Star Product and Its Application to MIC Kepler Problem

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Organization of Talk

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§1. Star Product

Consider the space of complex (resp. real) polynomials $\mathscr{P}(\boldsymbol{C}^{2n})$ (resp. real $\mathscr{P}(\boldsymbol{R}^{2n})$) of $2n$ variables $(\boldsymbol{u}, \boldsymbol{v}) = (u_1, \cdots, u_n, v_1, \cdots, v_n)$.

§1.1. Moyal Product

Biderivation (Poisson bracket) We consider biderivation

$$\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_0 \overrightarrow{\partial_{\boldsymbol{u}}} = \overleftarrow{\partial_{v_1}} \wedge_0 \overrightarrow{\partial_{u_1}} + \cdots + \overleftarrow{\partial_{v_n}} \wedge_0 \overrightarrow{\partial_{u_n}}$$

such that

$$f\left(\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_0 \overrightarrow{\partial_{\boldsymbol{u}}}\right) g = f\left(\overleftarrow{\partial_{v_1}} \wedge_0 \overrightarrow{\partial_{u_1}}\right) g + \cdots + f\left(\overleftarrow{\partial_{v_n}} \wedge_0 \overrightarrow{\partial_{u_n}}\right) g$$

where

$$\begin{aligned} f\left(\overleftarrow{\partial_{v_1}} \wedge_0 \overrightarrow{\partial_{u_1}}\right) g &= f\left(\overleftarrow{\partial_{v_1}}\overrightarrow{\partial_{u_1}} - \overleftarrow{\partial_{u_1}}\overrightarrow{\partial_{v_1}}\right) g \\ &= \partial_{v_1} f \partial_{u_1} g - \partial_{u_1} f \partial_{v_1} g, \ldots \quad \text{etc} \end{aligned}$$

Obvously, the operator is the canonical Poisson bracket of $\boldsymbol{C}^{2n}$ (resp. $\boldsymbol{R}^{2n}$).

Moyal product For polynomials f, $g \in \mathscr{P}(\boldsymbol{C}^{2n})$, the Moyal product $f * g$ is
given by

$$f * g = f \exp\left(\frac{i\hbar}{2}\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_0 \overrightarrow{\partial_{\boldsymbol{u}}}\right) g = f \sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{i\hbar}{2}\right)^N \left(\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_0 \overrightarrow{\partial_{\boldsymbol{u}}}\right)^N g$$
$$= fg + \left(\frac{i\hbar}{2}\right) f \left(\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_0 \overrightarrow{\partial_{\boldsymbol{u}}}\right) g + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 f (\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_0 \overrightarrow{\partial_{\boldsymbol{u}}})^2 g + \cdots$$

The product is associative, and is noncommutative in general.

Deformation Since $\lim_{\hbar \to 0} f * g = fg$ the Moyal product $*$ is considered as a deformation of the usual multiplication fg .

Canonical commutation relations

The generators $(u_1, \cdots, u_n, v_1, \cdots, v_n)$ satisfy the following caconical commutation relations (CCR) :

$$[u_j, u_k]_* = 0\ ,\ [v_j, v_k]_* = 0\ ,\ [u_j, v_k]_* = -i\hbar\delta_{jk} \quad (1 \leq {}^{\forall} j\, , k \leq n)$$

where $\quad [f, g]_* \equiv (f * g - g * f)$ is the commutator with respect to $*$.

Weyl algebra

Then the algebra $(\mathscr{P}(\boldsymbol{C}^{2n}), *)$ is isomorophic to the Weyl algebra, hence $(\mathscr{P}(\boldsymbol{C}^{2n}), *)$ is regarded as an polynomial expression of the Weyl algebra.

§1.2. Star products (Extended Moyal product)

Now we extend the Moyal product by using $2n \times 2n$ matrices.

We consider a $2n \times 2n$ matrix $\Gamma = (\gamma^{kl})$.

We denote the generators by

$$\tilde{u} = (\tilde{u}_1, \cdots, \tilde{u}_n, \tilde{u}_{n+1}, \cdots, \tilde{u}_{2n}) = (u_1, \cdots, u_n, v_1, \cdots, v_n).$$

Biderivation

Similar to the Moyal product, we consider a biderivation

$$\overleftarrow{\partial_{\tilde{\boldsymbol{u}}}} \wedge_{\Gamma} \overrightarrow{\partial_{\tilde{\boldsymbol{u}}}} = \sum_{k,l=1}^{2n} \gamma^{kl}\ \overleftarrow{\partial_{\tilde{u}_k}}\overrightarrow{\partial_{\tilde{u}_l}}$$

Remark when $\Gamma = J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, the biderivation $\overleftarrow{\partial_{\tilde{\boldsymbol{u}}}} \wedge_{\Gamma} \overrightarrow{\partial_{\tilde{\boldsymbol{u}}}}$ is equal to the biderivation $\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_0 \overrightarrow{\partial_{\boldsymbol{u}}}$ of the Moyal product.

Definition of Star product

Replacing the biderivation, we define a star product

$$\begin{aligned} f *_{\Gamma} g &= f \exp\left(\frac{i\hbar}{2}\overleftarrow{\partial_{\tilde{\boldsymbol{u}}}} \wedge_{\Gamma} \overrightarrow{\partial_{\tilde{\boldsymbol{u}}}}\right) g = f \sum_{N=0}^{\infty} \frac{1}{N!}\left(\frac{i\hbar}{2}\right)^N \left(\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_{\Gamma} \overrightarrow{\partial_{\boldsymbol{u}}}\right)^N g \\ &= fg + \left(\frac{i\hbar}{2}\right) f \left(\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_{\Gamma} \overrightarrow{\partial_{\boldsymbol{u}}}\right) g + \frac{1}{2!}\left(\frac{i\hbar}{2}\right)^2 f(\overleftarrow{\partial_{\boldsymbol{v}}} \wedge_{\Gamma} \overrightarrow{\partial_{\boldsymbol{u}}})^2 g + \cdots , \end{aligned}$$

for $f\,,\,g \in \mathscr{P}(\boldsymbol{C}^{2n})$.

We remark that if $\Gamma = J$, the star product $*_\Gamma$ coincides with the Moyal product. Also if we can realize several typical star products by putting Γ special matrics.

It is easy to see the following:

Theorem 1. For any matrix Γ, the star product is associative. Then we have an associative algebra $(\mathscr{P}(\boldsymbol{C}^{2n}), *_\Gamma)$.

Theorem 2. If the Γ and $\tilde{\Gamma}$ have have the same skew symmetric part, then $(\mathscr{P}(\boldsymbol{C}^{2n}), *_\Gamma)$ and $(\mathscr{P}(\boldsymbol{C}^{2n}), *_{\tilde{\Gamma}})$ are isomophic algebra. Thus the algebraic structure of $(\mathscr{P}(\boldsymbol{C}^{2n}), *_\Gamma)$ depends only on the skew symmetric part of Γ.

Theorem 3 If Γ is symmetric, the star product $*_\Gamma$ is commutative. The algebra $(\mathscr{P}(\boldsymbol{C}^{2n}), *_\Gamma)$ is isomorphic to the usual polynomial algebra $\mathscr{P}(\boldsymbol{C}^{2n})$.

§1.3. Example

We consider a simple matrix $\Gamma = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$ where ρ is the $(1,1)$ component. Γ is symmetric, then the star product is commutative, and witten as $p_1 *_\Gamma p_2 = p_1 \exp\left(\frac{i\hbar\rho}{2}\overleftarrow{\partial_{u_1}}\overrightarrow{\partial_{u_1}}\right) p_2$. This is essentially star product of one variable.

Then we put $w = \tilde{u}_1$. and we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$. The product is rewitten as a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2}\overleftarrow{\partial}_w \overrightarrow{\partial}_w} g(w)$$

By a direct calculation we have the star expnential function

$$\exp_{*_\tau} itw = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\tau}{2}\right)^n (iw)^n_{*_\tau} = \exp(itw - (\tau/4)t^2)$$

Theta functions Hence for τ with positive real part $\Re\tau > 0$, the star exponential $\exp_{*_\tau} niw = \exp(niw - (\tau/4)n^2)$ is rapidly decreasing with respect to integer n and then we can consider summations.

