Star Product and Star Exponential

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Abstract

We extend star products by means of complex symmetric matrices. We obtain a family of star products.

We consider star exponentials with respect to these star products, and we obtain several interesting identities.

Plan

① First we explain general setting; introducing the concept of q-number functions.

② Then we consider the example of star exponential and its application.

§1. A family of star products

§1.1. Moyal product, normal and anti-normal products

It is well known that the star products such as the Moyal product, normal product and the anti-normal product are obtained by fixing the orderings in the Weyl algebra.

These are products on polynimals and the obtained algebras are all isomorphic to the Weyl algebra.

§1.2, Extension

We extend these products and we obtain a family star products parametrized by the space of all complex symmetric matrices.

The intertwiners are also extended to these star products, and then all star product algebras are also mutually isomorphic and isomorophic to the Weyl algebra.

§1.3. Definition of star product

For simplicity, we consider star products of 2 variables (u_1, u_2). The general case for $(u_1, u_2, \cdots, u_{2m})$ is similar.

1. First we consider biderivation

For a complex matrix $\Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \in M_2(\boldsymbol{C})$, we consider a bi-derivation acting on complex polynoimals

$$p_1(u_1, u_2), p_2(u_1, u_2) \in \mathcal{P}(\boldsymbol{C}^2)$$

such that

$$\begin{aligned} p_1 \left(\overleftarrow{\partial} \Lambda \overrightarrow{\partial} \right) p_2 &= p_1 \left(\sum_{k,l=1}^{2} \lambda_{kl} \overleftarrow{\partial}_{u_k} \overrightarrow{\partial}_{u_l} \right) p_2 \\ &= \sum_{k,l=1}^{2} \lambda_{kl} \partial_{u_k} p_1 \partial_{u_l} p_2 \qquad (1) \end{aligned}$$

2. Star product

We fix the skew symmetric matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2}$$

For an arbitrary **complex symmetric matrix** $K \in \mathcal{S}_{\boldsymbol{C}}(2)$ we put

$$\Lambda = J + K$$

and we define a product $*_K$ on the space of complex polynomials $p_1(u_1, u_2), p_2(u_1, u_2) \in \mathcal{P}(\boldsymbol{C}^2)$;

$$\begin{aligned} p_1 *_K p_2 &= p_1 \exp\left(\frac{i\hbar}{2}\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right) p_2 \\ &= p_1 p_2 + \frac{i\hbar}{2} p_1 \left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right) p_2 \\ &\quad + \cdots + \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n p_1 \left(\overleftarrow{\partial}\Lambda\overrightarrow{\partial}\right)^n p_2 + \cdots \end{aligned} \tag{3}$$

3. Associativity

We have

Proposition 1 *For an* **arbitrary complex symmetric matrix** $K \in \mathcal{S}_{C}(2)$ *the product* $*_K$ *is associtaive on the space of all complex polynomials* $\mathcal{P}(\boldsymbol{C}^2)$*.*

4. Isomorphic to the Weyl algebra

CCR

For an artibrary $K \in \mathcal{S}_{C}(2)$, the product $*_K$ satisfies the canonical commutation relations

$$[u_k, u_l]_{*_K} = u_k *_K u_l - u_l *_K u_k = i\hbar\delta_{kl}, \quad k,l = 1,2. \tag{4}$$

and hence it follows that all algebras $(\mathcal{P}(\boldsymbol{C}^2), *_K)$ are isomorphic to the Weyl algebra W_2 of two generators u_1, u_2.

Intertwiners

The algebra isomorphis (intertwiners)

$$I_{K_1}^{K_2} : (\mathcal{P}(\boldsymbol{C}^2), *_{K_1}) \to (\mathcal{P}(\boldsymbol{C}^2), *_{K_2}) \tag{5}$$

are explicitly given by

$$I_{K_1}^{K_2}(p) = \exp\left(\frac{i\hbar}{4}(K_2 - K_1)\partial^2\right) p \tag{6}$$

where

$$(K_2 - K_1)\partial^2 = \sum_{kl=1}^{2} (K_2 - K_1)_{kl} \partial_{u_k} \partial_{u_l} \tag{7}$$

We have the relations

Proposition 2 *(i)* $I_{K_2}^{K_3} I_{K_1}^{K_2} = I_{K_1}^{K_3}$

(ii) $(I_{K_1}^{K_2})^{-1} = I_{K_2}^{K_1}$

Infinitesimal intertwiner

By differentiating the intertwiner with respect to K, we obtain the infinitesimal intertwiner at K

$$\nabla_\kappa(p) = \frac{d}{dt} I_K^{K+t\kappa}(p)_{|t=0} = \frac{i\hbar}{4}\kappa\partial^2 p \tag{8}$$

Then the infinitesimal intertwiner satisfies

$$\nabla_\kappa(p_1 *_K p_2) = \nabla_\kappa(p_1) *_K p_2 + p_1 *_K \nabla_\kappa(p_2) \tag{9}$$

for any $p_1(u_1, u_2), p_2(u_1, u_2) \in \mathcal{P}(\boldsymbol{C}^2)$.

