The Non-Uniqueness Problem of the Covariant Dirac Theory: “Conservative” vs. “Radical” Solutions

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Experimental context

- Quantum effects in the classical gravitational field are observed on Earth for neutrons (spin $\frac{1}{2}$ particles) & atoms:
  - COW effect: gravity-induced phase shift measured by neutron (1975) and atom (1991a) interferometry;
  - Sagnac effect: Earth-rotation-induced phase shift measured by neutron (1979) and atom (1991b) interferometry;
  - Granit effect: Quantization of the energy levels proved by threshold in neutron transmission through a thin horizontal slit (2002).
- These are the only observed effects of the gravity-quantum coupling! Motivates work on curved-spacetime Dirac equation (thus first-quantized theory).

State of the art

- (Generally-)covariant rewriting of the Dirac eqn:

$$\gamma^\mu D_\mu \Psi = -iM\Psi \qquad (M \equiv mc/\hbar). \tag{1}$$

γ^μ: Dirac 4×4 matrices. Verify anticommutation relation: $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\,\mathbf{1}_4$, $\mu,\nu \in \{0,...,3\}$, $\mathbf{1}_4 \equiv \mathrm{diag}(1,1,1,1)$. Here $(g^{\mu\nu}) \equiv (g_{\mu\nu})^{-1}$, with $g_{\mu\nu}$ the components of the Lorentzian metric $\boldsymbol{g}$ on the SpaceTime manifold V in a local chart $\chi : \mathrm{V} \supset \mathrm{U} \to \mathsf{R}^4$. Thus γ^μ depend on $X \in \mathrm{V}$.

Wave function ψ is a section of a vector bundle E ("spinor bundle") with base V. $\Psi : \mathrm{U} \to \mathsf{C}^4$: local expression of ψ in a local frame field $(e_a)_{a=0,\dots,3}$ on E over U.

$D_\mu \equiv \partial_\mu + \Gamma_\mu$, covariant derivatives. Γ_μ: 4×4 matrices.

State of the art (continued)

- For standard version (Dirac-Fock-Weyl, **DFW**): the field of the anticommuting Dirac matrices γ^μ is determined by an (orthonormal) *tetrad field* (u_α), i.e., $\mathrm{V} \ni X \mapsto u_\alpha(X) \in \mathrm{TV}_X \quad (\alpha = 0, ..., 3)$.

- The tetrad field (u_α) may be changed by a "local Lorentz transformation" $L : \mathrm{V} \to \mathsf{SO}(1,3)$, $\widetilde{u}_\beta = L^\alpha{}_\beta u_\alpha$. Lifted to a "spin transformation" $S : \mathrm{V} \to \mathsf{Spin}(1,3)$. S is smooth if V is topologically simple. Then the DFW eqn is covariant under changes of the tetrad field, thus the DFW eqn is unique.

- That covariance is got with the "spin connection" D on the spinor bundle E. This connection *depends on the field of the Dirac matrices* γ^μ, thus it depends on the tetrad field.

State of the art (end)

- DFW has been investigated in physical situations, notably
  - in a uniformly rotating frame in Minkowski SpaceTime
  - in a uniformly accelerating frame in Minkowski ST
  - in a static, or stationary, weak gravitational field.
- Differences with non-relativistic Schrödinger eqn with Newtonian potential: not currently measurable.
- First expected new effect with respect to non-relativistic Schrödinger eqn with Newtonian potential: "Spin-rotation coupling" in a rotating frame (Mashhoon 1988, Hehl-Ni 1990). Would affect the energy levels of a Dirac particle.

Covariant Dirac eqn: Alternative Versions

- Alternative versions of the covariant Dirac eqn (1) can be proposed (M.A., Found. Phys. 2008, M.A. & F. Reifler, Int. J. Geom. Meth. Mod. Phys. 2012), based on assuming any *fixed connection* on the spinor bundle E (in contrast with DFW). Price: Covariance under changes of the γ^μ field expressed by a system of quasilinear PDE's. (M.A. & F. Reifler, Braz. J. Phys. 2010)

NB. For a physically relevant spacetime V, there are two explicit realizations of a spinor bundle E :

- $\mathsf{E} = \mathrm{V} \times \mathsf{C}^4$ (wave function is a complex four- scalar)
- $\mathsf{E} = \mathrm{T}_{\mathsf{C}}\mathrm{V}$ (wave function is a complex *four-vector*).