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2niw = \sum_{n=-\infty}^{\infty} \exp\left(2niw - \tau\, n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niw},$$

where $q = e^{-\tau}$. This is Jacobi's theta function $\theta_3(w,\tau)$ Similarly other theta functions can be expressed by means of the star exponential functions.

Using the expression

$$\theta_3(w,\tau) = \sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2niw$$

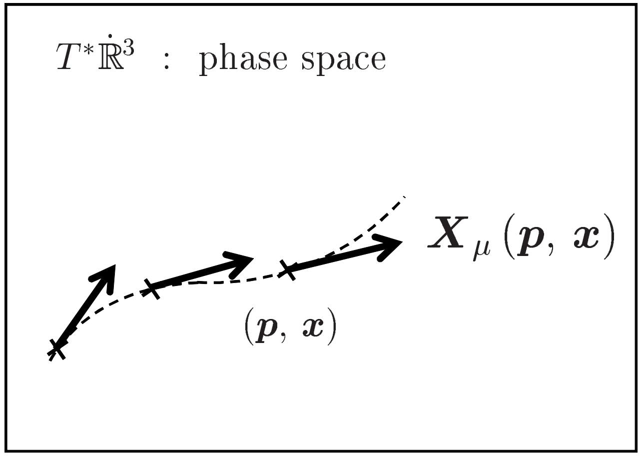
we can show several basic identities of theta functions; e.g., pseudo periodicity, imaginary transoformation, etc. See for example, Omor-Maeda-Miyazaki-Yoshioka [7], Iida-Yoshioka [8].

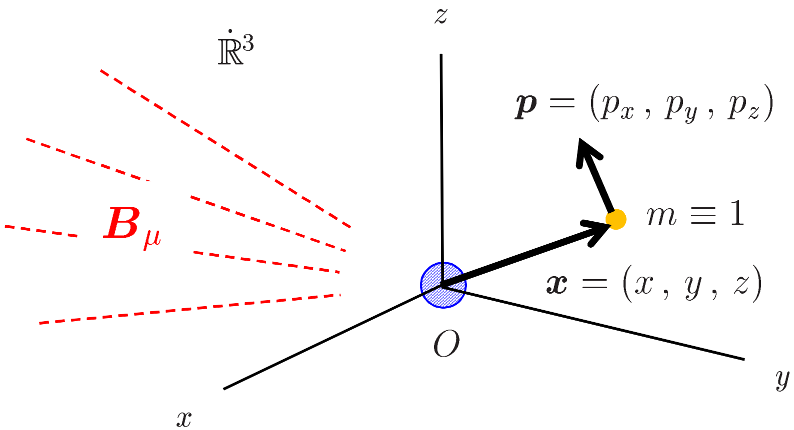
§2 MIC-Kepler Problem

MIC-Kepler problem is introduced by McIntosh and Cisneros, 1970.

The Hamiltonian formalism using the reduction method is investigated for classical mechanical system in T. Iwai & Y. Uwano (1986) [4]

The quantized system is also investigated by T. Iwai & Y. Uwano (1988) [5].



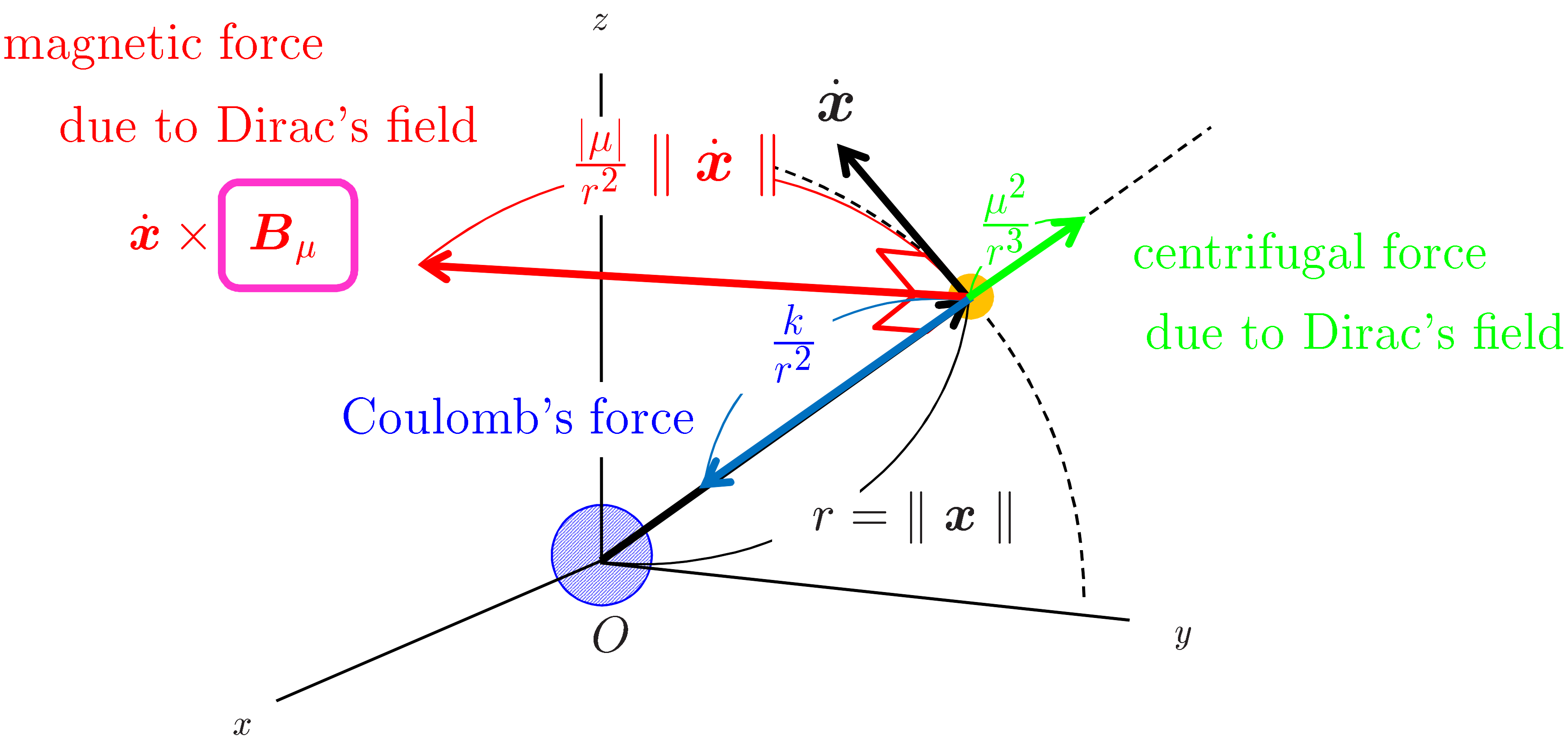


The equation of motion

$\Leftrightarrow$ vector field $\boldsymbol{X}_\mu = (\dot{\boldsymbol{p}} , \dot{\boldsymbol{x}})$

$$\begin{aligned} \dot{\boldsymbol{p}} &= \dot{\boldsymbol{x}} \times \left(-\frac{\mu}{r^3}\boldsymbol{x}\right) - grad\left(\frac{\mu^2}{2r^2} - \frac{k}{r}\right) \\ &= \dot{\boldsymbol{x}} \times \left(-\frac{\mu}{r^3}\boldsymbol{x}\right) + \left(\frac{\mu^2}{r^3} - \frac{k}{r^2}\right)\frac{\boldsymbol{x}}{r} \\ \dot{\boldsymbol{x}} &= \boldsymbol{p} \end{aligned}$$

where
$$\begin{cases} \boldsymbol{x}(t) = (x(t), y(t), z(t)) \neq \boldsymbol{O} \\ \boldsymbol{p}(t) = (p_x(t), p_y(t), p_z(t)) \\ r = \| \boldsymbol{x} \| = \sqrt{x^2 + y^2 + z^2} \\ \dot{\boldsymbol{x}} = \frac{d\boldsymbol{x}(t)}{dt} , \dot{\boldsymbol{p}} = \frac{d\boldsymbol{p}(t)}{dt} \\ k > 0 \quad \text{constant of Coulomb's potential} \\ \mu \in \mathbb{R} \quad \text{constant of magnetic field} \end{cases}$$



