Linear Connection Interpretation of the Non-linear Solutions of Extended Electrodynamics

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1. Linear connections

Linear connections ∇ are 1st-order differential operators in vector bundles. If such a connection ∇ is given and σ is a section of the bundle, then $\nabla\sigma$ is 1-form on the base space valued in the space of sections of the vector bundle, so if X is a vector field on the base space then $i(X)\nabla\sigma = \nabla_X\sigma$ is a new section of the same bundle. If f is a smooth function on the base space then $\nabla(f\sigma) = df \otimes \sigma + f\nabla\sigma$, which justifies the differential operator nature of ∇: the components of σ are differentiated and the basis vectors in the bundle space are linearly transformed.

Let e_a and $\varepsilon^b, a, b = 1, 2, \ldots, r$ be two dual local bases of the corresponding spaces of sections: $< \varepsilon^b, e_a >= \delta_a^b$, then we can write

$$\sigma = \sigma^a e_a, \quad \nabla = \mathbf{d} \otimes id + \Gamma^b_{\mu a} dx^\mu \otimes (\varepsilon^a \otimes e_b), \quad \nabla(e_a) = \Gamma^b_{\mu a} dx^\mu \otimes e_b,$$

so

$$\nabla(\sigma^m e_m) = \mathbf{d}\sigma^m \otimes e_m + \sigma^m \Gamma^b_{\mu a} dx^\mu < \varepsilon^a, e_m > \otimes e_b =$$
$$= \left[\mathbf{d}\sigma^b + \sigma^a \Gamma^b_{\mu a} dx^\mu\right] \otimes e_b,$$

and $\Gamma^b_{\mu a}$ are the components of ∇ with respect to the coordinates $\{x^\mu\}$ on the base space and with respect to the bases $\{e_a\}$ and $\{\varepsilon^b\}$.

Since the elements $(\varepsilon^a \otimes e_b)$ define a basis of the space of (local) linear maps of the local sections, it becomes clear that in order to define locally a linear connection it is sufficient to have some 1-form

θ on the base space and a linear map $\phi = \phi_a^b \varepsilon^a \otimes e_b$ in the space of sections. Then

$$\nabla(\sigma) = \mathbf{d}\sigma^a \otimes e_a + \theta \otimes \phi(\sigma)$$

defines a linear connection with components $\Gamma_{\mu a}^b = \theta_\mu \phi_a^b$ in these bases. So, locally, any linear connection ∇ may be written as

$$\nabla = \mathbf{d} \otimes (\varepsilon^a \otimes e_a) + \Psi_{\mu a}^b dx^\mu \otimes (\varepsilon^a \otimes e_b).$$

Let Ψ_1 and Ψ_2 be two 1-forms valued in the space of linear maps in a vector bundle. A map $(\Psi_1, \Psi_2) \to (\wedge, \odot)(\Psi_1, \Psi_2)$ is defined by (we shall write just $\odot$ for $(\wedge, \odot)$ and the usual $\circ$ will mean just composition)

$$\odot(\Psi_1, \Psi_2) = (\Psi_1)_{\mu a}^b (\Psi_2)_{\nu m}^n dx^\mu \wedge dx^\nu \otimes \left[\circ \, (\varepsilon^a \otimes e_b, \varepsilon^m \otimes e_n) \right] =$$

$$= (\Psi_1)_{\mu a}^b (\Psi_2)_{\nu m}^n dx^\mu \wedge dx^\nu \otimes \left[< \varepsilon^a, e_n > (\varepsilon^m \otimes e_b) \right] =$$

$$= (\Psi_1)_{\mu a}^b (\Psi_2)_{\nu m}^a dx^\mu \wedge dx^\nu \otimes (\varepsilon^m \otimes e_b), \quad \mu < \nu.$$

In the case of trivial vector bundles, the curvature of ∇ is given by

$$\left[\mathbf{d}(\Psi_{\mu a}^b dx^\mu) \right] \otimes (\varepsilon^a \otimes e_a) + \odot(\Psi, \Psi).$$

