Lorentz surfaces in pseudo-Riemannian space forms of neutral signature with horizontal reflector lifts

Kazuyuki HASEGAWA (Kanazawa University, Japan)

Joint work with Dr. K. Miura (Ochanomizu Univ. Japan)

0. Introduction.

$(\widetilde{M}, \widetilde{g})$: oriented n-dimensional pseudo-Riemannian manifold

(M, g) : oriented pseudo-Riemannian submanifold

$f : M \to \widetilde{M}$: isometric immersion

H : mean curvature vector field

Problem. : Classify $f : M \to \widetilde{M}$ with $H = 0$

• **Riemannian case** : When $\widetilde{M} = S^n(c)$, $M = S^2(K)$ of constant curvatures $c > 0$ and $K > 0$, a full immersion f with $H = 0$ is congruent to the standard immersion (E. Calabi, J. Diff. Geom., 1967). The **twistor space** of $\widetilde{M} = S^n(c)$ and the **twistor lift** play an important role.

• In this talk, we consider the problem above in the case of $\widetilde{M} = Q^n_s(c) =$ pseudo-Riemannian space form of constant curvature c and index s, and M =Lorentz surfaces, in particular, $n = 2m$ and $s = m$.

- **This talk is consists of**

1. **Examples of Lorentz surfaces in pseudo-Riemannian space forms with $H = 0$.**
2. **Reflector spaces and reflector lifts.**
3. **A Rigidity theorem.**
4. **Applications.**

1. Examples of Lorentz surfaces with $H = 0$.

• **In "K. Miura, Tsukuba J. math., 2007", pseudo-Riemannian submanifolds of constant curvature in pseudo-Riemannian space forms with $H = 0$ are constructed from the Riemannian standard immersion using " Wick rotations ".**

$\mathrm{K}[x] := \mathrm{K}[x_1, \ldots, x_{n+1}]$: polynomial algebra in $n + 1$ variables

$x_1, \ldots, x_{n+1}$ over K (K = R or C).

$\mathrm{K}_d[x] \subset \mathrm{K}[x]$: space of d-homogeneous polynomials

R^{n+1}_t : pseudo-Euclidean space of dim=$n + 1$ and index=t

$$\triangle_{\mathrm{R}_t^{n+1}} := -\sum_{i=1}^{n+1} \varepsilon_i \frac{\partial}{\partial x_i^2}, \text{ where } \varepsilon_i = \begin{cases} -1 & (1 \leq i \leq t) \\ 1 & (t+1 \leq i \leq n+1) \end{cases}$$

$$\mathcal{H}(\mathrm{R}_t^{n+1}) := \mathrm{Ker}\triangle_{\mathrm{R}_t^{n+1}}$$

$$\mathcal{H}_d(\mathrm{R}_t^{n+1}) := \mathcal{H}(\mathrm{R}_t^{n+1}) \cap \mathrm{K}_d[x]$$

We define $\rho_t : \mathrm{C}[x] \to \mathrm{C}[x]$ by

$$\rho_t(x_i) = \begin{cases} \sqrt{-1}x_i & (1 \leq i \leq t) \\ x_i & (t+1 \leq i \leq n+1) \end{cases}$$

- ρ_t is called "Wick rotation".

Set $\sigma_t := \rho_t \circ \rho_t$ and $P_t^{\pm} := \mathrm{Ker}(\sigma_t|_{\mathrm{R}[x]} \mp id_{\mathrm{R}[x]})$.

We define $\mathcal{H}^{\pm}_{d,t}(\mathrm{R}_0^{n+1}) := P_t^{\pm} \cap \mathcal{H}_d(\mathrm{R}_0^{n+1})$.

- $\mathcal{H}_d(\mathrm{R}_0^{n+1}) = \mathcal{H}_{d,t}^-(\mathrm{R}_0^{n+1}) \oplus \mathcal{H}_{d,t}^+(\mathrm{R}_0^{n+1})$

$\{u_i\}_i$:orthonormal basis $\mathcal{H}_d(\mathrm{R}_0^{n+1})$ s.t.

$$u_1, \ldots, u_l \in \mathcal{H}_{d,t}^-(\mathrm{R}_0^{n+1}),$$

$$u_{l+1}, \ldots u_{m+1} \in \mathcal{H}_{d,t}^+(\mathrm{R}_0^{n+1}),$$

$$\sum_{i=1}^{m+1} (u_i)^2 = (x_1^2 + \cdots + x_{n+1}^2)^2,$$

where $l = l(d,t) = \dim \mathcal{H}_{d,t}^-(\mathrm{R}_0^{n+1})$ and $m+1 = m(d,t)+1 = \dim \mathcal{H}_d(\mathrm{R}_0^{n+1})$.