Dirac's monopole field of strength $-\mu$

$$\boldsymbol{B}_\mu = \frac{-\mu}{r^3}\boldsymbol{x} \qquad \Rightarrow \quad \| \boldsymbol{B}_\mu \| = \frac{|\mu|}{r^2}$$

Theorem 3.1 (T. Iwai & Y. Uwano 1986)

The MIC (McIntosh and Cisneros, 1970) - Kepler problem is the Hamiltonian system ($T^*\dot{\mathbb{R}}^3$, σ_μ , H_μ) s.t.

$$\begin{cases} H_\mu(\boldsymbol{p}, \boldsymbol{x}) = \frac{1}{2} \| \boldsymbol{p} \|^2 + \frac{\mu^2}{2r^2} - \frac{k}{r} \\ \\ \sigma_\mu = dp_x \wedge dx + dp_y \wedge dy + dp_z \wedge dz + \Omega_\mu \end{cases}$$

$$\Longrightarrow \; -dH_\mu = \sigma_\mu \lrcorner \boldsymbol{X}_\mu$$

where Ω_μ : Dirac's monopole field of strength $-\mu$

$$\Omega_\mu = \frac{-\mu}{r^3} (x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$$

Remarks

c	velocity of light	are all set at unity i.e. $c = e = m = 1$
e	charge of electron	
m	mass of electron	

The MIC-Kepler problem as a reduced system

Hamiltonian $\quad H_\mu(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2m} \parallel \boldsymbol{p} \parallel^2 + \frac{\mu^2}{2mr^2} - \frac{k}{r}$

where $\begin{cases} -\mu \in \mathbb{R} & \text{constant for strength of Dirac's monopole field} \\ k > 0 & \text{constant for Coulomb's potential} \end{cases}$

Let E be the actual energy, the generalized Hamiltonian $\Phi(\boldsymbol{x}, \boldsymbol{p})$ is defined by

$$\begin{aligned} \Phi(\boldsymbol{x}\,, \boldsymbol{p}) &\equiv r(H_\mu - E) \\ &= r\left\{\frac{1}{2m}({p_x}^2 + {p_y}^2 + {p_z}^2) + \frac{\mu^2}{2mr^2} - \frac{k}{r} - E\right\} \\ &= \frac{1}{2m} r\left\{{p_x}^2 + {p_y}^2 + {p_z}^2 + \left(\frac{-\mu}{r}\right)^2\right\} - k - Er \end{aligned}$$

$E = H_\mu \Longleftrightarrow \Phi = 0$: equation of motion

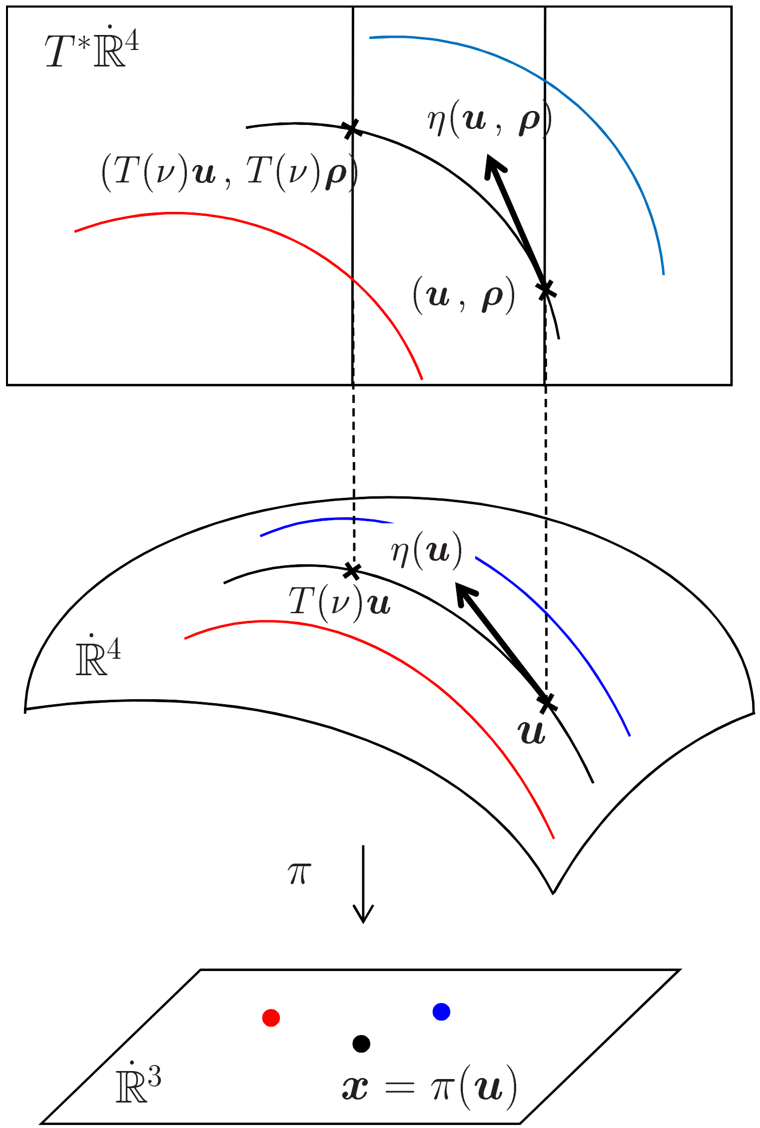
We consider a phase space $T^*\dot{\mathbb{R}}^4$ and the S^1 action defined by

$$t \in [0, 4\pi], \quad T^*\dot{\mathbb{R}}^4 \ni (\boldsymbol{u}\,,\, \boldsymbol{\rho}) \longmapsto (T(t)\boldsymbol{u}\,,\, T(t)\boldsymbol{\rho}) \in T^*\dot{\mathbb{R}}^4$$

where

$$T(t) = \begin{pmatrix} R(t) & O \\ O & R(t) \end{pmatrix}$$

$$R(t) = \begin{pmatrix} \cos\frac{t}{2} & -\sin\frac{t}{2} \\ \sin\frac{t}{2} & \cos\frac{t}{2} \end{pmatrix}$$



π is it's bundle projection, that is

$$\pi \quad : \quad \begin{array}{ccc} \dot{\mathbb{R}}^4 & \longrightarrow & \dot{\mathbb{R}}^3 \\ \boldsymbol{u} & \longmapsto & \pi(\boldsymbol{u}) = \boldsymbol{x} \end{array}$$

where
$$\begin{cases} x(\boldsymbol{u}) = 2(u_1u_3 + u_2u_4) \\ y(\boldsymbol{u}) = 2(-u_1u_4 + u_2u_3) \\ z(\boldsymbol{u}) = {u_1}^2 + {u_2}^2 - {u_3}^2 - {u_4}^2 \end{cases}$$

then $$u^2 \equiv {u_1}^2 + {u_2}^2 + {u_3}^2 + {u_4}^2 = r$$

Let

$$\begin{aligned} \eta \ : \ \dot{\mathbb{R}}^4 &\longrightarrow \ T_{\boldsymbol{u}}\dot{\mathbb{R}}^4 \\ \boldsymbol{u} &\longmapsto \eta(\boldsymbol{u}) = \tfrac{1}{2}\,(-u_2,\, u_1,\, -u_4,\, u_3) \end{aligned}$$

be the induced vector fied of the action. Then the momentum mapping $\psi(\boldsymbol{u}\,,\,\boldsymbol{\rho})$ is given by

$$\begin{aligned} -d\psi(\boldsymbol{u}\,,\,\boldsymbol{\rho}) &= (d\boldsymbol{\rho}\wedge d\boldsymbol{u}) \lrcorner\, \eta(\boldsymbol{u}\,,\,\boldsymbol{\rho}) \\ \Rightarrow \quad \psi(\boldsymbol{u}\,,\,\boldsymbol{\rho}) &= \frac{1}{2}(-u_2\rho_1 + u_1\rho_2 - u_4\rho_3 + u_3\rho_4) \end{aligned}$$

The conformal Kepler problem is a triple $(T^*\dot{\mathbb{R}}^4\,,\, d\boldsymbol{\rho}\wedge d\boldsymbol{u}\,,\, \boxed{H}\,)$,
defined by T. Iwai & Y. Uwano, where $H(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \frac{1}{2m}\left(\frac{1}{4u^2}\sum_{j=1}^{4}{\rho_j}^2\right)$.