§1.4. q-number polynomials

In the star product algebras $\left\{(\mathcal{P}(\boldsymbol{C}^2), *_K)\right\}_{K\in\mathcal{S}_{\boldsymbol{C}}(2)}$, the algebras $(\mathcal{P}(\boldsymbol{C}^2), *_{K_1})$ and $(\mathcal{P}(\boldsymbol{C}^2), *_{K_2})$ are mutually isomorphic by the intertwiner $I_{K_1}^{K_2}$ and the elements $p_1 \in (\mathcal{P}(\boldsymbol{C}^2), *_{K_1})$ and $p_2 \in (\mathcal{P}(\boldsymbol{C}^2), *_{K_2})$ are identified when

$$p_2 = I_{K_1}^{K_2}(p_1) \tag{10}$$

In order to give a geometric picture to the family of star product algebras $\left\{(\mathcal{P}(\boldsymbol{C}^2), *_K)\right\}_{K\in\mathcal{S}_{\boldsymbol{C}}(2)}$, we introduce an algebra bundle over $\mathcal{S}_{\boldsymbol{C}}(2)$ whose fibres consisit of the Weyl algebra in the following way.

1. Algebra bundle

We consider the the trivial bundle

$$\pi : \mathbb{P} = \mathcal{P}(\boldsymbol{C}^2) \times \mathcal{S}_{\boldsymbol{C}}(2) \to \mathcal{S}_{\boldsymbol{C}}(2) \tag{11}$$

whose fibre over $K \in \mathcal{S}_{\boldsymbol{C}}(2)$ consists of the star product algebra

$$\pi^{-1}(K) = (\mathcal{P}(\boldsymbol{C}^2), *_K) \tag{12}$$

2. Flat connection and parallel translation

On this bundle, we regard the infinitesimal intertwiner ∇ as a flat connection and the intertwiner $I_{K_1}^{K_2}$ as its parallel translation.

We consider a section $\tilde{p} \in \Gamma(\mathbb{P})$ of this bundle satisfying

$$\tilde{p}(K_2) = I_{K_1}^{K_2}(\tilde{p}(K_1)) \tag{13}$$

This means that $\tilde{p}$ is parallel

$$\nabla_\kappa \tilde{p}(K) = 0 \tag{14}$$

3. q-number polynomial

We denote by $\mathcal{P}(\mathbb{P})$ the space of all parallel sections, and call an element $\tilde{p} \in \mathcal{P}(\mathbb{P})$ q-**number polynomial**.

Due to the identies $I_{K_2}^{K_3} I_{K_1}^{K_2} = I_{K_1}^{K_3}$ and $(I_{K_1}^{K_2})^{-1} = I_{K_2}^{K_1}$ the intertwiners naturally induce the product $*$ on $\mathcal{P}(\mathbb{P})$. Then the algebra $(\mathcal{P}(\mathbb{P}), *)$ is regarded as a geometric realization of the Weyl algebra.

§2. q-number functions

We introduce a locally convex topology into the family of star product algebras by means of a system of semi-norms.

We take the completion of the algebras and then we can consider star exponentials.

1. Topology

We introduce a topology into $\mathcal{P}(\boldsymbol{C}^2)$ by a system of semi-norms in the following way.

Let ρ be a positive number. For every $s > 0$ we define a semi-norm for polynomials by

$$|p|_s = \sup_{u \in \boldsymbol{C}^2} |p(u_1, u_2)| \exp(-s|u|^\rho) \tag{15}$$

Then the system of semi-norms $\{|\cdot|_s\}_{s>0}$ defines a locally convex topology $\mathcal{T}_\rho$ on $\mathcal{P}(\boldsymbol{C}^2)$.

2. Fréchet space $\mathcal{E}_\rho(\boldsymbol{C}^2)$

Definition We take the completion of $\mathcal{P}(\boldsymbol{C}^2)$ with respect to the topology $\mathcal{T}_\rho$, we obtain a Fréchet space $\mathcal{E}_\rho(\boldsymbol{C}^2)$.

