Partial Differential Hamiltonian Systems

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XI International Conference
Geometry, Integrability and Quantization
Varna, June 5–10, 2009

Introduction

The very well known picture

Mechanics on Q	$\Longrightarrow$	Lagrangian Mechanics on TQ	$\Longrightarrow$	Hamiltonian Mechanics on T^*Q	$\Longrightarrow$	Hamiltonian Mechanics on (P, ω)

generalizes to

Field Theory on $\pi : E \to M$	$\Longrightarrow$	Lagrangian Field Theory on $J^1\pi$	$\Longrightarrow$	Hamiltonian Field Theory on $\mathscr{M}\pi$	$\Longrightarrow$	???

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Introduction: a naive remark

In hamiltonian mechanics the time line is assumed to be **parameterized**

motions = curves $\gamma : \mathbb{R} \longrightarrow P$
velocities = **tangent vectors** in TP

tangent vectors are naturally inserted into **differential forms**
Hamilton equations of motions: $\mathbf{i}_{\dot\gamma}\omega|_\gamma - (\mathbf{dH})|_\gamma = \mathbf{0}$

What if the time line is **unparameterized**?

motions = maps $\gamma : M \longrightarrow P$
velocities = **jets of maps** in $J^1(M, P)$

jets of maps are naturally inserted into **???**
equations of motion **???**

In field theory **space-time is unparameterized**!!!

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Special Forms on Fiber Bundles

Let $\alpha : P \longrightarrow M$ be a fiber bundle, $x^1, \ldots, x^n, y^1, \ldots, y^m$ adapted coordinates on P. On P consider special forms

$$\overline{\Lambda} := \{\sum_q \sum_{i_1,\ldots,i_q} f_{i_1,\ldots,i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} : f_{i_1,\ldots,i_q} \in C^\infty(P)\}$$

is a graded subalgebra in $\Lambda(P)$

$$\Lambda_p := \{\sum_{i_1,\ldots,i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge \omega_{i_1,\ldots,i_p} : \omega_{i_1,\ldots,i_p} \in \Lambda(P)\}$$

is a differential ideal in $\Lambda(P)$

Put $V\Lambda := \Lambda(P)/\Lambda_1$. $V\Lambda$ is endowed with the vertical differential

$$d^V : V\Lambda \longrightarrow V\Lambda$$

$$V\Lambda \ni \omega^V = \sum_p \sum_{i_1,\ldots,i_p} g_{i_1,\ldots,i_p} d^V y^{a_1} \wedge \cdots \wedge d^V y^{a_q},\ g_{i_1,\ldots,i_p} \in C^\infty(P)$$

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Affine Forms

Let $C(P,\alpha) := \{$(Ehresmann) connections in $\alpha : P \to M\}$.
$C(P,\alpha)$ is an **affine space** (modelled over $VD \otimes \overline{\Lambda}^1$).

Denote by $'\Omega^{k+1}$ the space of **affine maps** $C(P,\alpha) \longrightarrow V\Lambda^k \otimes \overline{\Lambda}^n$.

Linear parts of elements in $'\Omega^{k+1}$ live in $'\underline{\Omega}^{k+1} := V\Lambda^1 \otimes V\Lambda^k \otimes \overline{\Lambda}^{n-1}$.

In $'\Omega^{k+1}$ consider the subspace Ω^{k+1} made of **skew-symmetric elements**, i.e., elements whose linear parts live in $\underline{\Omega}^{k+1} := V\Lambda^{k+1} \otimes \overline{\Lambda}^{n-1}$.

$$\begin{array}{rccc} \omega : & \mathbf{C(E,\pi)} & \longrightarrow & \mathbf{V\Lambda} \otimes \overline{\Lambda}^n \\ & \nabla & \longmapsto & \mathbf{i}_\nabla \omega \end{array}$$

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Affine Forms and Differential Forms

Affine forms are just special differential forms!

Theorem: There are natural graded isomorphisms ...

$$\iota : \Lambda_{n-1} \longrightarrow \Omega \quad \text{and} \quad \underline{\iota} : \Lambda_{n-1}/\Lambda_n \longrightarrow \underline{\Omega}$$

such that the following diagram commutes

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For instance, $\omega \in \Omega^2$ is a differential form locally given by

$$\omega = \omega^i_{ab} dy^a \wedge dy^b \wedge d^{n-1}x_i - \omega_c dy^c \wedge d^n x \ ,$$

and, for $\nabla \in C(P, \alpha)$, $i_\nabla \omega = (\omega^i_{ab} \nabla^b_i - \omega_a) d^V y^a \otimes d^n x$.

