.



energy of C_{η_o}

Minima of the energy: $\lambda = 4$.



energy of $\beta(s)$

λ-elastic curves in $\mathbb{S}^2(1)$

Minima of the energy: $\lambda \geq 4$. We have computed the second variation formula of $\mathcal{F}^\lambda(\gamma)$ and showed that

- **C_{η_2} is always unstable .**
- **the once covered C_{η_1} is stable (multiple m-covers of this circle $C^m_{\eta_1}$ are stable provided that m is not too large)**
- **"eight figure" is stable ?**

5.2. λ-elastic curves in $\mathbb{H}^2(-1)$

We investigate minima of $\mathcal{F}^{\lambda}(\gamma) = \int_{\gamma}(\kappa - \lambda)^2 ds$ by following a procedure similar to previous one in $\mathbb{S}^2(1)$

- **We integrate explicitly the Euler-Lagrange equations in terms of the Jacobi Elliptic functions.**

- **The situation here is much reacher: new cases appear**

- **For each case, we choose coordinates systems adapted to the problem and establish the corresponding closedness conditions in terms of the progression angle**

λ-elastic curves in $\mathbb{H}^2(-1)$

- **We check numerically the closedness conditions**
- **We prove that they are satisfied**
- **We use the associated coordinate systems and numerical-graphical stuff to draw the critical points**
- **A rough stability analysis is made.**

λ-elastic curves in $\mathbb{H}^2(-1)$

For any $\lambda > 0$, $d \in (-\delta_2, 0)$**, the progression angle** $\Lambda^\lambda(d)$ **moves continuously in**

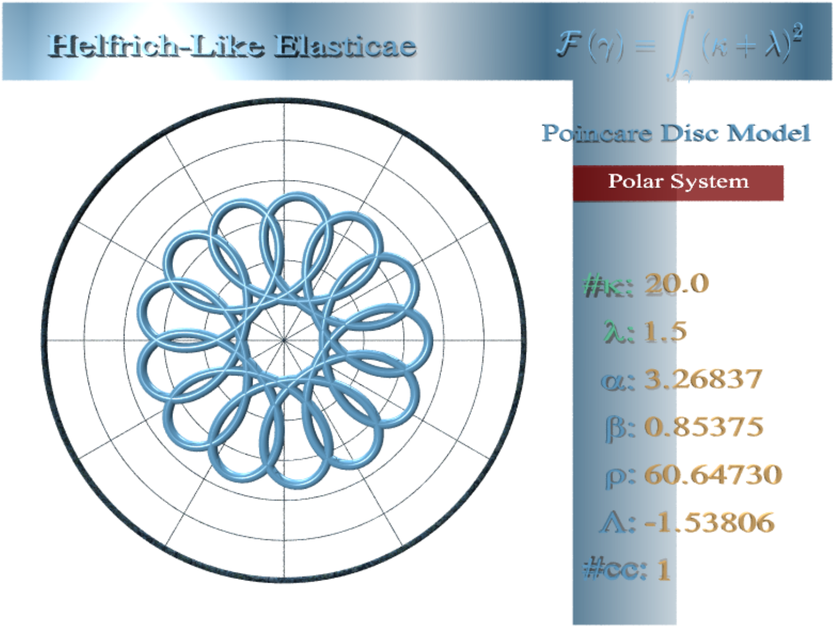
$$(-\delta_2, -16\lambda^2) \bigcup (-16\lambda^2, 0)$$

and, therefore, there exist infinite many closed critical curves of

$$\mathcal{F}^\lambda(\gamma) = \int_\gamma (\kappa - \lambda)^2 ds$$

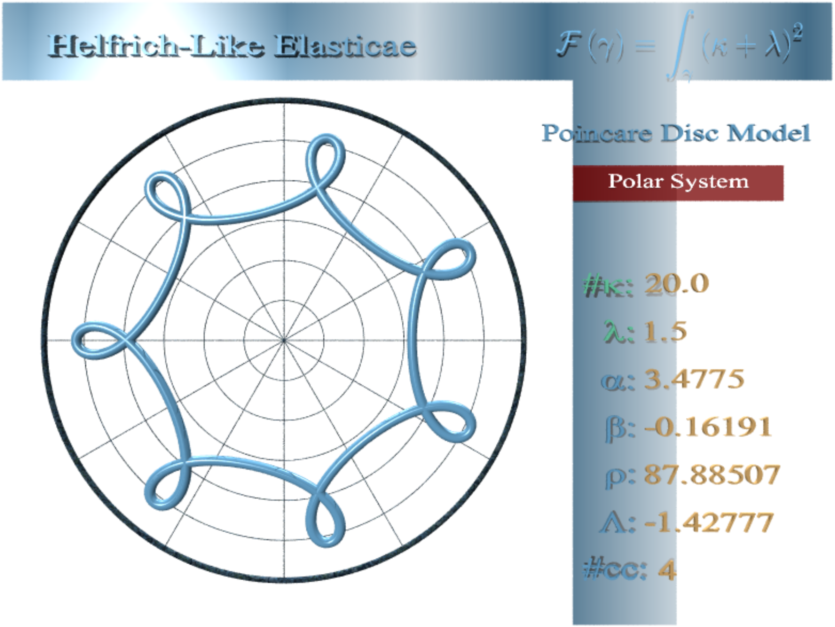
with rotational symmetry in the hyperbolic plane.