Proposition 3 *For a positive number ρ, the Fréchet space $\mathcal{E}_\rho$ consists of entire functions on the complex plane $\boldsymbol{C}^2$ with finite semi-norm for every $s > 0$, namely,*

$$\mathcal{E}_\rho(\boldsymbol{C}^2) = \Big\{ f \in \mathcal{H}(\boldsymbol{C}^2) \mid |f|_s < +\infty,\ \forall s > 0 \Big\} \qquad (16)$$

Continuity for the case $0 < \rho \leq 2$

As to the continuitiy of star products and intertwiners, the space $\mathcal{E}_\rho(\boldsymbol{C}^2)$, $0 < \rho \leq 2$ is very good, namely, we have the following

Theorem 1 *On $\mathcal{E}_\rho(\boldsymbol{C}^2)$, $0 < \rho \leq 2$ every product $*_K$ is continuous, and every intertwiner $I_{K_1}^{K_2} : (\mathcal{E}_\rho(\boldsymbol{C}^2), *_{K_1}) \to (\mathcal{E}_\rho(\boldsymbol{C}^2), *_{K_2})$ is continuous.*

Continuity as a bimodule for the case $\rho > 2$

As to the spaces $\mathcal{E}_\rho(\boldsymbol{C}^2)$ for $\rho > 2$, the situation is no so good, but still we have the following.

Theorem 2 *For $\rho > 2$, take $\rho' > 0$ such that*

$$\frac{1}{\rho'} + \frac{1}{\rho} = 1$$

then every star product $*_K$ defines a continuous bilinear product

$$*_K : \mathcal{E}_\rho(\boldsymbol{C}^2) \times \mathcal{E}_{\rho'}(\boldsymbol{C}^2) \to \mathcal{E}_\rho(\boldsymbol{C}^2),\ \mathcal{E}_{\rho'}(\boldsymbol{C}^2) \times \mathcal{E}_\rho(\boldsymbol{C}^2) \to \mathcal{E}_\rho(\boldsymbol{C}^2)$$

This means that $(\mathcal{E}_\rho(\boldsymbol{C}^2),\ *_K)$ is a continuous $\mathcal{E}_{\rho'}(\boldsymbol{C}^2)$-bimodule.

3. q-number functions

The case $0 < \rho \leq 2$

Due to the previous theorem, we can introduce a similar concept as q-number polynomials into the Fréchet spaces.

Namely, the star product $*_K$ is well defined on $\mathcal{E}_\rho(\boldsymbol{C}^2)$ and then we consider the trivial bundle

$$\pi : \mathbb{E}_\rho = \mathcal{E}_\rho(\boldsymbol{C}^2) \times \mathcal{S}_{\boldsymbol{C}}(2) \to \mathcal{S}_{\boldsymbol{C}}(2) \tag{17}$$

with fibre over the point $K \in \mathcal{S}_{\boldsymbol{C}}(2)$ consists of

$$\pi^{-1}(K) = (\mathcal{E}_\rho(\boldsymbol{C}^2), *_K) \tag{18}$$

The intertwiners $I_{K_1}^{K_2}$ are well defined for any $K_1, K_2 \in \mathcal{S}_{\boldsymbol{C}}(2)$ and then the bundle $\mathbb{E}_\rho$ has a flat connection ∇ and the parallel translation is the intertwiner.

The space of flat sections of the bundle denoted by $\mathcal{F}_\rho$ naturally has the product $*$ and can be regarded as a completion of the Weyl algebra W_2.

4. Remark to the case $\rho > 2$

For the case $\rho > 2$, at present it is not clear whether the intertwiners are well-defined and whether the product $*_K$ are well defined. However the flat connection ∇ is still well defined on $\pi : \mathbb{E}_\rho = \mathcal{E}_\rho(\boldsymbol{C}^2) \times \mathcal{S}_{\boldsymbol{C}}(2) \to \mathcal{S}_{\boldsymbol{C}}(2)$, so we can define a space $\mathcal{F}_\rho$ of all parallel sections of this bundle even for $\rho > 2$.

For $\rho > 2$, we are trying to extend the product $*_K$ and also the intertwiners $I_{K_1}^{K_2}$ by means of some regularizations, for example, Borel-Laplace transform, or finite part regularization. I hope to construct such a concept in near future.