2. Facts from Clasical electrodynamics and from Extended electrodynamics

We recall now some facts from Classical Electrodynamics (CED) and from Extended Electrodynamics (EED). The vector bundle under consideration is the (trivial) bundle $\Lambda^2(M)$ of 2-forms on the Minkowski space-time M. Recall that if $(F, *F)$ is a CED vacuum solution, i.e. $\mathbf{d}F = 0, \quad \mathbf{d}*F = 0$, then the combinations

$$\mathcal{F} = a\,F - b*F, \quad \mathcal{F}^* = b\,F + a*F,$$

where (a, b) are two arbitrary real numbers, also give a CED vacuum solution and, since on Minkowski space the corresponding Hodge star $*_2$ satisfies the relation $*^2 = -id_{\Lambda^2(M)}$, we obtain $\mathcal{F}^* = *\mathcal{F}$. The two corresponding energy tensors are related by

$$T(\mathcal{F}, \mathcal{F}^*) = (a^2 + b^2)\,T(F, *F).$$

Recall the real representation of complex numbers $z = aI + bJ$ where I is the unit matrix in $\mathbb{R}^2$ and J is the standard complex structure matrix in $\mathbb{R}^2$ with columns $(0, -1); (1, 0)$. So, we obtain an action of the linear group G of matrices $\alpha = aI + bJ$ on the CED vacuum solutions. This is a commutative group G and its Lie algebra $\mathcal{G}$ just adds the zero (2×2) matrix to G, and (I, J) define a natural basis of $\mathcal{G}$. So, having a CED vacuum solution, we have in fact a 2-parameter family of vacuum solutions. Hence, we can define a $\mathcal{G}$-valued 2-form Ω on M by $\Omega = F \otimes I + *F \otimes J$, and the equation $\mathbf{d}\Omega = 0$ is equivalent to $\mathbf{d}F = 0$, $\mathbf{d} * F = 0$.

Consider the new basis (I', J') in $\mathcal{G}$ given by

$$I' = (aI + bJ), \quad J' = (-bI + aJ).$$

Accordingly, the "new" solution Ω', i.e. the old solution in the new basis of $\mathcal{G}$, will be

$$\Omega' = F \otimes I' + *F \otimes J' = F \otimes (aI + bJ) + *F \otimes (-bI + aJ) =$$
$$= (a\,F - b * F) \otimes I + (b\,F + a * F) \otimes J = \mathcal{F} \otimes I + \mathcal{F}^* \otimes J.$$

In view of this we may consider this transformation as $nonessential$, i.e. we may consider $(F, *F)$ and $(\mathcal{F}, \mathcal{F}^*)$ as two different representations in corresponding bases of $\mathcal{G}$ of the same solution.

Such an interpretation is approporiate and useful if the field shows some $invariant\ properties$ with respect to this class of transformations. For example, if the Lorentz invariants

$$I_1 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = (\mathbf{B}^2 - \mathbf{E}^2), \quad I_2 = \frac{1}{2} F_{\mu\nu}(*F)^{\mu\nu} = 2\mathbf{E}.\mathbf{B},$$

where $\mathbf{E}$ and $\mathbf{B}$ are the corresponding electric and magnetic components of F, are zero: $I_1 = I_2 = 0$, (the so called "null field case") then all the above transformations keep unchanged these zero-values of I_1 and I_2. In fact, under such a transformation $(F, *F) \rightarrow (\mathcal{F}, \mathcal{F}^*)$ the two Lorentz invariants transform to (I_1', I_2') in the following way:

$$I_1' = (a^2 - b^2)\,I_1 + 2ab\,I_2, \quad I_2' = -2ab\,I_1 + (a^2 - b^2)\,I_2,$$

and the determinant of this transformation is $(a^2 + b^2)^2 \neq 0$. So, a null field, i.e. a field with zero invariants I_1 and I_2, stays a null field under these transformations. Moreover, NO non-null field can be transformed to a null field by means of these transformations, and, conversely, NO null field can be transformed to a non-null field in this way. Hence, the Lorentz invariance and the dual G-invariance of I_1 and I_2 hold simultanoiusly only in the null-field case. Further we are going to pay due repect to this invariance, keeping in mind the basic fact that only in this case the velocity of the energy propagation of the field is equal to "c" and follows straight lines, so this is intrinsic property of the field.