(M.A. & F. Reifler, Int. J. Geom. Meth. Mod. Phys. 2012)

Surprising recent results

- Ryder (Gen. Rel. Grav. 2008) considered uniform rotation w.r.t. inertial frame in Minkowski ST. Found in this particular case:

  Mashhoon's term in the DFW Hamiltonian operator H *is there for one tetrad field* (u_α)*, is not for another one* $(\widetilde{u}_\alpha)$.

- Independently we identified in the most general case the relevant scalar product for the covariant Dirac eqn (M.A. & F. Reifler, arXiv:0807.0570 (gr-qc)/ Braz. J. Phys. 2010). And:

  Hermiticity of H w.r.t. that scalar product *depends on the choice of the admissible field* γ^μ.

Surprising recent results (continued)

- This fact (instability of the hermiticity of H under admissible changes of the γ^μ field) led us to a general study of the non-uniqueness problem of the covariant Dirac theory.

- As for this fact, we did that study for DFW, *and* for alternative versions of the covariant Dirac eqn.

- Found that, for any of these versions (standard, alternative), in any given reference frame:
  - The Hamiltonian operator H is non-unique.
  - So is also the energy operator E (Hermitian part of H)
  - The **Dirac energy spectrum** (= of E) **is non-unique**.

Local similarity (or gauge) transformations

Recall: in a curved spacetime $(\mathrm{V}, \boldsymbol{g})$, the Dirac matrices γ^μ depend on $X \in \mathrm{V}$.

If one changes from one admissible field (γ^μ) to another one $(\widetilde{\gamma}^\mu)$, the new field obtains by a *local similarity transformation* (or local gauge transformation) :

$$\exists S = S(X) \in \mathsf{GL}(4, \mathsf{C}) : \qquad \widetilde{\gamma}^\mu(X) = S^{-1}\gamma^\mu(X)S, \quad \mu = 0, ..., 3. \tag{2}$$

For the standard Dirac eq (DFW), the gauge transformations are restricted to the spin group $\mathsf{Spin}(1,3)$, because they are got by lifting a local Lorentz transformation $L(X)$ applied to a tetrad field. For the alternative eqs, they are general: $S(X) \in \mathsf{GL}(4, \mathsf{C})$.

The general Dirac Hamiltonian

Rewriting the covariant Dirac eqn in the "Schrödinger" form:

$$i\frac{\partial \Psi}{\partial t} = \mathrm{H}\Psi, \qquad (t \equiv x^0), \tag{3}$$

gives the general explicit expression of the Hamiltonian operator H. (M.A., Phys. Rev. D 2006; M.A. & F. Reifler, Ann. der Phys. 2011)

• H depends on the coordinate system, or more exactly on the *reference frame* — an equivalence class of charts defined on a given open set $\mathrm{U} \subset \mathrm{V}$ and exchanging by

$$x'^0 = x^0, \quad x'^j = f^j((x^k)) \qquad (j,k = 1,2,3). \tag{4}$$

(M.A.& F. Reifler, Braz. J. Phys. 2010, Int. J. Geom. Meth. Mod. Phys. 2011. Thus a chart χ defines a reference frame: the equivalence class of χ.)

Invariance condition of the Hamiltonian under a local gauge transformation

When does a gauge transfo. $S(X)$, applied to the field of Dirac matrices γ^μ, leave H invariant? I.e., when do we have

$$\widetilde{\mathrm{H}} = S^{-1}\,\mathrm{H}\,S? \qquad (5)$$

E.g. if the Dirac eqn is covariant under the local gauge transformation S (case of DFW), it is easy to see that we have (5) iff $S(X)$ is time-independent, $\underline{\partial_0 S = 0}$, independently of the explicit form of H. (Other conditions for alternative eqs.)