Note that

$$\begin{aligned}\pi^*\Phi(\boldsymbol{u}\,,\,\boldsymbol{\rho}) &= \frac{1}{8m}({\rho_1}^2+{\rho_2}^2+{\rho_3}^2+{\rho_4}^2) - E(u_1^2+u_2^2+u_3^2+u_4^2) - k \\ &= r\left\{\boxed{\frac{1}{2m}\left(\frac{1}{4r}\sum_{j=1}^{4}{\rho_j}^2\right) - \frac{k}{r}} - E\right\} = r\left\{\boxed{H(\boldsymbol{u}\,,\,\boldsymbol{\rho})} - E\right\}\end{aligned}$$

Then the Kepler problem ($T^*\dot{\mathbb{R}}^3$, σ_μ , H_μ) is obtained from the conformal Kepler problem $(T^*\dot{\mathbb{R}}^4\,,\, d\boldsymbol{\rho}\wedge d\boldsymbol{u}\,,\, H)$ restricted to the submanifold $\psi^{-1}(\mu)$ by S^1 reduction. (Iwai-Uwano [4]).

Quantized MIC-Kepler problem

We consider a quantization of MIC-Kepler problem by means of the Moyal product. We calculate explicitly the energy and multiplicity. Also we can obtain the green function (section) of the Hamiltonian by means of star product.

Proposition [the conformal Kepler problem $H(\boldsymbol{u}\,,\,\boldsymbol{\rho})$]

It's eigenspace associated with the eigenvalue $E_n = \dfrac{-2mk^2}{\hbar^2\,(n+2)^2}$ $(n = 0, 1, 2, \cdots)$ is spanned by

$$f_n(\boldsymbol{u},\,\boldsymbol{\rho}) \;=\; f_0\,(-1)^n \sum_{\substack{n_1, n_2, n_3, n_4 \geq 0 \\ n_1+n_2+n_3+n_4=n}} L_{n_1}(4a_1^{+}a_1)\,L_{n_2}(4a_2^{+}a_2)\,L_{n_3}(4a_3^{+}a_3)\,L_{n_4}(4a_4^{+}a_4)$$

where $^{\forall}j = 1, 2, 3, 4$

$$\begin{cases} a_j \equiv \dfrac{1}{\sqrt{2}}\left(\sqrt{\dfrac{m\,w_n}{\hbar}}\,u_j + \dfrac{i}{\sqrt{m\hbar\,w_n}}\,\rho_j\right),\; a_j^{+} \equiv \dfrac{1}{\sqrt{2}}\left(\sqrt{\dfrac{m\,w_n}{\hbar}}\,u_j - \dfrac{i}{\sqrt{m\hbar\,w_n}}\,\rho_j\right) \\ \hbar\,w_n \equiv \dfrac{4k}{n+2} \\ f_0 \equiv f_{10}f_{20}f_{30}f_{40} = \dfrac{1}{(\pi\hbar)^4}\exp\!\left(-\dfrac{m\,w_n}{\hbar}\,\|\,\boldsymbol{u}\,\|^2 - \dfrac{1}{m\hbar\,w_n}\,\|\,\boldsymbol{\rho}\,\|^2\right) \\ L_{n_j}(4a_j^{+}a_j) = \displaystyle\sum_{l=0}^{n_j}(-1)^l \frac{n_j!}{(l!)^2(n_j-l)!}\cdot(4a_j^{+}a_j)^l \end{cases}$$

Reduction of
conformal Kepler problem
(4 - dim. oscillator)
by an S^1 action

Restricting the eigenspace of H to that of H_μ
i.e.,
restricting the eigenfunctions f_n , to $f_n|_{\psi^{-1}(\mu)}$

For this purpose, it is convenient to introduce the following functions

$$\begin{cases} b_1^+(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \dfrac{1}{\sqrt{2}}(a_1^+ - i a_2^+) \quad , \quad b_1(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \dfrac{1}{\sqrt{2}}(a_1 + i a_2) \\ b_2^+(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \dfrac{1}{\sqrt{2}}(a_3^+ - i a_4^+) \quad , \quad b_2(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \dfrac{1}{\sqrt{2}}(a_3 + i a_4) \\ b_3^+(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \dfrac{1}{\sqrt{2}}(a_1^+ + i a_2^+) \quad , \quad b_3(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \dfrac{1}{\sqrt{2}}(a_1 - i a_2) \\ b_4^+(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \dfrac{1}{\sqrt{2}}(a_3^+ + i a_4^+) \quad , \quad b_4(\boldsymbol{u}\,,\,\boldsymbol{\rho}) = \dfrac{1}{\sqrt{2}}(a_3 - i a_4) \end{cases}$$

Note that

$$[b_j \, * \, b_k \,] = [b_j^+ * b_k^+] = 0 \quad \text{commutative}$$

$$[b_j \, * \, b_k^+] = -\frac{i}{\hbar}\,\delta_{jk} \qquad \text{noncommutative}$$

Moreover, we introduce

$$\left\{\begin{array}{l} N_a \overset{\equiv}{\leftarrow} b_3^+ * b_3 \;=\; b_3 b_3^+ - \dfrac{1}{2} \\ \\ N_b \overset{\equiv}{\leftarrow} b_1^+ * b_1 \;=\; b_1 b_1^+ - \dfrac{1}{2} \\ \\ N_c \overset{\equiv}{\leftarrow} b_2^+ * b_2 \;=\; b_2 b_2^+ - \dfrac{1}{2} \\ \\ N_d \overset{\equiv}{\leftarrow} b_4^+ * b_4 \;=\; b_4 b_4^+ - \dfrac{1}{2} \end{array}\right. , \qquad \left\{\begin{array}{l} f_{a0} \overset{\equiv}{\leftarrow} \dfrac{1}{\pi\hbar} e^{-2b_3^+ b_3} \\ \\ f_{b0} \overset{\equiv}{\leftarrow} \dfrac{1}{\pi\hbar} e^{-2b_1^+ b_1} \\ \\ f_{c0} \overset{\equiv}{\leftarrow} \dfrac{1}{\pi\hbar} e^{-2b_2^+ b_2} \\ \\ f_{d0} \overset{\equiv}{\leftarrow} \dfrac{1}{\pi\hbar} e^{-2b_4^+ b_4} \end{array}\right.$$

We have

$$\boldsymbol{b} \cdot \boldsymbol{b}^+ = \sum_{j=1}^{4} b_j b_j^+ = \sum_{j=1}^{4} a_j a_j^+ = \boldsymbol{a} \cdot \boldsymbol{a}^+$$

$$\begin{aligned} \therefore\; N_a + N_b + N_c + N_d &= \boldsymbol{b} \cdot \boldsymbol{b}^+ - 2 \\ &= \boldsymbol{a} \cdot \boldsymbol{a}^+ - 2 \\ &= N_1 + N_2 + N_3 + N_4 \\ &= N \end{aligned}$$

