

# Partial Differential Hamiltonian Systems

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# Introduction

## The very well known picture



generalizes to



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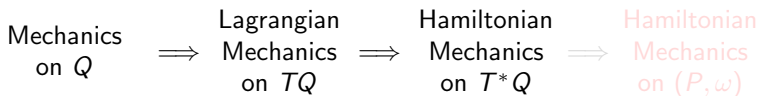


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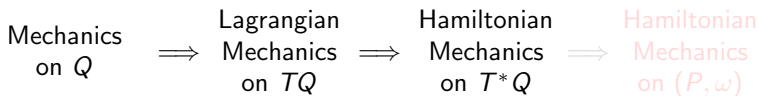


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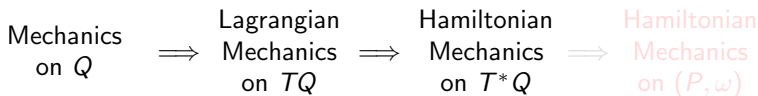


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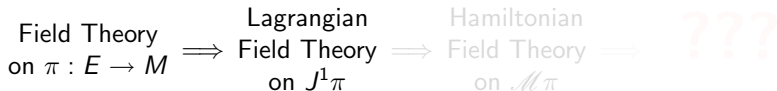


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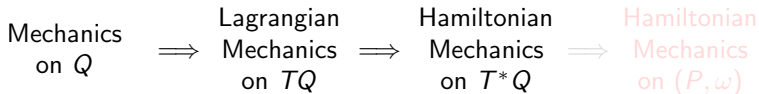


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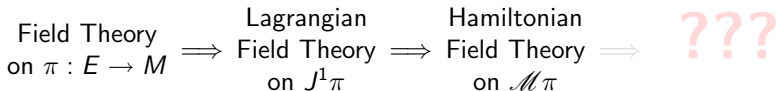


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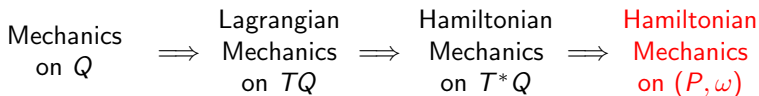
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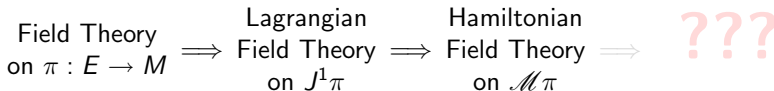


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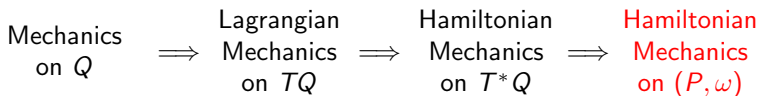


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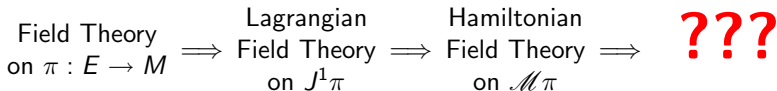


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## Introduction: a naive remark

In hamiltonian mechanics the time line is assumed to be **parameterized**

motions = curves  $\gamma : \mathbb{R} \rightarrow P$

velocities = **tangent vectors** in  $TP$

**tangent vectors** are naturally inserted into **differential forms**

Hamilton equations of motions:  $i_{\dot{\gamma}}\omega|_{\gamma} - (dH)|_{\gamma} = 0$

What if the time line is **unparameterized**?

motions = maps  $\gamma : M \rightarrow P$

velocities = **jets of maps** in  $J^1(M, P)$

**jets of maps** are naturally inserted into ???

equations of motion ???

In field theory **space-time is unparameterized!!!**

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- 1 Affine Forms on Fiber Bundles
- 2 Affine Form Calculus
- 3 PD Hamiltonian Systems
- 4 PD Noether Symmetries and Currents
- 5 Examples

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# Special Forms on Fiber Bundles

Let  $\alpha : P \rightarrow M$  be a fiber bundle,  $x^1, \dots, x^n, y^1, \dots, y^m$  adapted coordinates on  $P$ . On  $P$  consider special forms

$$\bar{\Lambda} := \left\{ \sum_q \sum_{i_1, \dots, i_q} f_{i_1, \dots, i_q} dx^{i_1} \wedge \dots \wedge dx^{i_q} : f_{i_1, \dots, i_q} \in C^\infty(P) \right\}$$

is a graded subalgebra in  $\Lambda(P)$

$$\Lambda_p := \left\{ \sum_{i_1, \dots, i_p} dx^{i_1} \wedge \dots \wedge dx^{i_p} \wedge \omega_{i_1, \dots, i_p} : \omega_{i_1, \dots, i_p} \in \Lambda(P) \right\}$$

is a differential ideal in  $\Lambda(P)$

Put  $V\Lambda := \Lambda(P)/\Lambda_1$ .  $V\Lambda$  is endowed with the vertical differential

$$d^V : V\Lambda \rightarrow V\Lambda$$

$$V\Lambda \ni \omega^V = \sum_p \sum_{i_1, \dots, i_p} g_{i_1, \dots, i_p} d^V y^{a_1} \wedge \dots \wedge d^V y^{a_p}, \quad g_{i_1, \dots, i_p} \in C^\infty(P)$$

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# Affine Forms

Let  $C(P, \alpha) := \{(\text{Ehresmann}) \text{ connections in } \alpha : P \rightarrow M\}$ .  
 $C(P, \alpha)$  is an **affine space** (modelled over  $VD \otimes \bar{\Lambda}^1$ ).