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Natural Operations with Affine Forms

- Insertion of a connection ∇, $\quad i_\nabla : \Omega \ni \omega \mapsto i_\nabla \omega \in \Omega$
- Insertion of an α-vertical vector field Y, $\quad i_Y : \Omega \ni \omega \mapsto i_Y \omega \in \Omega$
- Lie der. along an α-proj. vector field X, $\quad L_X : \Omega \ni \omega \mapsto L_X \omega \in \Omega$
- Differential $\quad \delta : \Omega \ni \omega \mapsto \delta\omega \in \Omega$
- Pull-back w.r.t. to a morphism $\begin{array}{ccc} P' & \xrightarrow{F} & P \\ & {\scriptstyle \alpha'} \searrow \quad \swarrow {\scriptstyle \alpha} & \\ & M & \end{array}$, $\quad F^* : \Omega \ni \omega \mapsto F^*(\omega) \in \Omega'$

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- Lie der. along an α-proj. vector field X, $\quad L_X : \Omega \ni \omega \mapsto L_X \omega \in \Omega$
- Differential $\quad \delta : \Omega \ni \omega \mapsto \delta \omega \in \Omega$
- Pull-back w.r.t. to a morphism $\begin{array}{ccc} P' & \xrightarrow{F} & P \\ & {\scriptstyle \alpha'} \searrow \quad \swarrow {\scriptstyle \alpha} & \\ & M & \end{array}$, $\quad F^* : \Omega \ni \omega \mapsto F^*(\omega) \in \Omega'$

Affine Form Cohomology

The sequence

$$0 \longrightarrow \overline{\Lambda}^{n-1} \xrightarrow{d} \Omega^1 \xrightarrow{\delta} \cdots \longrightarrow \Omega^{k+1} \xrightarrow{\delta} \cdots \longrightarrow \Omega^{n+m} \longrightarrow 0$$

is, clearly, a complex: $\delta^2 = 0$.

Cohomology $H(\Omega, \delta)$ depends on the topology of the bundle $\alpha : P \longrightarrow M$

Affine Form Poincaré Lemma

Let $\omega \in \Omega^{k+1}$, $k \geq 0$ be δ-closed, i.e., $\delta\omega = 0$, then ω is locally δ-exact, i.e., locally, $\omega = \delta\vartheta$, $\vartheta \in \Omega^k$.

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PD Hamiltonian Systems and PD Hamilton Equations

Definition

A **PD hamiltonian system (PDHS)** in the bundle $\alpha : P \longrightarrow M$ is a δ-closed affine 2-form $\omega \in \Omega^2$, i.e., $\delta\omega = 0$.

Locally, a PDHS is a δ-exact affine form and it is locally given by

$$\omega^i_{ab} dy^a \wedge dy^b \wedge d^{n-1}x_i - (\partial_a H + \partial_i \vartheta^i_a) dy^a \wedge d^n x, \quad \omega^i_{ab} = \partial_{[a}\vartheta^i_{b]}.$$

Definition.

PD Hamilton equations (PDHE) determined by ω are the equations: $i_{\dot\sigma}\omega|_\sigma = 0$ imposed on sections of $\sigma : M \longrightarrow P$ of α.

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Constraint Algorithm

Solutions to the PDHE can be searched in two steps:

1. search for connections $\nabla \in C(P, \alpha)$ such that $i_\nabla \omega = 0$,
2. search for flat sections with respect to ∇.

Problem 1 doesn't possess solutions in general. Therefore, we rather search for connections ∇' in some subbundle $P' \subset P$, such that $i_{\nabla'}\omega|_{P'} = 0$.

Theorem (existence of a Constraint Algorithm)

Let $P_{(s)} := \{\theta \in P : \exists\ \alpha\text{-horizontal } \Pi^n \subset T_\theta P_{(s-1)} \text{ s.t. } i_\Pi \omega_\theta = 0\}$.

Then $P_{(s)} = P'$ for $s \gg 1$, and P' is a maximal subbundle where $i_{\nabla'}\omega|_{P'} = 0$ for some connection ∇' in P'.

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PD Noether Theorem and PD Poisson Bracket

Definition

$Y \in VD$ and $f \in \overline{\Lambda}^{n-1}$ are a **PD Noether symmetry/current pair**, iff $i_Y \omega = df$.

Theorem (PD Noether)

Let $f \in \overline{\Lambda}^{n-1}$ be a PD Noether current and σ a solution of PDHE. Then $\int_\Sigma \sigma^*(f)$ is a *conserved quantity*, i.e., it is independent on the choice of $\Sigma^{n-1} \subset M$ in a homology class.

PD Noether symmetries and currents form Lie algebras.

If Y_1, f_1 and Y_2, f_2 are PD Noether symmetry/current pairs, then $[Y_1, Y_2], \{f_1, f_2\} := L_{Y_1} f_2$ is a well defined PD Noether symmetry/current pair and $f_1, f_2 \mapsto \{f_1, f_2\}$ is a Lie bracket.