We also introduce for $k_a\,, k_b\,, k_c\,, k_d = 0, 1, 2\cdots$

$$\begin{cases} f_{k_a} \overset{\equiv}{\longleftarrow} \dfrac{1}{k_a!}\underbrace{b_3^+ * \cdots * b_3^+}_{k_a} * f_{a0} * \underbrace{b_3 * \cdots * b_3}_{k_a} = \dfrac{1}{k_a!}(b_3^+ *)^{k_a}\, f_{a0}\,(*\, b_3)^{k_a} \\ f_{k_b} \overset{\equiv}{\longleftarrow} \dfrac{1}{k_b!}\underbrace{b_1^+ * \cdots * b_1^+}_{k_b} * f_{b0} * \underbrace{b_1 * \cdots * b_1}_{k_b} = \dfrac{1}{k_b!}(b_1^+ *)^{k_b}\, f_{b0}\,(*\, b_1)^{k_b} \\ f_{k_c} \overset{\equiv}{\longleftarrow} \dfrac{1}{k_c!}\underbrace{b_2^+ * \cdots * b_2^+}_{k_c} * f_{c0} * \underbrace{b_2 * \cdots * b_2}_{k_c} = \dfrac{1}{k_c!}(b_2^+ *)^{k_c}\, f_{c0}\,(*\, b_2)^{k_c} \\ f_{k_d} \overset{\equiv}{\longleftarrow} \dfrac{1}{k_d!}\underbrace{b_4^+ * \cdots * b_4^+}_{k_d} * f_{d0} * \underbrace{b_4 * \cdots * b_4}_{k_d} = \dfrac{1}{k_d!}(b_4^+ *)^{k_d}\, f_{d0}\,(*\, b_4)^{k_d} \end{cases}$$

$$f_n \overset{\equiv}{\longleftarrow} \sum_{\substack{k_a, k_b, k_c, k_d \geq 0 \\ k_a + k_b + k_c + k_d = n}} f_{k_a} * f_{k_b} * f_{k_c} * f_{k_d} \qquad (n = 0, 1, 2\cdots)$$

Similarly, we get the commutation relation

$$
\begin{cases}
[b_3 * f_{b0}] = [b_3 * f_{c0}] = [b_3 * f_{d0}] = [b_3^+ * f_{b0}] = [b_3^+ * f_{c0}] = [b_3^+ * f_{d0}] = 0 \\
[b_1 * f_{a0}] = [b_1 * f_{c0}] = [b_1 * f_{d0}] = [b_1^+ * f_{a0}] = [b_1^+ * f_{c0}] = [b_1^+ * f_{d0}] = 0 \\
[b_2 * f_{a0}] = [b_2 * f_{b0}] = [b_2 * f_{d0}] = [b_2^+ * f_{a0}] = [b_2^+ * f_{b0}] = [b_2^+ * f_{d0}] = 0 \\
[b_4 * f_{a0}] = [b_4 * f_{b0}] = [b_4 * f_{c0}] = [b_4^+ * f_{a0}] = [b_4^+ * f_{b0}] = [b_4^+ * f_{c0}] = 0
\end{cases}
$$

and then

$$
\begin{aligned}
& (N_a + N_b + N_c + N_d) * f_n = n\, f_n \\
\therefore\ & N * f_n = n\, f_n \ \Rightarrow\ (\boldsymbol{b} \cdot \boldsymbol{b}^+ - 2) * f_n = n\, f_n \\
\therefore\ & \boldsymbol{b} \cdot \boldsymbol{b}^+ * f_n = (n+2)\, f_n \ \Rightarrow\ \hbar w\, \boldsymbol{b} \cdot \boldsymbol{b}^+ * f_n = \hbar w\, (n+2) f_n \\
\therefore\ & \hbar w\, \boldsymbol{a} \cdot \boldsymbol{a}^+ * f_n = \hbar w\, (n+2) f_n \ \Rightarrow\ K * f_n = \hbar w\, (n+2) f_n
\end{aligned}
$$

As a result,

$$
\begin{aligned}
& K * f_n = \hbar w\, (n+2) f_n && (n = 0, 1, 2, \cdots) \\
& f_n * f_l = \frac{1}{(2\pi\hbar)^4}\, f_n\, \delta_{nl} && (n, l = 0, 1, 2, \cdots)
\end{aligned}
$$

We can rewrite the above-mentioned proposition as the following :

Proposition [the conformal Kepler problem $H(\boldsymbol{u}\,,\,\boldsymbol{\rho})$]

It's eigenspace associated with the eigenvalue $E_n = \dfrac{-2mk^2}{\hbar^2\,(n+2)^2}$ $(n = 0, 1, 2, \cdots)$ is spanned by

$$f_n(\boldsymbol{u},\,\boldsymbol{\rho}) \;=\; f_0\,(-1)^n \sum_{\substack{n_a, n_b, n_c, n_d \geq 0 \\ n_a+n_b+n_c+n_d = n}} L_{n_a}(4b_3^{+}b_3)\,L_{n_b}(4b_1^{+}b_1)\,L_{n_c}(4b_2^{+}b_2)\,L_{n_d}(4b_4^{+}b_4)$$

where

$$\begin{cases} f_0 \equiv f_{a0}f_{b0}f_{c0}f_{d0} = \dfrac{1}{(\pi\hbar)^4}\exp\left(-\dfrac{m\,w_n}{\hbar}\,\|\,\boldsymbol{u}\,\|^2 - \dfrac{1}{m\hbar\,w_n}\,\|\,\boldsymbol{\rho}\,\|^2\right) \\ \hbar\,w_n \equiv \dfrac{4k}{n+2} \\ \text{for all} \quad (\alpha\,,j) = (a\,,3)\,,\,(b\,,1)\,,\,(c\,,2)\,,\,(d\,,4)\,, \\ L_{n_\alpha}(4b_j^{+}b_j) = \displaystyle\sum_{l=0}^{n_\alpha}(-1)^l\frac{n_\alpha!}{(l!)^2(n_\alpha-l)!}\cdot(4b_j^{+}b_j)^l \end{cases}$$

also

$$
\begin{cases}
4b_3^{+}b_3 = \dfrac{mw}{\hbar}({u_1}^2 + {u_2}^2) + \dfrac{1}{m\hbar w}({\rho_1}^2 + {\rho_2}^2) + \dfrac{2}{\hbar}(u_1\rho_2 - u_2\rho_1) \\[2ex]
4b_1^{+}b_1 = \dfrac{mw}{\hbar}({u_1}^2 + {u_2}^2) + \dfrac{1}{m\hbar w}({\rho_1}^2 + {\rho_2}^2) - \dfrac{2}{\hbar}(u_1\rho_2 - u_2\rho_1) \\[2ex]
4b_2^{+}b_2 = \dfrac{mw}{\hbar}({u_3}^2 + {u_4}^2) + \dfrac{1}{m\hbar w}({\rho_3}^2 + {\rho_4}^2) - \dfrac{2}{\hbar}(u_3\rho_4 - u_4\rho_3) \\[2ex]
4b_4^{+}b_4 = \dfrac{mw}{\hbar}({u_3}^2 + {u_4}^2) + \dfrac{1}{m\hbar w}({\rho_3}^2 + {\rho_4}^2) + \dfrac{2}{\hbar}(u_3\rho_4 - u_4\rho_3)
\end{cases}
$$