5. Star expoenential

The space of q-number functions $\mathcal{F}_\rho$ is a complete topological algebra for $0 < \rho \leq 2$ (even $\rho > 2$ for future under some regularization). We can consider exponential element

$$\exp_* t\left(\frac{H}{i\hbar}\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{\frac{H}{i\hbar} * \cdots * \frac{H}{i\hbar}}_{n} \tag{19}$$

in this algebra.

For a q-number polynomial $H \in \mathcal{P}(\mathbb{P})$, we define the star exponenial $\exp_* t(H/i\hbar)$ by the differential equation

$$\frac{d}{dt}\exp_* t\left(\frac{H}{i\hbar}\right) = \frac{H}{i\hbar} * \exp_* t\left(\frac{H}{i\hbar}\right),\ \exp_* t\left(\frac{H}{i\hbar}\right)|_{t=0} = 1 \tag{20}$$

6. Remark

We set the Fréchet space

$$\mathcal{E}_{\rho+}(\boldsymbol{C}^2) = \cap_{\lambda>\rho}\mathcal{E}_\lambda(\boldsymbol{C}^2) \tag{21}$$

and we donote by $\mathfrak{E}_{\rho+}$ the correponding bundle and by $\mathcal{F}_{\rho+}$ the space of the flat sections of this bundle.

When $H \in \mathcal{P}(\mathbb{P})$ is a linear element, then $\exp_* t\left(\frac{H}{i\hbar}\right)$ belongs to the good space $\mathcal{F}_{1+}(\subset \mathcal{F}_2)$.

On the other hand, the most interesting case is given by quadratic form $H \in \mathcal{P}(\mathbb{P})$. In this case we can solve the differential equation explicitly, but the star exponential belongs to the space $\mathcal{F}_{2+}$, which is difficult to treat at present.

Although general theory related to the space $\mathcal{F}_{2+}$ is not yet established, we illustrate the concrete example of the star expoenential of the quadratic forms and its application.

§3. Star exponential of $\exp_* t(\frac{2u*v}{i\hbar})$

We very the parameter $K \in \mathcal{S}_C(2)$ and at some K we can obtain interesting identities in the algebra of $*_K$ product.

In this section, we construct a Clifford algebra by means of the star exponential $\exp_* t(\frac{2u*v}{i\hbar})$ for certain K. In what follows, we decsribe a rough sketch of construction.

First we consider a generic point in $\mathcal{S}_C(2)$

$$K = \begin{pmatrix} \tau' & \kappa \\ \kappa & \tau \end{pmatrix} \in \mathcal{S}_C(2)$$

In the star product $*_K$ algebra, we write the generator $u = u_1, v = u_2$ satisfying

$$[u, v]_{*_K} = -i\hbar$$

Star exponential

Then the star exponential of $H = 2u * v$ is explicitly given at a general point K as

$$\exp_{*_K} t\left(\frac{2u*v}{i\hbar}\right)$$

$$= \frac{2e^{-t}}{\sqrt{D}} \exp\left[\frac{e^t - e^{-t}}{i\hbar D}\Big((e^t - e^{-t})\tau u^2 + 2\triangle uv + (e^t - e^{-t})\tau' v^2\Big)\right]$$

where

$$D = \triangle^2 - (e^t - e^{-t})\tau'\tau,\ \triangle = e^t + e^{-t} - \kappa(e^t - e^{-t}) \quad (22)$$

Vacuum

In the sequel, **we assum** $\tau' = 0$, that is, we take a point

$$K = \begin{pmatrix} 0 & \kappa \\ \kappa & \tau \end{pmatrix} \tag{23}$$

We have a limit

$$\lim_{t \to -\infty} \varpi_{00} = \exp_{*_K} t \left(\frac{2u * v}{i\hbar} \right) = \tfrac{2}{1+\kappa} \exp \left(-\tfrac{1}{i\hbar(1+\kappa)} (2uv - \tfrac{\tau}{1+\kappa} u^2) \right) \tag{24}$$

which we call a **vacuum**.

Then we have

Lemma 1 *i)* $\varpi_{00} *_K \varpi_{00} = \varpi_{00}$

ii) $v *_K \varpi_{00} = \varpi_{00} *_K u = 0$.