In the general case $g_{\mu\nu,0} \neq 0$, any possible field γ^μ depends on t, and so does S. Thus **the Dirac Hamiltonian is not unique** and one also proves that **the energy operator and its spectrum are not unique.** (M.A. & F. Reifler, Ann. der Phys. 2011)

Basic reason for the non-uniqueness

- Thus, in a given general reference frame or even in a given coordinate system, the Hamiltonian and energy operators associated with the *generally-covariant* Dirac eqn depend on the choice of the *field* of Dirac matrices $X \mapsto \gamma^\mu(X)$.

- In contrast, in a given inertial reference frame or in a given Cartesian coordinate system, the Hamiltonian operator associated with the *original* Dirac eqn of special relativity is Hermitian and does *not* depend on the choice of the *constant* set of Dirac matrices $\gamma^{\sharp\alpha}$.
(M.A. & F. Reifler, Braz. J. Phys. 2008)

- Clearly, the non-uniqueness means there is too much choice for the field γ^μ — too much gauge freedom.

Tetrad fields adapted to a reference frame

- The data of a reference frame F fixes a unique four-velocity field v_{F}: the unit tangent vector to the world lines

$$X \in \mathrm{U}, \quad x^0(X) \text{ variable}, \qquad x^j(X) = \text{constant for } j = 1, 2, 3. \tag{6}$$

These world lines (invariant under an internal change (4)) are the trajectories of the particles constituting the reference frame ⇒ a chart *has* physical content after all!

- Natural to impose on the tetrad field (u_α) the condition: time-like vector of the tetrad = four-velocity of the reference frame: $\underline{u_0 = v_{\mathrm{F}}}$.

- Then the spatial triad $(u_p) \quad (p = 1, 2, 3)$ can only be *rotating* w.r.t. the reference frame. (Outline follows.)

Space manifold and spatial tensor fields

- Let F be a reference frame, with its domain $\mathrm{U} \subset \mathrm{V}$. The set M of the world lines (6) is endowed with a natural structure of differential manifold: for any chart $\chi \in \mathrm{F}$, its spatial part $\widetilde{\chi} : \mathrm{M} \ni x \mapsto (x^j)_{j=1,2,3}$ is a chart on M.

- Space manifold M is frame-dependent and is *not* a 3-D submanifold of the spacetime manifold V !
  (M.A. & F. Reifler, Int. J. Geom. Meth. Mod. Phys. 2011)

- One then defines spatial tensor fields depending on the spacetime position, e.g. a spatial vector field: $\mathrm{U} \ni X \mapsto \mathbf{u}(X) \in \mathrm{TM}_{x(X)}$, where, for $X \in \mathrm{U}$, $x(X)$ = unique world line $x \in \mathrm{M}$, s.t. $X \in x$. (See Eq. (6).)

Rotation rate tensor field of the spatial triad

- Again a reference frame F is given. $\forall X \in \mathrm{U}$, there is a canonical isomorphism between four-vectors $\perp v_{\mathrm{F}}$ and spatial vectors:

$$\mathrm{H}_X \equiv \{u_X \in \mathrm{TV}_X \, ; \, \boldsymbol{g}(u_X, v_{\mathrm{F}}(X)) = 0\} \rightleftharpoons \mathrm{TM}_{x(X)}, \quad (7)$$

u (with components $u^\mu, \mu = 0, ..., 3$ in some $\chi \in \mathrm{F}$)
$\mapsto \mathbf{u}$ (with components $u^j, j = 1, 2, 3$ in $\widetilde{\chi}$).
(Independent of $\chi \in \mathrm{F}$.)

- Then, $\exists$ one natural time-derivative for spatial vector fields. This allows one to geometrically define the rotation rate field $\boldsymbol{\Xi}$ of the spatial triad field $(\mathbf{u}_p)$ $(p = 1, 2, 3)$ associated with a tetrad field (u_α) $(\alpha = 0, ..., 3)$. MA, Ann. der Phys. 2011

Tetrad fields adapted to a reference frame (end)

- Two tetrad fields (u_α) and $(\tilde{u}_\alpha)$ s.t. $u_0 = \tilde{u}_0 = v_{\mathrm{F}}$, and with the same rotation rate $\boldsymbol{\Xi} = \widetilde{\boldsymbol{\Xi}}$, exchange by a time-independent Lorentz transformation.
  Hence they give rise in F to equivalent Hamiltonian operators and to equivalent energy operators.