We get

$$
\begin{aligned}
b_3^{+}b_3 - b_1^{+}b_1 - b_2^{+}b_2 + b_4^{+}b_4 &= \frac{1}{\hbar}\,(-u_2\rho_1 + u_1\rho_2 - u_4\rho_3 + u_3\rho_4) \\
&= \frac{2}{\hbar}\,\psi(\boldsymbol{u},\, \boldsymbol{\rho})
\end{aligned}
$$

$$
\therefore \psi(\boldsymbol{u},\, \boldsymbol{\rho}) = \frac{\hbar}{2}(b_3^{+}b_3 - b_1^{+}b_1 - b_2^{+}b_2 + b_4^{+}b_4) = \frac{\hbar}{2}(b_3^{+} * b_3 - b_1^{+} * b_1 - b_2^{+} * b_2 + b_4^{+} * b_4)
$$

$$\psi * f_n|_{\psi^{-1}(\mu)} = \mu \, f_n|_{\psi^{-1}(\mu)}$$

The conditional equation for reduction

$$\Longleftrightarrow \quad (b_3^+ * b_3 - b_1^+ * b_1 - b_2^+ * b_2 + b_4^+ * b_4) * f_n|_{\psi^{-1}(\mu)} = \frac{2}{\hbar} \mu \, f_n|_{\psi^{-1}(\mu)}$$

$$(b_3^+ * b_3 - b_1^+ * b_1 - b_2^+ * b_2 + b_4^+ * b_4) * f_n$$

$$= \sum_{\substack{n_a, n_b, n_c, n_d \geq 0 \\ n_a + n_b + n_c + n_d = n}} \frac{1}{n_a! n_b! n_c! n_d!} (b_3^+ * b_3 - b_1^+ * b_1 - b_2^+ * b_2 + b_4^+ * b_4) * \Big\{ (b_3^+ *)^{n_a} f_{a0} (* b_3)^{n_a} * (b_1^+ *)^{n_b} f_{b0} (* b_1)^{n_b} * (b_2^+ *)^{n_c} f_{c0} (* b_2)^{n_c} * (b_4^+ *)^{n_d} f_{d0} (* b_4)^{n_d} \Big\}$$

$$= \sum_{\substack{n_a, n_b, n_c, n_d \geq 0 \\ n_a + n_b + n_c + n_d = n}} \frac{n_a - n_b - n_c + n_d}{n_a! n_b! n_c! n_d!} (b_3^+ *)^{n_a} f_{a0} (* b_3)^{n_a} * (b_1^+ *)^{n_b} f_{b0} (* b_1)^{n_b} * (b_2^+ *)^{n_c} f_{c0} (* b_2)^{n_c} * (b_4^+ *)^{n_d} f_{d0} (* b_4)^{n_d}$$

$$\frac{2}{\hbar}\mu\, f_n$$

$$= \sum_{\substack{n_a, n_b, n_c, n_d \geq 0 \\ n_a + n_b + n_c + n_d = n}} \frac{2\mu/\hbar}{n_a! n_b! n_c! n_d!} (b_3^+ *)^{n_a} f_{a0} (* b_3)^{n_a} * (b_1^+ *)^{n_b} f_{b0} (* b_1)^{n_b} * (b_2^+ *)^{n_c} f_{c0} (* b_2)^{n_c} * (b_4^+ *)^{n_d} f_{d0} (* b_4)^{n_d}$$

$$\therefore \quad \boxed{(b_3^+ * b_3 - b_1^+ * b_1 - b_2^+ * b_2 + b_4^+ * b_4) * f_n = \frac{2}{\hbar}\mu\, f_n}$$ restricting

$$\Longrightarrow \; n_a - n_b - n_c + n_d = \frac{2}{\hbar}\mu \equiv l \quad (l \in \mathbb{Z})$$

$$\therefore \begin{cases} \boxed{\mu = \frac{l}{2}\hbar \quad (l \in \mathbb{Z})} \\ n_a + n_d = \frac{n+l}{2} \\ n_b + n_c = \frac{n-l}{2} \end{cases}$$

quantised

$|\, l \,| \leq n$

n and l are simultaneously even or odd

Finally, we reach the following proposition :

Proposition [the quantised MIC-Kepler problem H_μ]

It's eigenspace associated with the negative energy

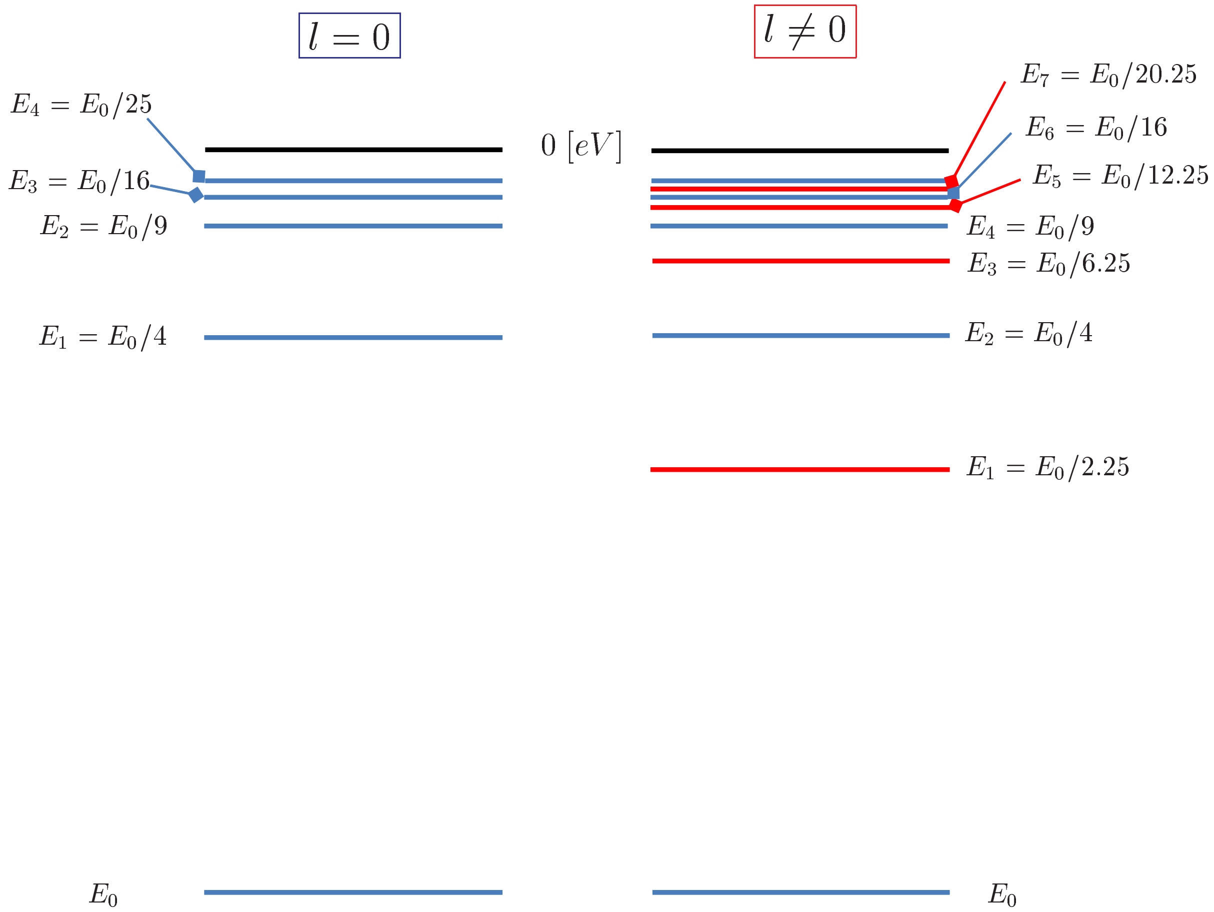
$$E_n = \frac{-2mk^2}{\hbar^2\,(n+2)^2} \quad (n = 0, 1, 2, \cdots) \text{ is spanned by}$$

$$f_n(\boldsymbol{u}, \boldsymbol{\rho}) = f_0\,(-1)^n \sum_{\substack{n_a, n_b, n_c, n_d \geq 0 \\ 2(n_a+n_d) = n+l \\ 2(n_b+n_c) = n-l}} L_{n_a}(4b_3^+ b_3)\, L_{n_b}(4b_1^+ b_1)\, L_{n_c}(4b_2^+ b_2)\, L_{n_d}(4b_4^+ b_4)$$

where

$$\begin{cases} |\,l\,| \leq n \\ \text{n and l are simultaneously even or odd} \end{cases}$$

※ This is the same conclusion as Theorem 5.1. given by T. Iwai & Y. Uwano (1988), except that they choose units where $\hbar = 1$ and m is set at unity : $m = 1$.