λ-elastic curves in $\mathbb{H}^2(-1)$



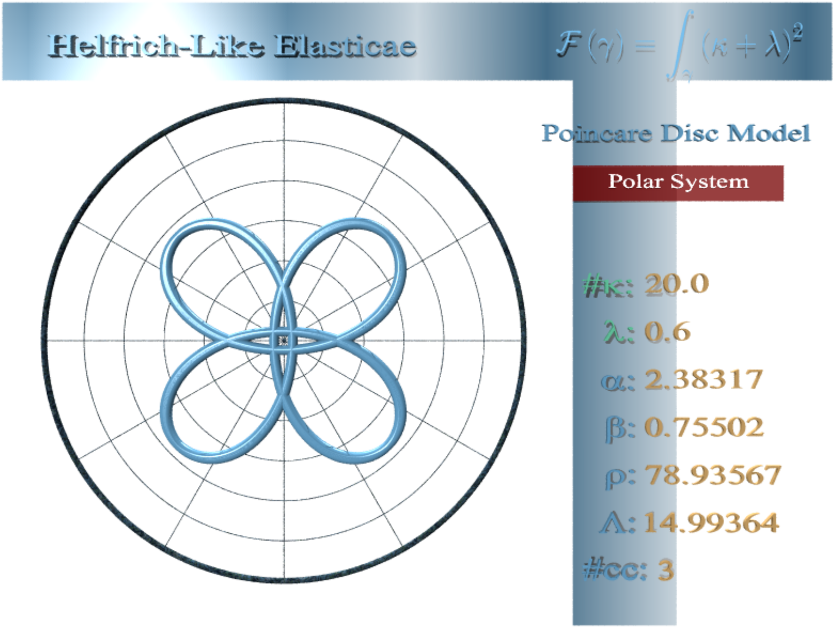
critical curve of the energy with rotational symmetry

λ-elastic curves in $\mathbb{H}^2(-1)$



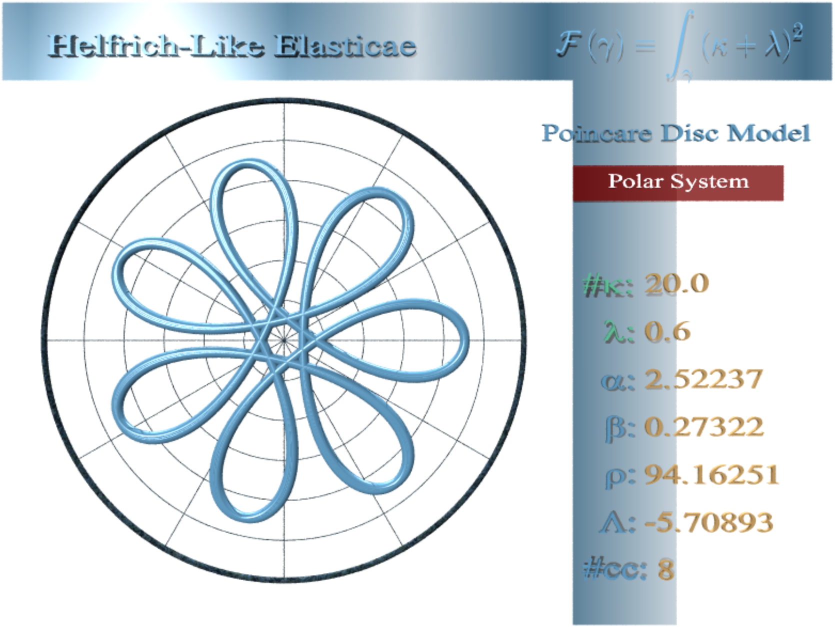
critical curve of the energy with rotational symmetry

λ-elastic curves in $\mathbb{H}^2(-1)$



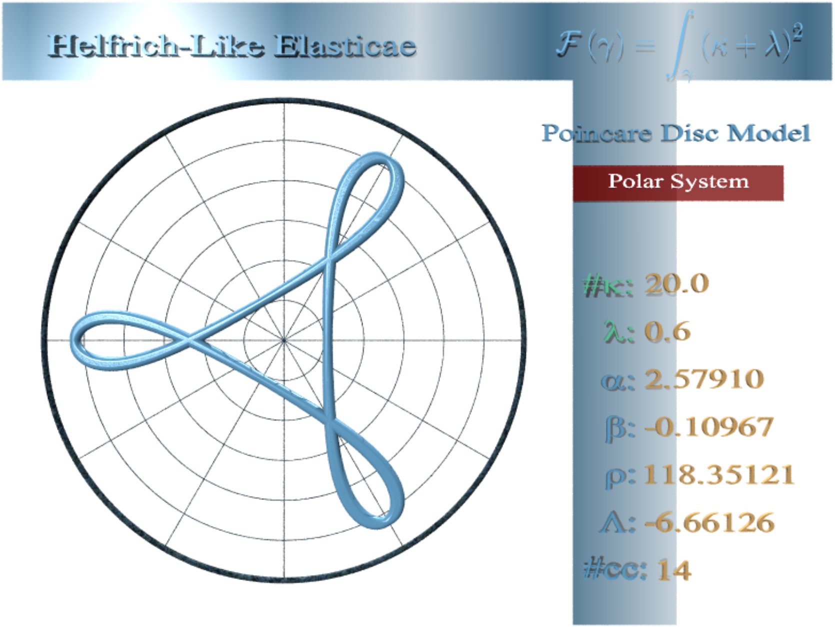
critical curve of the energy with rotational symmetry