Hereafter, we assume $n = 2$ and $t = 1$ (for simplicity). Then we see that $m = 2d + 1$ and $l = d$. We define

$$U_i := \mathrm{Im}\rho_t(u_i) \ \ (i = 1, \ldots, d)$$

and

$$U_i := \mathrm{Re}\rho_t(u_i) \ \ (i = d + 1, \ldots, 2d + 1)$$

Using these function, we define an immersion from $\phi : Q_1^2(1) \to Q_d^{2d}(\frac{d(d+1)}{2})$ by $\phi = (U_1, \ldots, U_{2d+1})$. Note that ϕ satisfies $H = 0$.

For example, when $d = 2$ ($n = 1$, $t = 1$), $\phi : Q_1^2(1) \to Q_2^4(3)$ is given by

$$U_1 = xy, U_2 = zx$$

$$U_3 = yz, U_4 = \frac{\sqrt{3}}{6}(\underline{+2x^2} + y^2 + z^2), U_5 = \frac{1}{2}(y^2 - z^2).$$

• **Composing homotheties and anti-isometries of $Q_1^2(1)$ and $Q_d^{2d}(\frac{d(d+1)}{2})$, we can obtain immersions from $Q_1^2(2c/d(d+1))$ to $Q_d^{2d}(c)$ with $H = 0$ ($c \neq 0$). We denote this immersion by $\underline{\phi_{d,c}}$.**

2. Reflector spaces and reflector lifts.

(See "G. Jensen and M. Rigoli, Matematiche (Catania) 45 (1990), 407-443. ")

$(\widetilde{M}, \widetilde{g})$: oriented $2m$-dim. pseudo-Riemannian manifold of neutral signature (index$=\frac{1}{2}\dim \widetilde{M}$)

$J_x \in \mathrm{End}(T_x\widetilde{M})$ s.t.

- $J_x^2 = I$,
- $(J_x)^*\widetilde{g}_x = -\widetilde{g}_x$,
- $\dim \mathrm{Ker}(J_x - I) = \dim \mathrm{Ker}(J_x + I) = m$,
- J_x preserves the orientation.

Z_x : the set of such $J_x \in \mathrm{End}(T_x\widetilde{M})$

$$\mathcal{Z}(\widetilde{M}) := \bigcup_{x \in \widetilde{M}} Z_x$$

(the bundle whose fibers are consist of para-complex structures)

• $\mathcal{Z}(\widetilde{M})$ is called the reflector space of $\widetilde{M}$, which is one of corresponding objects to the twistor space in Riemannian geometry.

(M, g) : oriented Lorentz surface with para-complex structure

$$J \in \mathcal{Z}(M).$$

$f : M \to \widetilde{M}$: isometric immersion

Def. : The map $\widetilde{J} : M \to \mathcal{Z}(\widetilde{M})$ satisfying $\widetilde{J}(x)|_{T_xM} = f_* \circ J_x$ is reflector lift of f.

Def. : The reflector lift is called horizontal if $\widetilde{\nabla}\widetilde{J} = 0$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{M}$.

- $\widetilde{\nabla}\widetilde{J} = 0 \Rightarrow H = 0$

- Surfaces with horizontal reflector lifts are corresponding to superminimal surfaces in Riemannian cases.

Prop. The immersion $\phi_{m,c} : Q_1^2(2c/m(m+1)) \to Q_m^{2m}(c)$ in §1 admits horizontal reflector lift.

3. A Rigidity theorem.

$(\widetilde{M}, \widetilde{g})$: oriented n-dim pseudo-Riemannian manifold

(M, g) : oriented pseudo-Riemannian submanifold

$f : M \to \widetilde{M}$: isometric immersion

$\widetilde{\nabla}$: Levi-Civita connection of $\widetilde{M}$.

We define $\widetilde{\nabla} X_1 := X_1$, $\widetilde{\nabla}(X_1, X_2) = \widetilde{\nabla}_{X_1} X_2$ and inductively for $k \geq 3$

$$\widetilde{\nabla}(X_1, \ldots, X_k) = \widetilde{\nabla}_{X_1} \widetilde{\nabla}(X_2, \ldots, X_k),$$

where $X_i \in \Gamma(TM)$.

We define

$$\mathrm{Osc}^k_x(f) := \{(\widetilde{\nabla}(X_1,\ldots,X_k))_x \mid X_i \in \Gamma(TM), 1 \le i \le k\}$$

and

$$\mathrm{Osc}^k(f) := \bigcup_{x\in M} \mathrm{Osc}^k_x(f).$$

- $TM = \mathrm{Osc}^1(f) \subset \mathrm{Osc}^2(f) \subset \cdots \subset \mathrm{Osc}^k(f) \subset \cdots \subset f^{\#}(T\widetilde{M})$.
- **There exists the maximum number** m **such that** $\mathrm{Osc}^1(f),\ldots,\mathrm{Osc}^m(f)$ **are subbundles of** $f^{\#}(T\widetilde{M})$ **and the induced metrics of all subbundles** $\mathrm{Osc}^1(f),\ldots,\mathrm{Osc}^m(f)$ **are nondegenerate. Then** f **is called**

nondegenerately nicely curved

up to m**.**

Thm. : Let f, $\bar{f} : M \to Q^{2m}_m(c)$ be an isometric immersions from a connected Lorentz surface M with horizontal reflector lifts. If both immersions f and $\bar{f}$ are nondegenerately nicely curved up to m, then there exist an isometry Φ of $Q^{2m}_m(c)$ such that $\bar{f} = \Phi \circ f$.