In order to come to the equations of EED we recall that every bi-linear map $\varphi : \mathcal{G} \times \mathcal{G} \rightarrow W$, where W is some linear space with basis $\{e_i\}, i = 1, 2, \ldots$, defines corresponding product in the $\mathcal{G}$-valued differential forms by means of the relation

$$\varphi(\Omega_1^i \otimes e_i, \Omega_2^j \otimes e_j) = \Omega_1^i \wedge \Omega_2^j \otimes \varphi(e_i, e_j).$$

Recall now the following identity in Minkowski space:

$$I_1 \delta_\mu^\nu \equiv \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \delta_\mu^\nu = F_{\sigma\mu} F^{\sigma\nu} - (*F)_{\sigma\mu}(*F)^{\sigma\nu},$$

and the standard energy-tensor Q_μ^ν of electromagnetic field:

$$Q_\mu^\nu = -\frac{1}{2}\Big[F_{\sigma\mu} F^{\sigma\nu} + (*F)_{\sigma\mu}(*F)^{\sigma\nu} \Big].$$

We see that under $I_1 = 0$ the two fields F and $*F$ carry the same energy-momentum during propagation. Moreover, there is no inter-action energy-momentum between F and $*F$ as it is seen from the expression for Q_μ^ν. Hence, F and $*F$ may interact only in regime of *dynamical equilibrium*, i.e. any energy-momentum loss of $F / * F$ should be compensated by equal gain of $*F/F$: $F \rightleftarrows *F$.

EED makes use of these facts assuming that the corresponding dy-namical equations must have local energy-momentum exchange physical sense, so the symmetry $F \rightleftarrows *F$ must be respected.

Now, let $\varphi = \vee$, where "$\vee$" is the symmetrized tensor product in $\mathcal{G}$. We consider the expression $\vee(\Omega, *\mathbf{d}\Omega)$.

$$\vee(\Omega, *\mathbf{d}\Omega) = (F \wedge *\mathbf{d}F) \otimes I \vee I + (*F \wedge *\mathbf{d} * F) \otimes J \vee J+$$

$$+(F \wedge *\mathbf{d} * F \otimes + * F \wedge *\mathbf{d}F) \otimes I \vee J.$$

The vacuum EED equations are $\vee(\Omega, *\mathbf{d}\Omega) = 0$, or equavalently,

$$F \wedge *\mathbf{d}F = 0, \quad (*F) \wedge *\mathbf{d} * F = 0, \quad F \wedge *\mathbf{d} * F + (*F) \wedge *\mathbf{d}F = 0.$$

In terms of the codifferential $\delta = *\mathbf{d}*$ these equations look like

$$\delta * F \wedge F = 0, \quad \delta F \wedge *F = 0, \quad \delta F \wedge F - \delta * F \wedge *F = 0.$$

In components we obtain correspondingly

$$\frac{1}{2}F^{\alpha\beta}(\mathbf{d}F)_{\alpha\beta\mu} \equiv (*F)_{\mu\nu}(\delta * F)^\nu = 0,$$

$$\frac{1}{2}(*F)^{\alpha\beta}(\mathbf{d} * F)_{\alpha\beta\mu} \equiv F_{\mu\nu}(\delta F)^\nu = 0;$$

$$\frac{1}{2}(*F)^{\alpha\beta}(\mathbf{d}F)_{\alpha\beta\mu} + \frac{1}{2}F^{\alpha\beta}(\mathbf{d}*F)_{\alpha\beta\mu} \equiv (\delta * F)^\nu F_{\nu\mu} + (\delta F)^\nu(*F)_{\nu\mu} = 0.$$

Basic property of the nonlinear solutions:
All nonlinear solutions to these EED vacuum equations, i.e. those satisfying $\mathbf{d}F \neq 0, \mathbf{d} * F \neq 0$, have zero invariants: $I_1 = I_2 = 0$, so, they minimize the quantity $I_1^2 + I_2^2 \geq 0$. Moreover, for any nonlinear solution defined by F there exists a canonical coordinate system on M, called further F-adapted, in which F and $*F$ look as follows:

$$F = A \wedge \zeta+, \quad *F = A^* \wedge \zeta,$$

$$A = udx + pdy, \quad A^* = pdy - udz, \quad \zeta = \varepsilon dz + d\xi, \quad \varepsilon \pm 1,$$