- Two natural ways to fix the tensor field $\boldsymbol{\Xi}$ are: i) $\boldsymbol{\Xi} = \boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is the unique *rotation rate field of the given reference frame* F, and ii) $\boldsymbol{\Xi} = \mathbf{0}$.

- Either choice, i) or ii), thus provides a solution to the non-uniqueness problem. These two solutions are not equivalent, so that experiments would be required to decide between the two. Moreover, each solution is valid **only in a given reference frame**.

Getting unique Hamiltonian & energy operators in any reference frame at once?

- The invariance condition of the Hamiltonian H after a gauge transfo. for DFW: $\partial_0 S = 0$, is coordinate-dependent. This condition implies also the invariance of the energy operator E for DFW.

- $\Rightarrow$ The stronger condition $\partial_\mu S = 0$ $(\mu = 0, ..., 3)$ implies the invariance of both H and E simultaneously in any chart (hence in any reference frame), for DFW.

Getting unique Hamiltonian & energy operators in any reference frame at once? (continued)

- Alternative versions of covariant Dirac eqn: the invariance conditions of H and E contain $D_\mu S$. But, for the "QRD–0" version, we define the connection matrices to be

$$\Gamma_\mu = 0 \quad \text{in the canonical frame field } (E_a) \text{ of } \mathrm{V} \times \mathbb{C}^4, \qquad (8)$$

so we have by construction $\partial_\mu S = D_\mu S$ for QRD–0.

- Thus, if we succeed in restricting the choice of the γ^μ field so that any two choices exchange by a **constant** gauge transfo. ($\partial_\mu S = 0$), we solve the non-uniqueness problem simultaneously in any reference frame — for both DFW and QRD–0, and only for them.

Fixing one tetrad field in each chart

In a chart, a tetrad (u_α) is defined by a matrix $a \equiv ({a^\mu}_\alpha)$, s.t. $u_\alpha = {a^\mu}_\alpha \partial_\mu$. Orthonormality of the tetrad in the metric with matrix $G \equiv (g_{\mu\nu}) = G(X) \quad (X \in \mathrm{V})$:

$$b^T \eta b = G \qquad [b \equiv a^{-1}, \quad \eta \equiv \mathrm{diag}(1, -1, -1, -1)]. \tag{9}$$

Generalized Cholesky decomposition (Reifler 2008): $\exists!\ b = C$: lower triangular solution of (9) with ${C^\mu}_\mu > 0,\ \mu = 0, ..., 3$.

$\rightarrow$ a unique tetrad in a given chart: "Cholesky prescription". One other known prescription (Kibble 1963) has this property. Both coincide for a "diagonal metric": $G = \mathrm{diag}(d_\mu) \ \Rightarrow \ u_\alpha \equiv \delta^\mu_\alpha\, \partial_\mu / \sqrt{|d_\mu|}$, "diagonal tetrad".

The reference frame, not the chart, is physically given

- What is physically given is the reference frame:
a three-dimensional congruence of time-like world lines.

- Given a reference frame F, there remains a whole
functional space of different choices for a chart $\chi \in \mathrm{F}$.

Fixing one tetrad field in each chart is not enough

- Consider a prescription (e.g. "Cholesky"): $\chi \mapsto a \mapsto (u_\alpha)$. For two different charts $\chi, \chi' \in \mathrm{F}$, we get two tetrad fields $(u_\alpha),\ (u'_\alpha)$ with matrices a, a'. We have $u'_\beta = L^\alpha{}_\beta u_\alpha$, with

$$L = b\,P\,a', \quad b \equiv a^{-1}, \qquad P^\mu{}_\nu \equiv \frac{\partial x^\mu}{\partial x'^\nu}. \tag{10}$$

- b and a' depend on $t \equiv x^0 = x'^0$ as do G and G'. Since $\chi, \chi' \in \mathrm{F}$, the matrix P doesn't depend on t, Eq. (4). In general, the dependences on t of b and a' don't cancel each other in Eq. (10).