$n = 7\,(l = \pm 1, \pm 3, \pm 5, \pm 7)$

$E_7 = E_0/20.25$

$n = 6\,(\boxed{l = 0}, \pm 2, \pm 4, \pm 6)$

$E_6 = \boxed{E_0/16}$

$n = 5\,(l = \pm 1, \pm 3, \pm 5)$

$E_5 = E_0/12.25$

$n = 4\,(\boxed{l = 0}, \pm 2, \pm 4)$

$E_4 = \boxed{E_0/9}$

$n = 3\,(l = \pm 1, \pm 3)$

$E_3 = E_0/6.25$

$n = 2\,(\boxed{l = 0}, \pm 2)$

$E_2 = \boxed{E_0/4}$

$n = 1\,(l = \pm 1)$

$E_1 = E_0/2.25$

$n = 0\,(\boxed{l = 0})$

$E_0 = \boxed{E_0/1}$

Definition [a $*$-unitary evolution function, a "$*$-exponential"]

$$-i\hbar \frac{\partial U_*}{\partial t} = H * U_* = U_* * H$$

where

$$\begin{cases} H(\boldsymbol{x}, \boldsymbol{p}) \ : \ \text{Hamiltonian} \\ U_*(\boldsymbol{x}, \boldsymbol{p}\,;\, t) \\ = 1 + \dfrac{it}{\hbar}H + \dfrac{1}{2!}\left(\dfrac{it}{\hbar}\right)^2 H * H + \cdots + \dfrac{1}{N!}\left(\dfrac{it}{\hbar}\right)^N \overbrace{H * H * \cdots * H}^{N} + \cdots \\ \underset{\longrightarrow}{\equiv}\ e_*^{\frac{it}{\hbar}H} \end{cases}$$

For the Hamiltonian of n-dimensional harmonic oscillator K

$$K(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2m}\| \boldsymbol{p} \|^2 + \frac{1}{2}mw^2\| \boldsymbol{x} \|^2 \quad ,$$

we get the following proposition :

Proposition [The $*$-exponential of n-dim. harmonic oscillator]

differential equation :

$$-i\hbar\frac{\partial}{\partial t}e_*^{\frac{it}{\hbar}K} = K * e_*^{\frac{it}{\hbar}K} = e_*^{\frac{it}{\hbar}K} * K$$

$$= \left(K - \frac{\hbar^2 w^2}{4} n \frac{\partial}{\partial K} - \frac{\hbar^2 w^2}{4} K \frac{\partial^2}{\partial K^2}\right) e_*^{\frac{it}{\hbar}K}$$

solution :

$$e_*^{\frac{it}{\hbar}K} = \left(\cos\frac{wt}{2}\right)^{-n} \exp\left(i\frac{2K}{\hbar w}\tan\frac{wt}{2}\right)$$

$$^{\forall}l \in \mathbb{Z} \quad , \quad \frac{wt}{2} \neq \left(l + \frac{1}{2}\right)\pi$$

Since this evolution function $e_*^{\frac{it}{\hbar}K}$ has singularities on real axis t ($t \geq 0$), there is an attempt to shift variable from t to z', $z' \equiv t + iy'$ ($y' \neq 0$).

$$\begin{cases} -i\hbar\dfrac{\partial}{\partial z'}\ e_*^{\frac{iz'}{\hbar}K} = K * e_*^{\frac{iz'}{\hbar}K} = e_*^{\frac{iz'}{\hbar}K} * K \\ e_*^{\frac{iz'}{\hbar}K} = \left(\cos\dfrac{wz'}{2}\right)^{-n} \exp\left(i\,\dfrac{2K}{\hbar w}\tan\dfrac{wz'}{2}\right) \end{cases}$$

Let $n = 4$ i.e. Four dimensional oscillator :

$$K(\boldsymbol{u}, \boldsymbol{\rho}) = \frac{1}{2m}\| \boldsymbol{\rho} \|^2 + \frac{1}{2}mw^2\| \boldsymbol{u} \|^2 = \frac{1}{2m}\rho^2 + \frac{1}{2}mw^2u^2 \quad .$$

When $y' > 0$, we get the inverse Fourier-transform of the following $*$-exponential ($\boldsymbol{u}_i$: initial point , $\boldsymbol{u}_f$: final point).

$$\begin{aligned} &\mathscr{F}^{-1}\Big[e_*^{\frac{iz'}{\hbar}K(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\boldsymbol{\rho})}\Big] \\ =\ &\mathscr{F}^{-1}\left[\left(\cos\frac{wz'}{2}\right)^{-4}\exp\left(i\,\frac{2}{\hbar w}K\left(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\,\boldsymbol{\rho}\right)\tan\frac{wz'}{2}\right)\right] \\ =\ &\frac{-m^2w^2}{4\pi^2\hbar^2}\,\frac{1}{\sin^2(wz')}\exp\left[-i\,\frac{mw}{2\hbar}\frac{1}{\sin(wz')}\{(u_i^2+u_f^2)\cos(wz') - 2\boldsymbol{u}_i\cdot\boldsymbol{u}_f\}\right] \end{aligned}$$

This expression resembles its ‘propagator’ in appearance.

We've made another attempt to calculate its 'Green function' as follows:

$$
\begin{aligned}
& \operatorname{Im} \lim_{z' \to +0} \mathscr{L}\left[\mathscr{F}^{-1}\left[e_*^{\frac{iz'}{\hbar}K(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\boldsymbol{\rho})}\right]\right] \\
= & \lim_{y' \to +0} \frac{i}{\hbar}\int_0^\infty \mathscr{F}^{-1}\left[e_*^{\frac{i}{\hbar}(t+iy')K(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\boldsymbol{\rho})}\right] e^{-\frac{i}{\hbar}4k(t+iy')}\,dt \\
= & \frac{-im^2w^2}{4\pi^2\hbar^3} \cdot \\
& \lim_{y' \to +0} \int_0^\infty e^{-\frac{i}{\hbar}4kt} \frac{e^{\frac{4k}{\hbar}y'}}{\sin^2(wt+iwy')} \cdot \\
& \qquad \exp\left[-i\,\frac{mw}{2\hbar}\frac{1}{\sin(wt+iwy')}\{(u_i^2+u_f^2)\cos(wt+iwy') - 2\boldsymbol{u}_i\cdot\boldsymbol{u}_f\}\right] dt \\
& \qquad\qquad \dots\dots\dots\dots\dots\dots \quad (1)
\end{aligned}
$$

It remains to be seen if this expression could make sense mathematically.

Furthermore, we've tried to reduce the Green function of 4-dim. harmonic oscillator to that of MIC-Kepler problem by an S^1 action.