Putting $t = \pi i$, we have the identity

$$\exp_{*_K} \pi i \left(\frac{2u * v}{i\hbar} \right) = 1 \tag{25}$$

Using

$$v *_K (u *_K v) = (v *_K u) *_K v = (u *_K v + i\hbar) *_K u$$

we see that the star exponential satisfies

$$\begin{aligned} v *_K \exp_{*_K} t \left(\frac{2u * v}{i\hbar} \right) &= \exp_{*_K} t \left(\frac{2v * u}{i\hbar} \right) *_K v \\ &= \exp_{*_K} t \left(\frac{2u * v + 2i\hbar}{i\hbar} \right) *_K v \\ &= e^{2t} \exp_{*_K} t \left(\frac{2u * v}{i\hbar} \right) *_K v \end{aligned}$$

Then the integral $\frac{1}{2}\int_{-\infty}^{0} \exp_{*_K} t(\frac{2v*u}{i\hbar})dt$ converges and then we define

$$\frac{1}{2}\int_{-\infty}^{0} \exp_{*_K} t\left(\frac{2v*u}{i\hbar}\right)dt = (v*_K u)_{+}^{-1} \tag{26}$$

and

$$\overset{\circ}{v} = u *_K (v *_K u)_{+}^{-1}. \tag{27}$$

Then we have

Lemma 2 *The element* $\overset{\circ}{v}$ *is the right inverse of* v *satisfying*

$$v *_K \overset{\circ}{v} = 1, \quad \overset{\circ}{v} *_K v = 1 - \varpi_{00}$$

Now we fix an integer l. By putting

$$t = t_l = \frac{\pi i}{2^l}$$

we obtain 2^l roots of the unity

$$\Omega_l = \exp_{*_K} \frac{\pi i}{2^l}\left(\frac{2u{*}v}{i\hbar}\right), \;\; \varpi_l = \exp 2\left(\frac{\pi i}{2^l}\right) \qquad (28)$$

such that

$$\Omega_{l *_K}^{2^l} = \underbrace{\Omega_l *_K \cdots *_K \Omega_l}_{2^l} = 1, \;\; \varpi_l^{2^l} = 1$$

Then we have

Lemma 3 *These satisfy*

$$\Omega_{l *_K}^{k} *_K u_{*_K}^{m} *_K \varpi_{00} *_K v_{*_K}^{m} = \varpi_l^{km} u_{*_K}^{m} *_K \varpi_{00} *_K v_{*_K}^{m}$$

Now we take appropriate complex numbers $a_0, a_1, \cdots, a_{2^l-1}$ so that an element

$$E = \sum_{k=0}^{2^l-1} a_k \Omega_{l*_K}^k$$

satisfies the identies

$$E *_K u_{*_K}^m *_K \varpi_{00} *_K v_{*_K}^m = \begin{cases} *_K u_{*_K}^m *_K \varpi_{00} *_K v_{*_K}^m & \cdots 0 \le m \le 2^{l-1} - 1 \\ 0 & \cdots 2^{l-1} \le m \le 2^l - 1 \end{cases}$$

We see this is equivalent to

$$\sum_{k=0}^{2^l-1} a_k \varpi_l^{km} = \begin{cases} 1 \cdots 0 \le m \le 2^{l-1} - 1 \\ 0 \cdots 2^{l-1} \le m \le 2^l - 1 \end{cases}$$

The complex numbers $a_0, a_1, \cdots, a_{2^l-1}$ are uniquely determined by these equations.

Then we have

Lemma 4 *The element* E *satisfies*

$$E *_K E = 1$$

and the element $F = 1 - E$ *satisfies*

$$F *_K F = 1, \; E *_K F = F *_K E = 0$$

Further we have

Lemma 5

$$E *_K (v)^{2^{l-1}}_{*_K} = (v)^{2^{l-1}}_{*_K} *_K F, \quad (\overset{\circ}{v})^{2^{l-1}}_{*_K} *_K F = E * (\overset{\circ}{v})^{2^{l-1}}_{*_K}$$

where $(v)^{2^{l-1}}_{*_K} = \underbrace{v *_K \cdots *_K v}_{2^{l-1}}$ *and* $(\overset{\circ}{v})^{2^{l-1}}_{*_K} = \underbrace{\overset{\circ}{v} *_K \cdots *_K \overset{\circ}{v}}_{2^{l-1}}$

Now we set

$$\xi = E *_K (v)^{2^{l-1}}_{*_K}, \ \eta = (\overset{\circ}{v})^{2^{l-1}}_{*_K} *_K F$$

Then we have

Theorem 3 *The elements* ξ *and* η *of the* $*_K$ *product algebra satisfies the identities*

$$\xi *_K \xi = \eta *_K \eta = 0$$
$$\xi *_K \eta + \xi *_K \eta = 1$$

Thank you!