- Thus in general the Lorentz transformation L depends on t. $\Rightarrow$ L is lifted to a gauge transformation S depending on t. $\Rightarrow$ H and H' not equivalent: The non-uniqueness still there.

The case with a diagonal metric

- Consider the Cholesky prescription applied to a "diagonal metric": $G = \mathrm{diag}(d_\mu) \quad (d_0 > 0, \quad d_j < 0, j = 1, 2, 3)$.
  Some algebra gives us

$$\frac{\partial}{\partial t}\left(L^p{}_3\right) \propto P^p{}_3 (P^j{}_3)^2 \frac{\partial}{\partial t}\left(\frac{d_j}{d_p}\right) \quad (\text{no sum on } p = 1, 2, 3), \tag{11}$$

  with a non-zero proportionality factor. Thus in general $\frac{\partial}{\partial t}\left(L^p{}_3\right) \neq 0$, non-uniqueness of H and E still there.

- Exception: $d_j(X) = d_j^0\, h(X)$ with d_j^0 constant ($d_0^j < 0$ with $h > 0$). Then after changing $x'^j = x^j \sqrt{-d_j^0}$, we get $d'_j = -h \; (j = 1, 2, 3)$, or

$$\underline{\overline{G \equiv (g_{\mu\nu}) = \mathrm{diag}(f, -h, -h, -h)}}, \qquad f > 0, \; h > 0. \tag{12}$$

Space-isotropic diagonal metric

Theorem (M.A., arXiv:1205.3386). *Let the metric have the space-isotropic diagonal form (12) in some chart* χ. *Let* χ' *belong to the same reference frame* R.

(i) *The metric has the* **form** *(12) also in* χ', *iff* $(x^j) \mapsto (x'^j)$ *is a constant rotation, combined with a constant homothecy.*

(ii) *If one applies the "diagonal tetrad" prescription in each of the two charts, the two tetrads obtained thus are related together by a* **constant** *Lorentz transformation* L, *hence give rise,* **in any reference frame** F, *to equivalent Hamiltonian operators as well to equivalent energy operators — for the DFW and QRD–0 versions of the Dirac equation.*

Uniformly rotating frame in flat spacetime

Let $\chi' : X \mapsto (ct', x', y', z')$ be a Cartesian chart in the Minkowski spacetime, thus $g'_{\mu\nu} = \eta_{\mu\nu}$. Defines inertial frame F'.

Go from χ' to $\chi : X \mapsto (ct, x, y, z)$ defining uniformly rotating ref. frame F ($\omega = \mathrm{constant}$):

$$t = t',\ x = x' \cos\omega t + y' \sin\omega t,\ y = -x' \sin\omega t + y' \cos\omega t,\ z = z'. \tag{13}$$

With $\rho \equiv \sqrt{x^2 + y^2}$, the Minkowski metric writes in the chart χ:

$$g_{00} = 1 - \left(\frac{\omega\rho}{c}\right)^2, \quad g_{01} = -g_{02} = \frac{\omega}{c}, \quad g_{03} = 0, \quad g_{jk} = -\delta_{jk}. \tag{14}$$

4-velocity of F : $v = \partial_0/\sqrt{g_{00}} \Rightarrow \boldsymbol{g}(v, \partial_j) \neq 0$.
Each of Ryder's (2008) two tetrads has $u_0 = v' \neq v$:
Each is adapted to the inertial frame, *not* to the rotating frame.

A tetrad adapted to the rotating frame

Adopt the "rotating cylindrical" chart χ°, also belonging to the rotating frame F. Related to the "rotating Cartesian" chart (13):

$$\chi^\circ : X \mapsto (ct, \rho, \varphi, z) \quad \text{with} \;\; x = \rho\cos\varphi, \quad y = \rho\sin\varphi. \tag{15}$$

Define $u_0 \equiv v,\ u_p \equiv \Pi\partial_p / \parallel \Pi\partial_p \parallel$, where $\Pi =\perp$ projection onto the hyperplane $\perp v$. This is an *orthonormal* tetrad adapted to F, because for the chart χ° we have $\boldsymbol{g}(u_p, u_q) = 0,\ 1 \leq\ p \neq q \leq 3$.