λ-elastic curves in $\mathbb{H}^2(-1)$



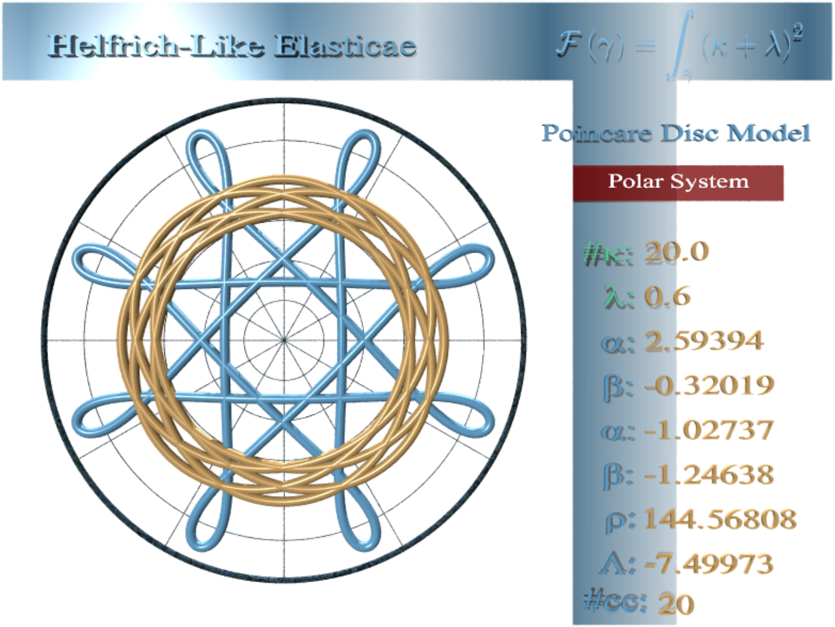
critical curve of the energy with rotational symmetry

λ-elastic curves in $\mathbb{H}^2(-1)$



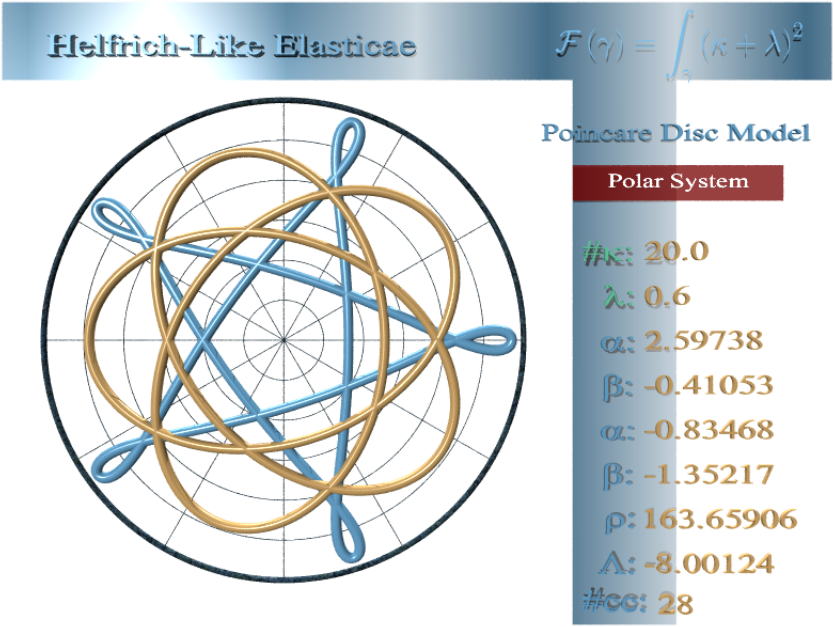
critical curve of the energy with rotational symmetry

λ-elastic curves in $\mathbb{H}^2(-1)$



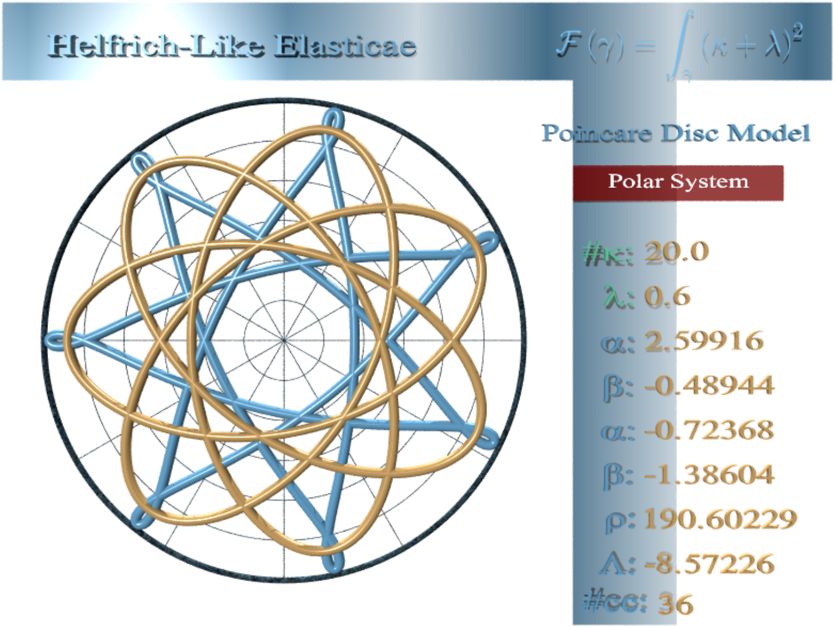
critical curve of the energy with rotational symmetry