and (u, p) are two functions on M, satisfying the equation

$$u(u_\xi - \varepsilon u_z) + p(p_\xi - \varepsilon p_z) = 0$$

As for the energy-momentum tensor $T_{\mu\nu}$ of the vacuum solutions, considered as a symmetric 2-form on M, it is defined in terms of Ω as follows:

$$T(X,Y) = \frac{1}{2} * g\left[i(X)\Omega, *i(Y)\Omega\right] =$$

$$= -\frac{1}{2} X^{\mu} Y^{\nu}\left[F_{\mu\sigma} F_{\nu}{}^{\sigma} + (*F)_{\mu\sigma}(*F)_{\nu}{}^{\sigma}\right] = X^{\mu} Y^{\nu} T_{\mu\nu},$$

where (X,Y) are two arbitrary vector fields on M, g is the metric in $\mathcal{G}$ defined by $g(\alpha, \beta) = \frac{1}{2} tr(\alpha.\beta^*)$, and β^* is the transposed to β. Note that $g(I, J) = 0$, which elliminates the corresponding coefficient which reads $F_{\mu\sigma}(*F)^{\nu\sigma} + (*F)_{\mu\sigma} F^{\nu\sigma} = \frac{1}{2} F_{\alpha\beta}(*F)^{\alpha\beta} \delta_{\mu}^{\nu}$, so, in a g-NONorthogonal basis of $\mathcal{G}$ this coefficient will appear.

Finally, recall the generalization of Lie derivative $\mathcal{L}_K$ with respect to the k-vector K, acting in the exterior algebra of differential forms according to the formula $\mathcal{L}_K = i(K)\mathbf{d} - (-1)^k \mathbf{d} i(K)$. Then, in view of the relations $F_{\mu\nu} F^{\mu\nu} = (*F)_{\mu\nu} F^{\mu\nu} = 0$, the above equations acquire the form

$$\mathcal{L}_{\bar{F}} F = 0, \quad \mathcal{L}_{*\bar{F}}(*F) = 0, \quad \mathcal{L}_{\bar{F}}(*F) + \mathcal{L}_{(*\bar{F})} F = 0,$$

where $\bar{F}$ and $*\bar{F}$ are the η-corresponding 2-vectors. In terms of Ω and $\bar{\Omega} = \bar{F} \otimes e_1 + *\bar{F} \otimes e_2$ these three equations can be united in one as follows

$$\mathcal{L}_{\bar{\Omega}}^{\vee} \Omega = \mathcal{L}_{\bar{F}} F \otimes e_1 \vee e_1 + \mathcal{L}_{*\bar{F}} *F \otimes e_2 \vee e_2 + (\mathcal{L}_{\bar{F}} *F + \mathcal{L}_{*\bar{F}} F) \otimes e_1 \vee e_2 = 0.$$

3. Linear connection interpretation of the nonlinear part of EED

If $\mathcal{I}$ is the identity in $\Lambda^2(M)$ and $\mathcal{J} = *$ is the complex structure in $\Lambda^2(M)$ then a representation ρ of G in $\Lambda^2(M)$ is given by

$$\rho(\alpha) = a\mathcal{I} + b\mathcal{J}.$$

Also, a representation ρ' of the corresponding Lie algebra $\mathcal{G}$ is defined by the same relation. So, if $\alpha : M \to \mathcal{G}$ is a map then $\rho'(\alpha)$ is a linear

map in $\Lambda^2(M)$, and recalling our 1-form $\zeta = \varepsilon dz + d\xi$ we define a linear connection ∇ in $\Lambda^2(M)$ by

$$\nabla = \mathbf{d} \otimes id_{\Lambda^2(M)} + \zeta \otimes \rho'(\alpha(u,p)) =$$

$$= \mathbf{d} \otimes id_{\Lambda^2(M)} + \zeta \otimes (u\mathcal{I} + p\mathcal{J}), \quad \alpha \in \mathcal{G}.$$

Two other connections $\bar{\nabla}$ and ∇^* are defined by

$$\rho'(\bar{\alpha}(u,p)) = \rho'(\alpha(u,-p)) = u\mathcal{I} - p\mathcal{J},$$

$$(\rho')^*(\alpha(u,p)) = \rho'(\alpha.J) = \rho'(\alpha(-p,u)) = -p\mathcal{I} + u\mathcal{J},$$

and we shall denote:

$$\chi = u\mathcal{I} + p\mathcal{J}, \ \bar{\chi} = u\mathcal{I} - p\mathcal{J}, \ \chi^* = -p\mathcal{I} + u\mathcal{J}.$$