Denote by  $'\Omega^{k+1}$  the space of **affine maps**  $C(P, \alpha) \longrightarrow V\Lambda^k \otimes \bar{\Lambda}^n$ .

Linear parts of elements in  $'\Omega^{k+1}$  live in  $'\underline{\Omega}^{k+1} := V\Lambda^1 \otimes V\Lambda^k \otimes \bar{\Lambda}^{n-1}$ .

In  $'\Omega^{k+1}$  consider the subspace  $\underline{\Omega}^{k+1}$  made of **skew-symmetric elements**, i.e., elements whose linear parts live in  $\underline{\Omega}^{k+1} := V\Lambda^{k+1} \otimes \bar{\Lambda}^{n-1}$ .

$$\begin{array}{ccc} \omega : C(E, \pi) & \longrightarrow & V\Lambda \otimes \bar{\Lambda}^n \\ \nabla & \longmapsto & i_{\nabla} \omega \end{array}$$

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# Affine Forms and Differential Forms

Affine forms are just special differential forms!

Theorem: There are natural graded isomorphisms ...

$$\iota : \Lambda_{n-1} \longrightarrow \Omega \quad \text{and} \quad \underline{\iota} : \Lambda_{n-1}/\Lambda_n \longrightarrow \underline{\Omega}$$

such that the following diagram commutes

$$\begin{array}{ccc} \Lambda_{n-1} & \longrightarrow & \Lambda_{n-1}/\Lambda_n \\ \iota \downarrow & & \downarrow \underline{\iota} \\ \Omega & \longrightarrow & \underline{\Omega} \end{array}$$

For instance,  $\omega \in \Omega^2$  is a differential form locally given by

$$\omega = \omega_{ab}^i dy^a \wedge dy^b \wedge d^{n-1}x_i - \omega_c dy^c \wedge d^n x,$$

and, for  $\nabla \in C(P, \alpha)$ ,  $i_{\nabla} \omega = (\omega_{ab}^i \nabla_i^b - \omega_a) d^V y^a \otimes d^n x$ .

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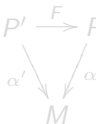
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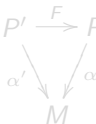
# Natural Operations with Affine Forms

- Insertion of a connection  $\nabla$ ,  $i_{\nabla} : \Omega \ni \omega \mapsto i_{\nabla}\omega \in \Omega$
- Insertion of an  $\alpha$ -vertical vector field  $Y$ ,  $i_Y : \Omega \ni \omega \mapsto i_Y\omega \in \Omega$
- Lie der. along an  $\alpha$ -proj. vector field  $X$ ,  $L_X : \Omega \ni \omega \mapsto L_X\omega \in \Omega$
- Differential  $\delta : \Omega \ni \omega \mapsto \delta\omega \in \Omega$
- Pull-back w.r. to a morphism  $P' \xrightarrow{F} P$ ,  $F^* : \Omega \ni \omega \mapsto F^*(\omega) \in \Omega'$



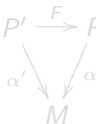
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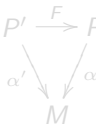
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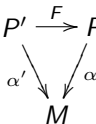
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# Affine Form Cohomology

The sequence

$$0 \longrightarrow \bar{\Lambda}^{n-1} \xrightarrow{d} \Omega^1 \xrightarrow{\delta} \dots \longrightarrow \Omega^{k+1} \xrightarrow{\delta} \dots \longrightarrow \Omega^{n+m} \longrightarrow 0$$

is, clearly, a complex:  $\delta^2 = 0$ .

Cohomology  $H(\Omega, \delta)$  depends on the topology of the bundle  $\alpha : P \longrightarrow M$

## Affine Form Poincaré Lemma

Let  $\omega \in \Omega^{k+1}$ ,  $k \geq 0$  be  $\delta$ -closed, i.e.,  $\delta\omega = 0$ , then  $\omega$  is locally  $\delta$ -exact, i.e., locally,  $\omega = \delta\vartheta$ ,  $\vartheta \in \Omega^k$ .

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A **PD hamiltonian system (PDHS)** in the bundle  $\alpha : P \longrightarrow M$  is a  $\delta$ -closed affine 2-form  $\omega \in \Omega^2$ , i.e.,  $\delta\omega = 0$ .