λ-elastic curves in $\mathbb{H}^2(-1)$



critical curve of the energy with rotational symmetry

λ-elastic curves in $\mathbb{H}^2(-1)$



critical curve of the energy with rotational symmetry

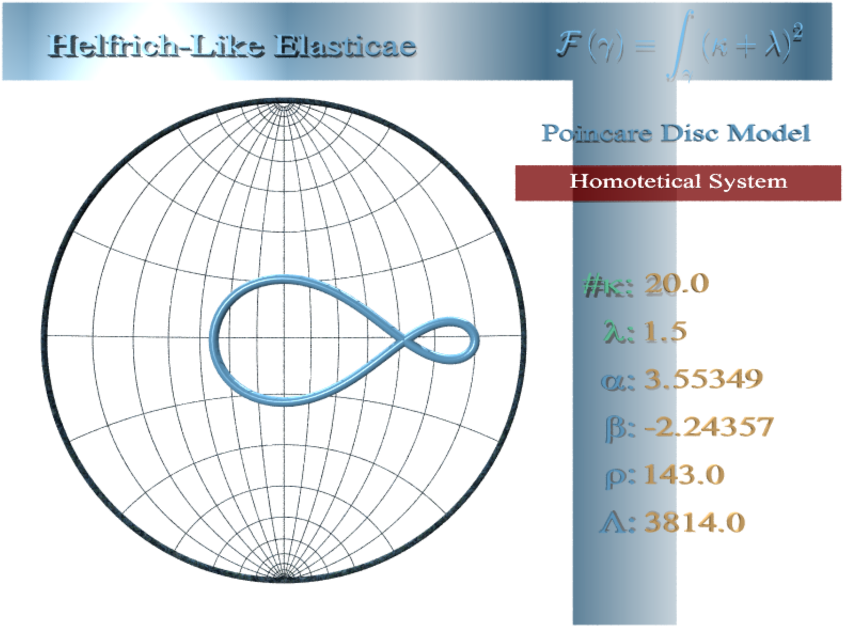
λ-elastic curves in $\mathbb{H}^2(-1)$

For any $\lambda > 1$, $d > 0$), the progression angle $\Lambda^\lambda(d)$ reaches the zero value exactly once, and, therefore, there is a closed "eight figure" critical curve of

$$\mathcal{F}^\lambda(\gamma) = \int_\gamma (\kappa - \lambda)^2 ds$$

in the hyperbolic plane.

λ-elastic curves in $\mathbb{H}^2(-1)$



the only critical curve of the energy with translational symmetry

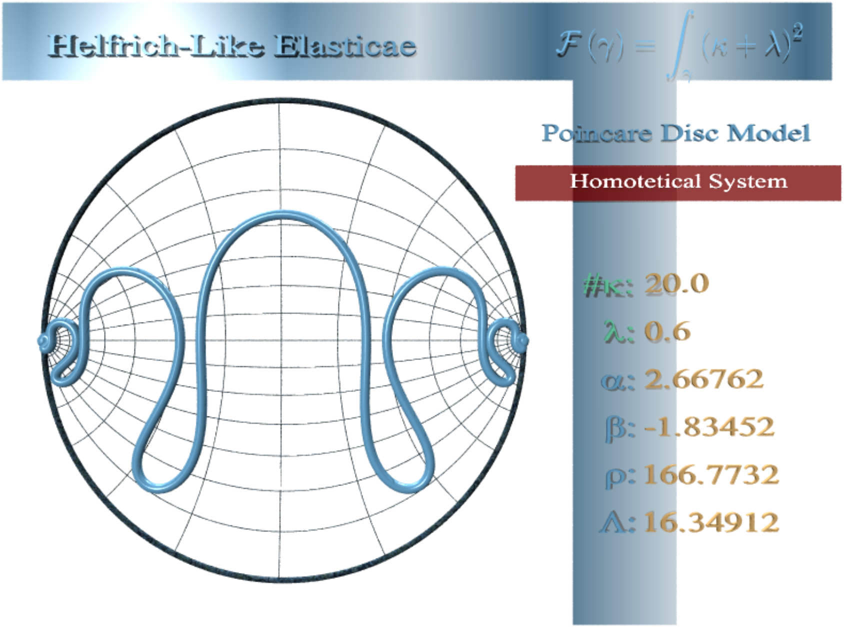
λ-elastic curves in $\mathbb{H}^2(-1)$

For any $\lambda > 0$, there exist periodic critical curves of

$$\mathcal{F}^{\lambda}(\gamma) = \int_{\gamma} (\kappa - \lambda)^2 ds$$

in the hyperbolic plane with horocyclical symmetry, but they never close up.

λ-elastic curves in $\mathbb{H}^2(-1)$



critical curve of the energy with horocyclical symmetry

6. Some applications

We shall investigate some applications of the elastic curves results previously considered to:

- Membranes and vesicles;
- Chen-Willmore submanifolds.

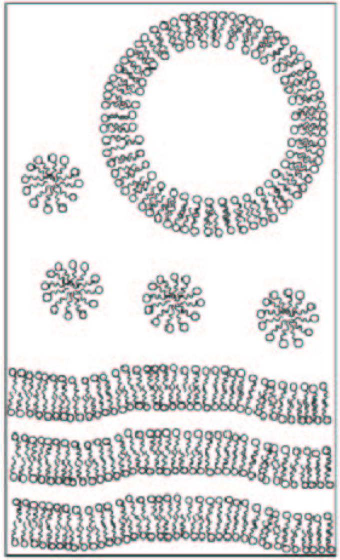
6.1. Membranes and vesicles

- **Investigation of surfaces which are extremal for a free energy which is quadratic in the principal curvatures is relevant in the study of many physical and biophysical problems.**

- **Example: The theoretical description of amphiphilic systems. Well known classes of amphiphiles are: tensides or surfactants (used for washing and cleaning purposes) and lipids (the basic components of biomembranes)**

Membranes and vesicles

The physics of amphiphilic systems is mostly determined by their interfaces.
In binary systems, amphiphiles self-assemble into bilayer structures which are fluid membranes .



Thus, embedded surfaces in Euclidean space $\mathbb{R}^3$ are considered not so much as a geometric object but as an idealized model for the interfaces or middle surfaces occurring in real materials: open or closed lipid bilayers and surfactant films, thin elastic plates, etc...