Denoting $\Psi = \zeta \otimes \chi, \ \bar{\Psi} = \zeta \otimes \bar{\chi}, \ \Psi^* = \zeta \otimes \chi^*$, we note that (since $\zeta \wedge \zeta = 0$)

$$\odot(\Psi, \Psi) = \odot(\Psi, \bar{\Psi}) = \odot(\Psi, \Psi^*) = 0.$$

Now, since

$$\Psi = u\zeta \otimes \mathcal{I} + p\zeta \otimes \mathcal{J},$$
$$\bar{\Psi} = u\zeta \otimes \mathcal{I} - p\zeta \otimes \mathcal{J},$$
$$\Psi^* = -p\zeta \otimes \mathcal{I} + u\zeta \otimes \mathcal{J}$$

for the corresponding curvatures we obtain

$$\mathcal{R} = \mathbf{d}(u\zeta) \otimes \mathcal{I} + \mathbf{d}(p\zeta) \otimes \mathcal{J},$$

$$\bar{\mathcal{R}} = \mathbf{d}(u\zeta) \otimes \mathcal{I} - \mathbf{d}(p\zeta) \otimes \mathcal{J},$$

$$\mathcal{R}^* = \mathbf{d}(-p\zeta) \otimes \mathcal{I} + \mathbf{d}(u\zeta) \otimes \mathcal{J}.$$

By direct calculation we obtain:

$$*\frac{1}{6}Tr\left[\odot(\bar{\Psi}, *\mathbf{d}\Psi)\right] = -\varepsilon\left[u(u_\xi - \varepsilon u_z) + p(p_\xi - \varepsilon p_z)\right]dz -$$

$$- \left[u(u_\xi - \varepsilon u_z) + p(p_\xi - \varepsilon p_z)\right]d\xi;$$

$$\frac{1}{6}Tr\left[\odot(\Psi^*, *\mathbf{d}\Psi)\right] = \varepsilon\left[p(u_\xi - \varepsilon u_z) - u(p_\xi - \varepsilon p_z)\right]dx \wedge dy \wedge dz+$$

$$+\left[p(u_\xi - \varepsilon u_z) - u(p_\xi - \varepsilon p_z)\right]dx \wedge dy \wedge d\xi =$$

$$= \delta F \wedge F = \delta * F \wedge *F = \varepsilon\frac{1}{6}Tr\left[\odot(\Psi, *\mathbf{d}\Psi^*)\right],$$

where δ is the coderivatie. Denoting by $|\mathcal{R}|^2$ the quantity $\frac{1}{6}|*Tr\left[\odot(\mathcal{R} \wedge *\bar{\mathcal{R}})\right]|$ we obtain (in the F-adapted coordinate system)

$$|\mathcal{R}|^2 = \frac{1}{6}|*Tr\left[\odot(\mathbf{d}\Psi, *\mathbf{d}\bar{\Psi})\right]| =$$

$$= (u_\xi - \varepsilon u_z)^2 + (p_\xi - \varepsilon p_z)^2 = |\delta F|^2 = |\delta * F|^2.$$

Finally we note the relations

$$\frac{1}{6}tr(\chi) = \frac{1}{6}tr(u\mathcal{I} + p\mathcal{J}) = u, \quad \text{and}$$

$$\frac{1}{6}tr\left[(\chi \circ \bar{\chi})\right] = \frac{1}{6}tr\left[(u\mathcal{I} + p\mathcal{J}) \circ (u\mathcal{I} - p\mathcal{J})\right] = u^2 + p^2.$$

These relations allow to introduce two characteristic functions for any nonlinear solution: *phase-function* ψ and *scale factor* $\mathcal{L}$ according to:

$$\psi = \arccos\frac{\frac{1}{6}tr\chi}{\sqrt{\frac{1}{6}tr(\chi \circ \bar{\chi})}}, \quad \mathcal{L} = \frac{\sqrt{\frac{1}{6}tr(\chi \circ \bar{\chi})}}{\sqrt{\frac{1}{6}|\mathcal{R}|}} = \frac{\sqrt{tr(\chi \circ \bar{\chi})}}{|\mathcal{R}|}.$$