Locally, a PDHS is a  $\delta$ -exact affine form and it is locally given by

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Solutions to the PDHE can be searched in two steps:

- 1 search for connections  $\nabla \in C(P, \alpha)$  such that  $i_{\nabla}\omega = 0$ ,
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Problem 1 doesn't possess solutions in general. Therefore, we rather search for connections  $\nabla'$  in some subbundle  $P' \subset P$ , such that  $i_{\nabla'}\omega|_{P'} = 0$ .

Theorem (existence of a Constraint Algorithm)

Let  $P_{(s)} := \{\theta \in P : \exists \alpha\text{-horizontal } \Pi^n \subset T_{\theta}P_{(s-1)} \text{ s.t. } i_{\Pi}\omega_{\theta} = 0\}$ .

Then  $P_{(s)} = P'$  for  $s \gg 1$ , and  $P'$  is a maximal subbundle where  $i_{\nabla'}\omega|_{P'} = 0$  for some connection  $\nabla'$  in  $P'$ .

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$Y \in VD$  and  $f \in \bar{\Lambda}^{n-1}$  are a **PD Noether symmetry/current pair**, iff  $i_Y \omega = df$ .

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Let  $f \in \bar{\Lambda}^{n-1}$  be a PD Noether current and  $\sigma$  a solution of PDHE. Then  $\int_{\Sigma} \sigma^*(f)$  is a *conserved quantity*, i.e., it is independent on the choice of  $\Sigma^{n-1} \subset M$  in a homology class.

PD Noether symmetries and currents form Lie algebras.

If  $Y_1, f_1$  and  $Y_2, f_2$  are PD Noether symmetry/current pairs, then  $[Y_1, Y_2], \{f_1, f_2\} := L_{Y_1} f_2$  is a well defined PD Noether symmetry/current pair and  $f_1, f_2 \mapsto \{f_1, f_2\}$  is a Lie bracket.

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Let  $\omega$  be an unconstrained PDHS. A PD Noether symmetry  $Y$  such that  $i_Y\omega = 0$  is naturally interpreted as a gauge symmetry. Gauge symmetries should be quotiented out via reduction.

## Remark

Gauge symmetries span an involutive  $\alpha$ -vertical distribution  $G$ .

Denote by  $\tilde{P}$  the bundle of leaves of  $G$  and by  $p : P \rightarrow \tilde{P}$  the projection.

## Theorem (gauge reduction of PDHSs)

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## A Non-Degenerate Example

In the bundle  $(x^1, x^2, u, u_1, u_2) \mapsto (x^1, x^2)$  consider the PDHS

$$\omega := T^{-1}(\delta^{ij} - T^{-2}u^i u^j) du_i \wedge (du \wedge d^{n-1}x_j - u_j d^n x), \quad T := \sqrt{1 + \delta^{ij} u_i u_j}$$

The PDHE are

$$\begin{cases} (\delta^{ij} - T^{-2}u^i u^j) u_{i,j} = 0 \\ u_{,i} = u_i \end{cases}$$

which is equivalent to the minimal surface equation.

### Proposition

$Y, f$  is a PD Noether symmetry current pair iff

$$Y = U \frac{\partial}{\partial u}, \quad f = U T^{-1} (u_2 dx^1 - u_1 dx^2) + dB,$$

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## A Degenerate, Unconstrained Example

In  $\alpha : P \ni (\dots, x^\mu, \dots, A_\mu, \dots, A_{[\mu|\nu]}, \dots) \mapsto (\dots, x^\mu, \dots) \in \mathbb{M}$  consider the PDHS  $\omega := 2dA^{[\nu|\mu]} \left( \frac{1}{2} A_{[\mu|\nu]} d^n x - dA_\mu d^{n-1} x_\nu \right)$ . The PDHE are

$$\begin{cases} A^{[\mu|\nu]},{}_{,\nu} = 0 & \text{which are Maxwell equations} \\ A_{[\mu|\nu]} = A_{[\nu|\mu]} & \text{for the potential} \end{cases}$$

$G := \left\langle \dots, \frac{\partial}{\partial A_{\mu|\nu}} + \frac{\partial}{\partial A_{\nu|\mu}}, \dots \right\rangle$  and the **gauge reduction of  $P$**  is

$$\mathfrak{p} : (\dots, x^\mu, \dots, A_\mu, \dots, A_{\mu|\nu}, \dots) \mapsto (\dots, x^\mu, \dots, A_\mu, \dots, F_{\mu\nu}, \dots)$$

$$F_{\mu\nu} = -F_{\nu\mu} \text{ and } \mathfrak{p}^*(F_{\mu\nu}) := 2A_{[\nu|\mu]}.$$

$$\tilde{\omega} = dF^{\mu\nu} \left( \frac{1}{4} F_{\mu\nu} d^n x - dA_\mu d^{n-1} x_\nu \right)$$

and the reduced PDHE

$$\begin{cases} F^{\mu\nu},{}_{,\nu} = 0 & \text{which are Maxwell equations} \\ A_{[\mu|\nu]} = \frac{1}{2} F_{\mu\nu} & \text{for the field strength} \end{cases}$$

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