Membranes and vesicles

- **The free energy of an amphiphilic system can be written as a functional of its interfacial geometry.**
- **The shape of the membrane is determined by the mechanical equilibrium of the free energy.**
- **Their elastic properties suggest that the free energy of S is controlled not only by the tension, but also by the curvature.**

Linear elasticity theory: Hooke's law suggest that the Free energy of a surface, $\mathcal{E}(S)$, is quadratic in the principal curvatures. We may assume $\widetilde{\Phi}(\kappa_1, \kappa_2) = \Phi(H, K)$

- $\Phi(H, K) = a + b(H - c_o)^2 - c\,K$**;**
- K **is the Gaussian curvature;**
- H **is the Mean curvature.**

S. Germain, 1810; S.D. Poisson, 1812; G.R. Kirchhoff, 1850; A.E.H. Love, 1906; P.B. Canhman, 1970; W. Helfrich, 1973 T. Thomsem, H. Hopf, T.J. Willmore,...

So the free energy is

$$\mathcal{E}(S) = \int_S (a + b(H - c_o)^2 - c\,K) \cdot dA\,, \tag{6.28}$$

- $a, b, c \in \mathbf{R}$ **are material constants (surface tension, elastic moduli,...)**
- H, K **are the mean and Gaussian curvatures of** S**.**
- c_o **is the spontaneous curvature related to**
  - **initial state.**
  - **asymmetry in the two faces of the bilayer.**

The static equilibrium shape of our interface S is determined by the condition that S be energy minimizing or, more generally and less restrictive, that S be an stationary for the energy functional $\mathcal{E}(S)$.

S must be a solution of the variational problem:

$$\delta\mathcal{E} = 0\,.$$

Membranes and vesicles

For suitable choices of the parameters, membranes family includes important classes of surfaces

- **minimal surfaces** (soap films).
- **constant mean curvature surfaces** (soap bubbles)
- **Willmore surfaces** (vesicles)
- **bimomembranes and vesicles**, etc...

This variational problem leads not only to the Euler-Lagrange equation,

$$b\{\triangle H + 2H(H^2 - K)\} - 2(a + b\,c_o^2)H + 2b\,c_o\,K = 0, \quad (6.29)$$

where $\triangle$ is the Laplacian of S, but also to certain specific intrinsic, or natural, boundary conditions.

$$\begin{aligned} -b\frac{\partial H}{\partial n} - c\{\frac{\partial \tau}{\partial s} + \frac{\partial^2 \vartheta}{\partial s^2}\}, \\ b(H - c_o) - c\kappa_n, \\ -a + b(H - c_o)^2 c_o\,K, \end{aligned} \qquad (6.30)$$

where κ_n, n are normal curvature and interior normal of ∂S in S; τ is the torsion of ∂S in $\mathbb{R}^3$; and $\vartheta = \angle(N, n)$.

Often the interface separates two media of prescribed volumes: volume constraint.

- **The E-L equation is now,**

$$
\begin{aligned}
& b\{\triangle H + 2H(H^2 - K)\} - \\
& \qquad - 2(a + bc_o^2)H + 2b\, c_o\, K - d = 0,
\end{aligned}
\tag{6.31}
$$

- **Obviously, the boundary conditions will have to be complemented as well.**

Euler-Lagrange equation (6.31) is a nonlinear partial differential equation of fourth order for x, the position vector of S. Using the Beltrami's equation

$$\triangle x = 2H\mathbb{N} \tag{6.32}$$

$\mathbb{N}$ the unit normal to S, it can be written in the form of four differential equations of second order (three, namely (6.32), for x, and one, namely (6.31), for the mean curvature H.)

BOUNDARY VALUE PROBLEMS FOR VARIATIONAL INTEGRALS:

determination of minimizing or stationary surfaces for the energy functional in the class of all surfaces of a prescribed topological type (subject or not to a volume constraint) and with boundaries on fixed curves (Plateau type) or on prescribed surfaces (free boundary).

Membranes and vesicles

- **symmetry in the bilayer, $c_o = 0$, and no volume constraint $d = 0$: Minimal surfaces.**
- **asymmetric bylayer, $c_o \neq 0$, and no volume constraint $d = 0$: Constant mean curvature surfaces.**
- **symmetry in the bilayer $c_o = 0$, no area constraint $a = 0$ and no volume constraint $d = 0$: Willmore surfaces.**

Membranes and vesicles

For mathematicians the most central question is the existence proof of stationary surfaces.

- **The existence and uniqueness of minimizers of $\mathcal{E}(S)$ of a certain topological class is still unknown.**
- **It is also not known whether the minimizer is symmetric in any sense.**
- **On the mathematical level the attending problems are formidable.**

Physicists are more interested in analytical solutions of the Euler-Lagrange equation (6.31)

$$
\begin{aligned}
b\{\triangle H + 2H(H^2 - K)\} - \\
- 2(a + bc_o^2)H + 2b\, c_o\, K - d = 0
\end{aligned}
$$

since they can be used to derive physical properties of the corresponding system.