Since the nonlinear solutions of the two equations

$$\delta F \wedge *F = 0, \ \delta * F \wedge F = 0,$$

i.e. those satisfyiing $\delta F \neq 0$ and $\delta * F \neq 0$, are parametrized by two functions (u, p) and satisfy the relations

$$u(u_\xi - \varepsilon u_z) + p(p_\xi - \varepsilon p_z) = 0, \quad (u_\xi - \varepsilon u_z) \neq 0, \quad (p_\xi - \varepsilon p_z) \neq 0,$$

in the corresponding F-adapted coordinate system, we obtain that on those solutions the following relation holds:

$$Tr\left[\odot(\bar{\Psi}, *\mathbf{d}\Psi)\right] = 0,$$

and the equation
$$\delta F \wedge F = \delta * F \wedge *F$$
is equivalent to
$$Tr\left[\odot(\Psi, *\mathbf{d}\Psi^*)\right] = \varepsilon\, Tr\left[\odot(\Psi^*, *\mathbf{d}\Psi)\right].$$
It can be shown that the non-zero value of the squared curvature invariant $|\mathcal{R}|^2$ guarantees availability of rotational component of propagation.

4. Conclusion

The linear connection ∇ is defined through $\zeta \otimes \rho'(\alpha(u, p))$. We note that the 2-form $F_o = dx \otimes \zeta$ gives the possibility to consider a nonlinear solution $F(u, p)$ as an appropriately defined linear map
$$\rho'(\alpha(u, p)) = u\mathcal{I} + p\mathcal{J}$$
in $\Lambda^2(M)$, since the action of $\rho'(\alpha(u, p))$ on F_o gives F.

This special importance of ζ is based on the fact that it intrinsically defines the translational part of the dynamical befavior of the solution, and its uniqueness is determined by the fact that all nonlinear solutions of EED equations have zero invariants: $F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}(*F)^{\mu\nu} = 0$. As for the rotational part of the dynamical behavior of the solution it is available only if the curvature $\mathcal{R}$ iz nonzero and is locally represented by any of the two 3-forms $F \wedge \delta F = *F \wedge \delta * F \neq 0$. For all nonlinear solutions we have $\delta F \neq 0$ and $\delta * F \neq 0$, and all finite nonlinear solutions have finite energy density:
$$0 < \phi^2 = \frac{1}{6}tr(F \circ \bar{F}) = \frac{1}{6}tr(\chi \circ \bar{\chi}) = (u^2 + p^2) < \infty.$$
The nonzero finite scale factor $0 < \mathcal{L} < \infty$ separates those finite nonlinear solutions which carry spin momentum, and this happens only when $|\delta F| = |\mathcal{R}| \neq 0$. The spin momentum is carried by any of the two 3-forms $\delta F \wedge F = \delta * F \wedge *F$, determining the energy-momentum exchange between F and $*F$. Clearly, on the linear Maxwell solutions these 3-forms are zero.

Hence, in terms of curvature we can say that *the nonzero curvature invariant $|\mathcal{R}|$, is responsible for availability of rotational component of propagation*, in other words, *the spin properties of a non-linear solution require non-zero curvature.*

From physical viewpoint the corresponding dynamical process that generates these spin properties is the mutual energy-momentum exchange between the two components F and $*F$ during propagation. It has the following three characteristic properies:

-it is *permanent*, i.e. it occurs constantly during propagation,

-it is *simultanious* in the both directions: $F \rightleftarrows *F$,

-it is in *equal quantities.*

It follows that F and $*F$ live in a *permanent dynamical equilibrium*. They carry always the same quantities of energy-momentum:

$$\left[F_{\mu\nu}F^{\mu\nu} = 0\right] \;\Rightarrow\; F_{\mu\sigma}F^{\nu\sigma} = (*F)_{\mu\sigma}(*F)^{\nu\sigma}.$$

This dynamical equilibrium is quantitatively described by the equation

$$\delta F \wedge F = \delta * F \wedge *F,$$

or by

$$Tr\left[\odot(\Psi, *\mathbf{d}\Psi^*)\right] = \varepsilon \, Tr\left[\odot(\Psi^*, *\mathbf{d}\Psi)\right].$$