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Gauge Reduction of PD Hamiltonian Systems

Let ω be an unconstrained PDHS. A PD Noether symmetry Y such that $i_Y\omega = 0$ is naturally interpreted as a gauge symmetry. Gauge symmetries should be quotiented out via reduction.

Remark

Gauge symmetries span an involutive α-vertical distribution G.

Denote by $\widetilde{P}$ the bundle of leaves of G and by $\mathfrak{p} : P \to \widetilde{P}$ the projection.

Theorem (gauge reduction of PDHSs)

There exists a unique PDHS $\widetilde{\omega}$ in $\widetilde{P}$ such that
1) ω doesn't possess gauge symmetries, 2) $\omega = \mathfrak{p}^*(\widetilde{\omega})$, 3) a section σ of P is a solution of the PDHE determined by ω iff $\mathfrak{p} \circ \sigma$ is a solution of the PDHE determined by $\widetilde{\omega}$.

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A Non-Degenerate Example

In the bundle $(x^1, x^2, u, u_1, u_2) \longmapsto (x^1, x^2)$ consider the PDHS

$$\omega := T^{-1}(\delta^{ij} - T^{-2} u^i u^j) du_i \wedge (du \wedge d^{n-1}x_j - u_j d^n x), \quad T := \sqrt{1 + \delta^{ij} u_i u_j}$$

The PDHE are

$$\begin{cases} (\delta^{ij} - T^{-2} u^i u^j) u_{i,j} = 0 \\ u_{,i} = u_i \end{cases}$$

which is equivalent to the minimal surface equation.

Proposition

Y, f is a PD Noether symmetry current pair iff

$$Y = U \frac{\partial}{\partial u}, \quad f = U\, T^{-1} \left(u_2 dx^1 - u_1 dx^2 \right) + dB,$$

where $U = \mathrm{const}$, $B = B(x^1, x^2)$. The PD Poisson bracket is trivial.

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A Degenerate, Unconstrained Example

In $\alpha : P \ni (\ldots, x^\mu, \ldots, A_\mu, \ldots, A_{\mu|\nu}, \ldots) \mapsto (\ldots, x^\mu, \ldots) \in \mathbb{M}$ consider the PDHS $\omega := 2dA^{[\nu|\mu]}\left(\frac{1}{2}A_{[\mu|\nu]}d^n x - dA_\mu d^{n-1}x_\nu\right)$. The PDHE are

$$\begin{cases} A^{[\mu|\nu]}{}_{,\nu} = 0 \\ A_{[\mu,\nu]} = A_{[\mu|\nu]} \end{cases}$$

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$G := \left\langle \ldots, \frac{\partial}{\partial A_{\mu|\nu}} + \frac{\partial}{\partial A_{\nu|\mu}}, \ldots \right\rangle$ and the **gauge reduction of** P is

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$$\begin{cases} F^{\mu\nu}{}_{,\nu} = 0 \\ A_{[\mu,\nu]} = \frac{1}{2}F_{\mu\nu} \end{cases}$$

which are Maxwell equations for the field strenght

A Degenerate, Unconstrained Example

In $\alpha : P \ni (\ldots, x^\mu, \ldots, A_\mu, \ldots, A_{\mu|\nu}, \ldots) \mapsto (\ldots, x^\mu, \ldots) \in \mathbb{M}$ consider the PDHS $\omega := 2dA^{[\nu|\mu]} \left(\frac{1}{2} A_{[\mu|\nu]} d^n x - dA_\mu d^{n-1} x_\nu\right)$. The PDHE are

$$\begin{cases} A^{[\mu|\nu]}{}_{,\nu} = 0 \\ A_{[\mu,\nu]} = A_{[\mu|\nu]} \end{cases}$$

which are Maxwell equations for the potential

$G := \left\langle \ldots, \frac{\partial}{\partial A_{\mu|\nu}} + \frac{\partial}{\partial A_{\nu|\mu}}, \ldots \right\rangle$ and the **gauge reduction of** P is

$$\mathfrak{p} : (\ldots, x^\mu, \ldots, A_\mu, \ldots, A_{\mu|\nu}, \ldots) \mapsto (\ldots, x^\mu, \ldots, A_\mu, \ldots, F_{\mu\nu}, \ldots)$$

$F_{\mu\nu} = -F_{\nu\mu}$ and $\mathfrak{p}^*(F_{\mu\nu}) := 2A_{[\nu|\mu]}$.

$$\widetilde{\omega} = dF^{\mu\nu} \left(\frac{1}{4} F_{\mu\nu} d^n x - dA_\mu d^{n-1} x_\nu\right)$$

and the reduced PDHE

$$\begin{cases} F^{\mu\nu}{}_{,\nu} = 0 \\ A_{[\mu,\nu]} = \frac{1}{2} F_{\mu\nu} \end{cases}$$

which are Maxwell equations for the field strenght

Bibliographic Reference

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