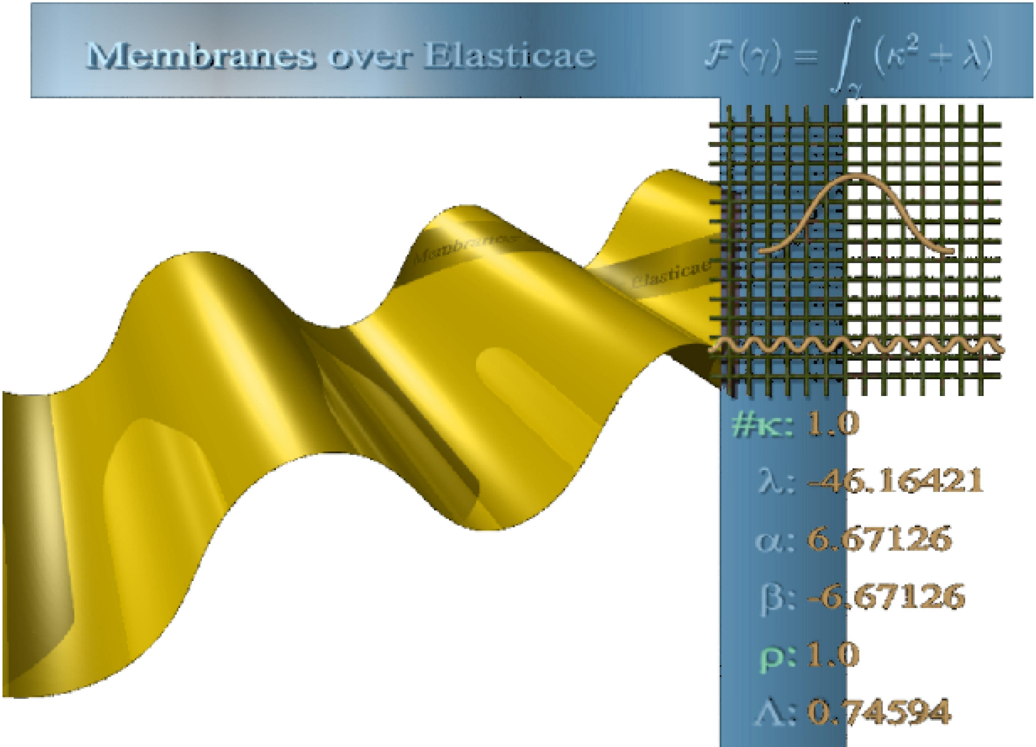
Membranes and vesicles

- **Very few analytical solutions are known today.**
- **As far as closed surfaces are concerned, we have of course the spheres and certain anchor rings.**
- **There are extensive numerical investigations of the solution surfaces of (6.31) generally restricted to surfaces with rotational symmetry.**
- **Seifert, Lipowsky, Michalef, Bensimon, Julicher, Mladenov, etc...**

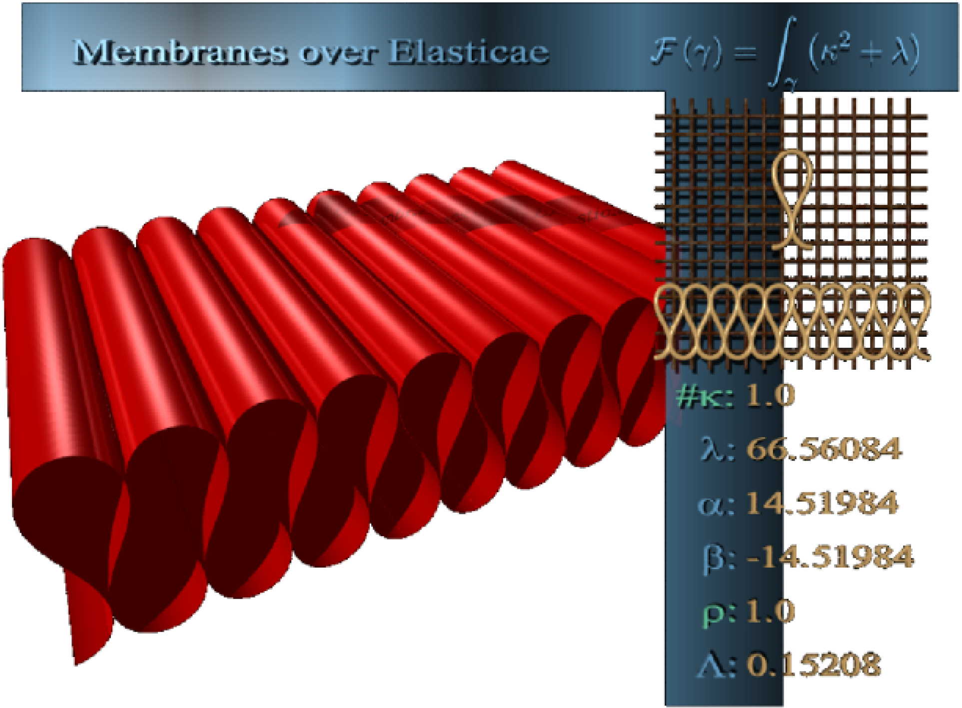
Membranes and vesicles

- **The 1-dimensional version of membranes are the elastic curves.**
- **Under certain boundary conditions, cylindrical membranes in $\mathbb{R}^3$ are cylinders shaped on plane elastic curves (J.C.C. Nitsche, (1999)).**

Membranes and vesicles



Membranes and vesicles



The simplest type of elastic energy is the bending energy or Willmore energy

Willmore surfaces: Critical points of the bending energy

$$\mathcal{E}(S) = \int_S H^2 \cdot dA \,,$$

The Willmore energy is a conformal invariant.

Willmore surfaces

- **In 1978 J.L. Weiner showed that minimal surfaces of real space forms are examples of Willmore surfaces.**

- **Consequently, he used the conformal invariance, the stereographic projection and the Lawson minimal examples in $\mathbb{S}^3$, to produce Willmore surfaces of any genus in $\mathbb{R}^3$.**

Willmore surfaces

$$\textbf{Surfaces} \begin{cases} \textbf{cones} \\ \textbf{cylinders} \\ \textbf{surfaces of revolution} \end{cases}$$

which are Willmore membranes, have to be shaped on elastic curves of $\begin{cases} \mathbb{S}^2(1) \\ \mathbb{R}^2 \\ \mathbb{H}^2(-1) \end{cases}$

(Hertich-Jeromin, (2003)).