Rotation rate tensor of $(\mathbf{u}_p)$: $\Xi_{pq} = -c\frac{\mathrm{d}\tau}{\mathrm{d}t}\gamma_{pq0}$. Here $\Xi_{pq} = 0$ except for

$$\Xi_{21} = -\Xi_{12} = \frac{\omega}{\sqrt{1-(\omega\rho)^2/c^2}}. \tag{16}$$

Differs from rotation rate tensor $\boldsymbol{\Omega}$ of the *rotating frame* F only by $O(V^2/c^2)$ terms ($V \equiv \omega\rho \ll c$).

Explicit expression of the Dirac Hamiltonian operator

Hamiltonian operator for the generally-covariant Dirac eqn (1):

$$\mathrm{H} = mc^2\alpha^0 - i\hbar c\alpha^j D_j - i\hbar c\Gamma_0, \tag{17}$$

where

$$\alpha^0 \equiv \gamma^0/g^{00}, \qquad \alpha^j \equiv \gamma^0\gamma^j/g^{00}. \tag{18}$$

Spin connection matrices with an orthonormal tetrad field (u_α):

$$\Gamma^\sharp_\epsilon = \frac{1}{8}\gamma_{\alpha\beta\epsilon}\left[\gamma^{\sharp\alpha}, \gamma^{\sharp\beta}\right]. \qquad (\gamma^{\sharp\alpha} = \text{“flat” Dirac matrices}) \tag{19}$$

Spin connection matrices with the natural basis $(\partial_\mu = b^\alpha{}_\mu u_\alpha)$:

$$\Gamma_\mu = b^\alpha{}_\mu \Gamma^\sharp_\alpha. \tag{20}$$

Hamiltonian for adapted rotating tetrad

Using the foregoing expressions, it is straightforward to compute H in the rotating frame F with the adapted rotating tetrad. We find that the spin connection matrices Γ_μ do involve spin operators made with the Pauli matrices σ^j. In particular, we have for $V \equiv \omega\rho \ll c$:

$$\Gamma_0 = -\frac{i}{2}\frac{\omega}{c}\Sigma^3\left[1 + O\left(\frac{V}{c}\right)\right], \qquad \Sigma^j \equiv \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}, \tag{21}$$

for which $-i\hbar c\Gamma_0$ is the usual *"spin-rotation coupling" term* in H.

Also the Γ_j matrices ($j = 1, 2, 3$) contain spin operators. Likely to come from the fact that the adapted rotating tetrad involves projecting the natural tetrad of the rotating coordinates.

H for rotating frame with γ^μ matrices from Minkowski tetrad ("gauge freedom restricted" sol$^{\mathrm{n}}$)

Defining the γ^μ matrices from the "diagonal tetrad" prescription in the Cartesian chart χ', and transforming them to the rotating chart χ, gives after a simple calculation:

$$\mathrm{H} = \mathrm{H}' - i\hbar\omega(y\partial_x - x\partial_y) = \mathrm{H}' - \boldsymbol{\omega}.\mathbf{L}, \qquad (22)$$

with $\mathrm{H}' \equiv$ special-relativistic Dirac Hamiltonian in the inertial frame F', and $\mathbf{L} \equiv \mathbf{r} \wedge (-i\hbar\nabla)$: angular momentum operator.

NB. The same H applies, whether DFW or QRD–0 is chosen. (The spin connection matrices are zero.)

Thus, *there is no spin-rotation coupling with the "gauge freedom restriction" solution* of the non-uniqueness problem.

Conclusion

- Non-unique Hamiltonian and energy operators in covariant Dirac theory: due to gauge freedom in choice of γ^μ matrices. (Yet standard covariant Dirac *eqn* is unique by construction.)

- "Conservative" way of restricting the gauge freedom: fix vector u_0, then fix rotation rate of triad (u_p). Applies to a given reference frame. Uneasy to implement. Spin-rotation coupling.

- "Radical" way: arrange that same gauge freedom applies as in special relativity — constant gauge transformations. Needs diagonal space-isotropic metric. (Always valid in "scalar ether theory". Other metrics?) Applies independently of reference frame. Easy to implement. No spin-rotation coupling.