Let

$$\begin{cases} U_+ \overset{\longleftarrow}{=\!=} \{\boldsymbol{x} \in \dot{\mathbb{R}}^3 \text{ without negative } z \text{ axis}\} \\ U_- \overset{\longleftarrow}{=\!=} \{\boldsymbol{x} \in \dot{\mathbb{R}}^3 \text{ without positive } z \text{ axis}\} \end{cases},$$

then we define two kinds of local coordinate as follows.

$$\pi \quad : \quad \pi^{-1}(U_+) \ni \boldsymbol{u}(r, \theta, \phi, \nu) \longmapsto \boldsymbol{x}(r, \theta, \phi) \in U_+$$

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}, \quad \begin{cases} u_1 = \sqrt{r} \cos\dfrac{\theta}{2} \cos\dfrac{\nu+\phi}{2} \\ u_2 = \sqrt{r} \cos\dfrac{\theta}{2} \sin\dfrac{\nu+\phi}{2} \\ u_3 = \sqrt{r} \sin\dfrac{\theta}{2} \cos\dfrac{\nu-\phi}{2} \\ u_4 = \sqrt{r} \sin\dfrac{\theta}{2} \sin\dfrac{\nu-\phi}{2} \end{cases} \quad \begin{pmatrix} r > 0 \\ 0 \le \theta < \pi \\ 0 \le \phi \le 2\pi \\ 0 \le \nu \le 4\pi \end{pmatrix}$$

$$\pi \quad : \quad \pi^{-1}(U_-) \ni \boldsymbol{u}(\tilde r, \tilde\theta, \tilde\phi, \tilde\nu) \longmapsto \boldsymbol{x}(\tilde r, \tilde\theta, \tilde\phi) \in U_-$$

$$\begin{cases} x = \tilde r \sin\tilde\theta \cos\tilde\phi \\ y = \tilde r \sin\tilde\theta \sin\tilde\phi \\ z = -\tilde r \cos\tilde\theta \end{cases} , \quad \begin{cases} u_1 = \sqrt{\tilde r} \sin\dfrac{\tilde\theta}{2} \cos\dfrac{\tilde\nu + \tilde\phi}{2} \\ u_2 = \sqrt{\tilde r} \sin\dfrac{\tilde\theta}{2} \sin\dfrac{\tilde\nu + \tilde\phi}{2} \\ u_3 = \sqrt{\tilde r} \cos\dfrac{\tilde\theta}{2} \cos\dfrac{\tilde\nu - \tilde\phi}{2} \\ u_4 = \sqrt{\tilde r} \cos\dfrac{\tilde\theta}{2} \sin\dfrac{\tilde\nu - \tilde\phi}{2} \end{cases} \quad \begin{pmatrix} \tilde r > 0 \\ 0 \le \tilde\theta < \pi \\ 0 \le \tilde\phi \le 2\pi \\ 2\pi \le \tilde\nu \le 6\pi \end{pmatrix}$$

For all $\boldsymbol{u} \in \left(\pi^{-1}(U_+) \cap \pi^{-1}(U_-)\right)$,

$$\begin{aligned} {}^{\exists}g_{+-} \quad : \quad \boldsymbol{u}(r, \theta, \phi, \nu) \longmapsto g_{+-}(\boldsymbol{u}) &= (\tilde r, \tilde\theta, \tilde\phi, \tilde\nu) \\ &= (r, \pi - \theta, \phi, \nu + 2\pi) \end{aligned}$$

As an example, when $\boldsymbol{u}_i$, $\boldsymbol{u}_f \in \left(\pi^{-1}(U_+) \cap \pi^{-1}(U_-)\right)$, we get

$$\int_0^{4\pi} \mathscr{L}\left[\mathscr{F}^{-1}\left[e_*^{\frac{i}{\hbar}(t+iy')K(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\boldsymbol{\rho})}\right]\right] e^{i\,l\,\frac{\nu_i-\nu_f}{2}}\,d\nu_i \qquad \left(l=\frac{2\mu}{\hbar}\in\mathbb{Z}\right)$$

$$= (-1)^{\frac{\mu}{\hbar}}\,\frac{-i\,m^2w^2}{16\pi\hbar^3}\cdot$$

$$\lim_{y'\to+0}\int_0^{\infty} e^{-\frac{i}{\hbar}4kt}\frac{e^{\frac{4k}{\hbar}y'}}{\sin^2(wt+iwy')}\,e^{-i\,\frac{2\mu}{\hbar}\,\frac{\Theta}{2}}$$

$$\times\exp\left[-i\,\frac{mw}{2\hbar}(r_i+r_f)\cot(wt+iwy')\right]$$

$$\times J_{\frac{2\mu}{\hbar}}\left(\frac{mw}{2\hbar}\sqrt{2\boldsymbol{x}_i\cdot\boldsymbol{x}_f+2r_i\,r_f}\,\operatorname{cosec}(wt+iwy')\right)dt\,, \qquad (2)$$

where $\quad \dfrac{\Theta}{2} \equiv \tan^{-1}\left[\dfrac{x_i\,y_f-y_i\,x_f}{r_i\,z_f-r_f\,z_i}\cdot\dfrac{z_i\,z_f-\sqrt{(r_i^2-z_i^2)(r_f^2-z_f^2)}+r_i\,r_f}{z_i\,z_f-\sqrt{(r_i^2-z_i^2)(r_f^2-z_f^2)}-\boldsymbol{x}_i\cdot\boldsymbol{x}_f}\right].$

In addition to this, we get

$$\int_{2\pi}^{6\pi} \mathscr{L}\left[\mathscr{F}^{-1}\left[e_*^{\frac{i}{\hbar}(t+iy')K(\frac{\boldsymbol{u}_i+\boldsymbol{u}_f}{2},\boldsymbol{\rho})}\right]\right] e^{i\,l\,\frac{\tilde{\nu}_i-\tilde{\nu}_f}{2}}\, d\tilde{\nu}_i \qquad \left(l=\frac{2\mu}{\hbar}\in\mathbb{Z}\right)$$

$$= (-1)^{\frac{\mu}{\hbar}}\,\frac{-i\,m^2w^2}{16\pi\hbar^3}\cdot$$

$$\lim_{y'\to+0}\int_0^\infty e^{-\frac{i}{\hbar}4kt}\frac{e^{\frac{4k}{\hbar}y'}}{\sin^2(wt+iwy')}\,e^{i\,\frac{2\mu}{\hbar}\,\frac{\tilde{\Theta}}{2}}$$

$$\times\exp\left[-i\,\frac{mw}{2\hbar}(\tilde{r}_i+\tilde{r}_f)\cot(wt+iwy')\right]$$

$$\times J_{\frac{2\mu}{\hbar}}\left(\frac{mw}{2\hbar}\sqrt{2\boldsymbol{x}_i\cdot\boldsymbol{x}_f+2\tilde{r}_i\,\tilde{r}_f}\,\operatorname{cosec}(wt+iwy')\right)dt\,, \qquad (3)$$

where $\quad \dfrac{\tilde{\Theta}}{2}\equiv\tan^{-1}\left[\dfrac{y_i\,x_f-x_i\,y_f}{\tilde{r}_i\,z_f+\tilde{r}_f\,z_i}\cdot\dfrac{z_i\,z_f-\sqrt{(\tilde{r}_i^2-z_i^2)(\tilde{r}_f^2-z_f^2)}+\tilde{r}_i\,\tilde{r}_f}{z_i\,z_f-\sqrt{(\tilde{r}_i^2-z_i^2)(\tilde{r}_f^2-z_f^2)}-\boldsymbol{x}_i\cdot\boldsymbol{x}_f}\right].$

With g_{+-} , we can show that

$\tan\dfrac{\tilde{\Theta}}{2}=-\tan\dfrac{\Theta}{2}$ i.e. $\tilde{\Theta}=-\Theta \implies$ (2) and (3) are equivalent.

Note (Problems to be solved)

I : It remains to be seen if expression (1) could make sense mathematically.

II: It also remains to be seen if expression (2) and (3) could make sense with mathematical precision.

III: The other cases of $(\boldsymbol{u}_i\ ,\ \boldsymbol{u}_f)$:

$$\begin{cases} \boldsymbol{u}_i \in \pi^{-1}(U_+)/(\pi^{-1}(U_+) \cap \pi^{-1}(U_-)) \\ \text{and} \\ \boldsymbol{u}_f \in \pi^{-1}(U_-)/(\pi^{-1}(U_+) \cap \pi^{-1}(U_-)) \end{cases}$$

$$\begin{cases} \boldsymbol{u}_i \in \pi^{-1}(U_-)/(\pi^{-1}(U_+) \cap \pi^{-1}(U_-)) \\ \text{and} \\ \boldsymbol{u}_f \in \pi^{-1}(U_+)/(\pi^{-1}(U_+) \cap \pi^{-1}(U_-)) \end{cases}$$

may require further consideration.

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