Willmore surfaces

- **The Willmore energy is a conformal invariant.**
- **By combining this with the Palais' Symmetric Criticality Principle, we obtain a method to produce exact solutions of the Euler-Lagrange equations for membranes and vesicles.**

Palais' Principle: Take a manifold $\mathcal{N}$ and a group G which acts by diffeomorphisms.

Consider a functional $\mathcal{W} : \mathcal{N} \to \mathbb{R}$ which is G-invariant

$$\mathcal{W}(a \cdot \varphi) = \mathcal{W}(\varphi), \quad \forall a \in G.$$

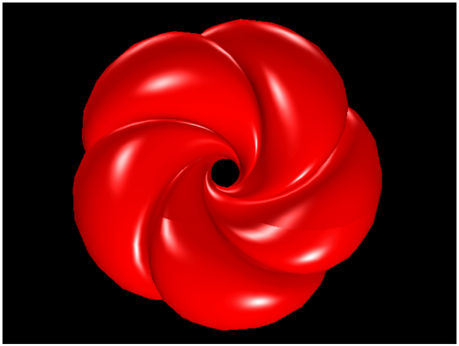
Consider the following sets:

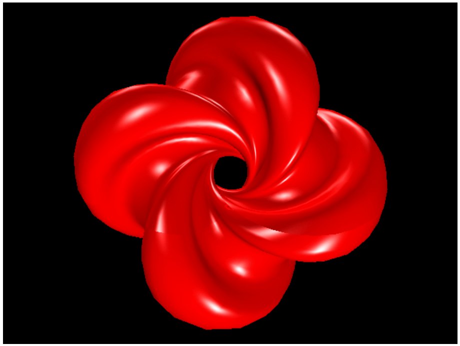
- **Symmetric points**
  $\mathcal{N}_G = \{\varphi \in \mathcal{N} \, : \, a \cdot \varphi = \varphi, \forall a \in G\}.$
- **Critical points** Σ **of** $\mathcal{W} : \mathcal{N} \to \mathbb{R}$**.**
- **Critical points** Σ_G **of the restriction of** $\mathcal{W}$ **to the set** $\mathcal{N}_G$ **of symmetric points.**
- **If** G **is compact, then** $\mathcal{N}_G$ **is a submanifold of** $\mathcal{N}$**.**
- **Under this assumption, we have**

$$\Sigma \cap \mathcal{N}_G = \Sigma_G,$$

Palais' Symmetric Criticality Principle.

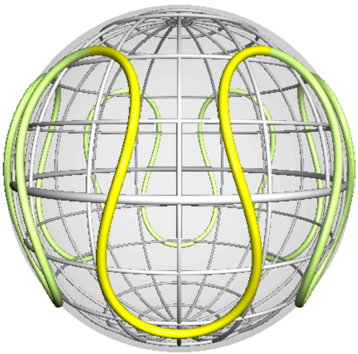
First known examples of Willmore membranes in $\mathbb{R}^3$ which did not come from minimal surfaces of $\mathbb{S}^3(1)$ were constructed using Hopf Tori shaped on the elastic curves of $\mathbb{S}^2(1/2)$ (U. Pinkal, (1985)).

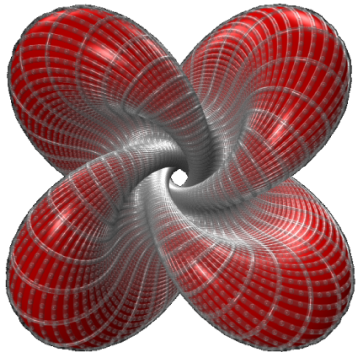




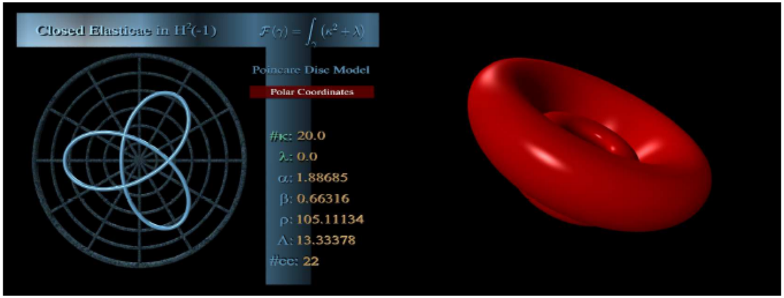
Willmore surfaces

In a similar way closed vesicles in $\mathbb{S}^3$ may be produced by lifting closed elasticae in $\mathbb{S}^2$ which are circular at rest (J. Arroyo and O.J. Garay (2001)).

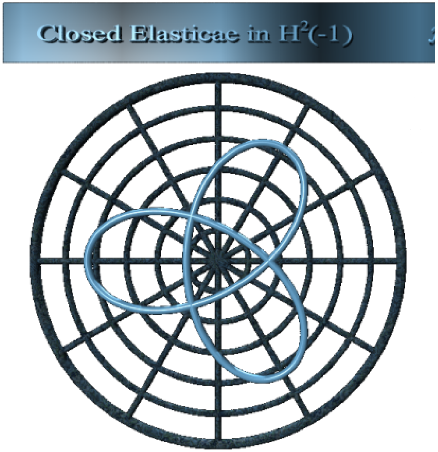


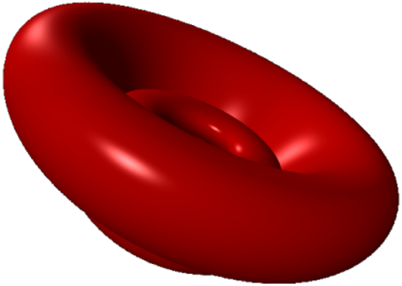


The Surfaces of Revolution in $\mathbb{R}^3$ which are Willmore membranes are precisely those shaped on the elastic curves of $\mathbb{H}^2(-1)$ (J. Langer, D. Singer, (1985)).