Remark. The immersion $\phi_{m,c} : Q^2_1(2c/m(m+1)) \to Q^{2m}_m(c)$ in §1 is nondegenerately nicely curved up to m.

Cor. : Let $f : M \to Q^{2m}_m(c)$ be an isometric immersion from a connected Lorentz surface M with horizontal reflector lift. If f is nondegenerately nicely curved up to m and the Gaussian curvature K is constant, then $K = 2c/m(m+1)$ and f is locally congruent to $\phi_{m,c}$.

4. Applications.

• When f is called nondegenerately nicely curved up to m, we can define the $(k-1)$-th normal space N^{k-1} which is defined the orthogonal complement subspace of $\mathrm{Osc}^{k-1}(f)$ in $\mathrm{Osc}^{k}(f)$.

Def. : When f is nondegenerately nicely curved up to m, we define $(k+1)$-th fundamental form α^{k+1} as follows :

$$\alpha^{k+1}(X_1, \ldots, X_{k+1}) := (\widetilde{\nabla}(X_1, \ldots, X_{k+1})^{N^k},$$

where $X_i \in \Gamma(TM)$ $(1 \leq i \leq k+1)$ and $1 \leq k \leq m-1$

• **If $\widetilde{M} = Q^n_s(c)$, α^{k+1} is symmetric.**

V, W : **vector spaces with inner products** $\langle\ ,\ \rangle$

$\beta : V \times V \cdots \times V \to W$: **symmetric** k**-multilinear map to** W.

For the sake of simplification, we set

$$\beta(X^k) := \beta(\underbrace{X, \ldots, X}_{k})$$

for $X \in V$.

Def. : We say that $\boldsymbol{\beta}$ is **spacelike (resp. timelike) isotropic** if $\langle \boldsymbol{\beta}(\boldsymbol{u}^k), \boldsymbol{\beta}(\boldsymbol{u}^k)\rangle$ is independent of the choice of all spacelike (resp. timelike) unit vectors $\boldsymbol{u}$. The number $\langle \boldsymbol{\beta}(\boldsymbol{u}^k), \boldsymbol{\beta}(\boldsymbol{u}^k)\rangle$ is called spacelike (resp. timelike) **isotropic constant** of $\boldsymbol{\beta}$.

- $\boldsymbol{\beta}$ is spacelike isotropic with spacelike isotropic constant $\boldsymbol{\lambda} \iff \boldsymbol{\beta}$ is timelike isotropic with timelike isotropic constant $(-1)^k\boldsymbol{\lambda}$.

Def. : We say that the k-th fundamental form α^k is **spacelike (resp. timelike) isotropic** if α_p^k is spacelike (reps. timelike) isotropic at each point $p \in M$. The function $\lambda_k : M \to \mathrm{R}$ defined by $\lambda_k(p) :=$ spacelike isotropic constant of α_p^k is called the spacelike (resp. timelike) isotoropic function. If the spacelike (resp. timelike) isotoropic function λ_k is constant, then k-th fundamental form α^k is called **constant spacelike (resp. timelike) isotropic** .

Prop. : **Let $f : M \to Q^n_s(c)$ be an isometric immersion with $H = 0$. If the higher fundamental forms α^k are spacelike isotropic for $2 \leq k \leq m$ and their spacelike isotropic functions are everywhere nonzero on M. Then we have $n \geq 2m$ and $s \geq m$. In particular, if $n = 2m$, then $s = m$ and f admits horizontal reflector lift.**

Cor. : If $f : M \to Q_s^{2m}(c)$ be an isometric immersion with $H = 0$ such that α^k are spacelike isotropic for $2 \leq k \leq m$ whose isotropic functions are everywhere nonzero and M has a constant Gaussian curvature K, then we have $s = m$, $K = 2c/m(m+1)$ and f is locally congruent to the immersion $\phi_{m,c}$

Consider the following condition :

(†) An isometric immersion f maps each <u>null</u> geodesic of M into a totally isotropic and totally geodesic submanifold L of $\widetilde{M}$.

- **L is totally isotropic : $\iff$ $\widetilde{g}|_L = 0$.**
- **L is totally geodesic : $\iff$ $\widetilde{\nabla}_X Y \in \Gamma(TL)$ for all $X, Y \in \Gamma(TL)$.**

<u>Remark.</u> The immersion $\phi_{m,c}$ satisfies the condition (†).

Cor. : **Let $f : M \to Q^{2m}_s(c)$ be an isometric immerison with $H = 0$. If $K = 2c/m(m+1)$ and the condition (†) holds, then $s = m$ and f is locally congruent to the immersion $\phi_{m,c}$.**