Willmore surfaces





6.3. Chen-Willmore submanifolds

In the early seventies, B-Y Chen extended the Thomsem-Willmore functional to any submanifold M of any Riemannian manifold N. He defined (Chen-Willmore functional):

$$\mathcal{CW}(M) = \int_M \left(H^2 - \tau_e\right)^{\frac{n}{2}} dv,$$

- H and τ_e being the mean curvature and the extrinsic scalar curvature of M, respectively;
- It is conformally invariant.
- its critical points are known as Chen-Willmore submanifolds

Chen-Willmore submanifolds

- **When $n = 2$ and $N = \mathrm{R}^3$ it coincides with the Willmore functional.**

- **Examples of Chen-Willmore tori in spheres and complex projective spaces have been given by : Barros, Chen, Garay, Singer,...**

- **Z. Guo, H. Li and Ch. Wang (2001) have shown that, in contrast with the surfaces case, a minimal submanifold of the sphere is not necessarily a Chen-Willmore submanifold. They also determined the Riemannian products of standard spheres which are Chen-Willmore hypersurfaces of S^{n+1} (standard examples).**

Chen-Willmore submanifolds

- **A quite general procedure to construct Chen-Willmore submanifolds in warped product Riemannian manifolds has been described by Arroyo, Barros, Garay (1999).**

Theorem

Let $(M, g) = M_1 \times_f M_2$ be a warped product where (M_2, g_2) is a compact homogeneous space of dimension n_2. Let γ be a closed curve immersed in (M_1, g_1). The submanifold $N = \gamma \times_f M_2$ is a Willmore-Chen submanifold in (M, g) if and only if γ is a n_2-generalized elastica in $(M_1, \frac{1}{f^2} g_1)$.

Chen-Willmore submanifolds

The main point is that we can relate this variational problem to that of hyperelastic curves in the conformal structure on the base space.

It explains

- **The Willmore cylinders shaped on plane elastica.**
- **The Willmore Hopf Tori shaped on spherical elastica.**
- **The Willmore surfaces of revolution shaped on hyperbolic elastica.**

6.4. Chen-Willmore hypersurfaces

In (2003), we produced the first examples of Chen-Willmore hypersurfaces of $\mathbb{R}^{n+1}$ and $\mathbb{S}^{n+1}$, which are not in the conformal class of the standard examples.

We use the conformal invariance of the Chen-Willmore functional and the Palais' symmetric criticality principle,to characterize the Chen-Willmore rotational hypersurfaces of $\mathbb{R}^{n+1}$ and $\mathbb{S}^{n+1}$ in terms of the closed free n-elastic curves of the hyperbolic plane $\mathbb{H}^2(-1)$.

Chen-Willmore hypersurfaces

We prove that there exist periodic solutions to the Euler-Lagrange equation.

We also have a qualitative description of the non-constant curvature closed n-elastic curves, they are convex curves travelling along ϵ_n, which oscillate between two concentric circles and close up after an integer number of trips around ϵ_n.

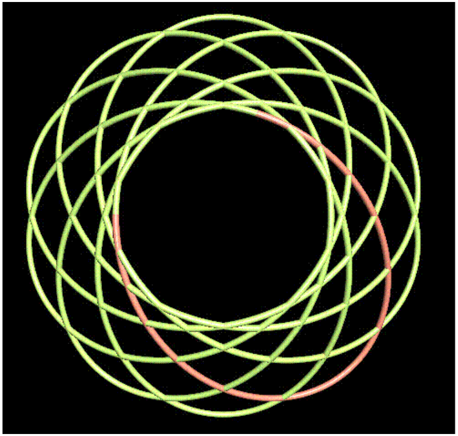
Getting concrete examples would require first to solve explicitly the Euler-Lagrange equations and then to quantify the closure condition.

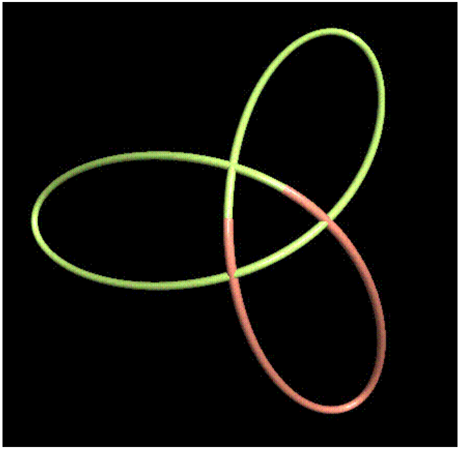
Chen-Willmore hypersurfaces

Although this task does not seem to be possible in general, it has been done for $n = 2$ by J. Langer and D. Singer (1987) and for $n = 3$ by J. Arroyo, M. Barros, O.J. Garay, (2002) .

- **Euler-Lagrange equation of 3-elastic curves in $\mathbb{H}^2(-1)$ can be explicitly integrated and the corresponding Frenet equations can be integrated by quadratures.**
- **We found a rationally dependent family of curves which fulfilled the closure condition. They provide the required examples.**

This gives explicit examples of Chen-Willmore hypersurfaces in $\mathbb{R}^4$.





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All pictures and animations are due to Prof. J. Arroyo.

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