Harmonic analysis on Lagrangian manifolds of integrable Hamiltonian systems

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Algebraic Integrable Systems

Integrability in Kowalewska sense

Any solution of a system admits a holomorphic continuation in time.
In other words, any solution is associated with a Riemann surface $\mathcal{R}$.

Integrable systems on orbits of a loop group obey equations of Lax type

$$\frac{dL(\lambda)}{dt} = [A, L(\lambda)], \qquad L(\lambda) = \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \gamma(\lambda) & -\alpha(\lambda) \end{pmatrix}$$

$$L \in \widetilde{\mathfrak{g}}^*, \quad \widetilde{\mathfrak{g}} = \mathfrak{sl}(2, \mathbb{C}) \times \mathcal{P}(\lambda, \lambda^{-1})$$

$$\alpha(\lambda) = \sum_{j=0}^{N} \alpha_j \lambda^j, \quad \beta(\lambda) = \sum_{j=0}^{N} \beta_j \lambda^j, \quad \gamma(\lambda) = \sum_{j=0}^{N} \gamma_j \lambda^j.$$

The spectrum of L does not change $\Rightarrow$ There exists the spectral curve

$$\mathcal{R} = \{\det(L(\lambda) - w) = 0\}.$$

Phase Space Structure

A finite gap phase space of the system is a coadjoint orbit $\mathcal{O}^N$ of the loop group generated by $\widetilde{\mathfrak{g}}$:

$$\mathcal{O}^N = \{\operatorname{Tr} L^k(\lambda) = \text{const}\}.$$

The complex Liouville torus of the system coincides with the **generalized Jacobian** of a Riemann surface $\mathcal{R}$ (which is the spectral curve):

$$\widetilde{\text{Jac}}(\mathcal{R}) = \text{Symm}_N \underbrace{\mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}}_{N}, \qquad N > g,$$

where g is the genus of $\mathcal{R}$.

Previato E. Hyperelliptic quasi-periodic and soliton solution of the nonlinear Schrodinger equation, *Duke Math. J.*, **52** (1985), 323–332.

Separation of Variables

Phase space $\{\gamma_0, \ldots, \gamma_{N-1}, \alpha_0, \ldots, \alpha_{N-1}\}$ $\Longrightarrow$ Canonically conjugated variables $\{\lambda_1, \ldots, \lambda_N, w_1, \ldots, w_N\}$

The equations of orbit eliminate the variables $\{\beta_0, \ldots, \beta_{N-1}\}$:

$$\begin{matrix} f_k(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = c_k \\ k=1, \ldots, N \end{matrix} \quad \Rightarrow \quad \begin{matrix} \beta_j = \beta_j(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{c}) \\ j=0, \ldots, N-1 \end{matrix} \quad \Rightarrow$$

$$\Rightarrow \quad h_k(\boldsymbol{\gamma}, \boldsymbol{\alpha}, \boldsymbol{c}) = h_k(\boldsymbol{\lambda}, \boldsymbol{w}, \boldsymbol{c}) \qquad \text{for Hamiltonians} \quad h_1, \ldots, h_N.$$

If one requires (λ_k, w_k) be a point of the spectral curve $\mathcal{R}$ then $\gamma(\lambda_k) = 0$:

$$\det(L(\lambda_k) - w_k) = 0 \qquad \Rightarrow \qquad \gamma(\lambda_k) = 0.$$

Bernatska J., Holod P. On Separation of Variables for Integrable Equations of Soliton Type, *Journal of Nonlinear Math. Phys.*, **14** (2007), 353–374.

Canonical Quantization

Lagrangian manifold

This is a half-dimensional submanifold in the phase space such that the exterior form specifying the symplectic structure on the phase space vanishes identically on it.

In terms of the canonical coordinates $\{\lambda_1, \ldots, \lambda_N, w_1, \ldots, w_N\}$ a submanifold parameterized by $\{\lambda_1, \ldots, \lambda_N\}$ is a Lagrangian manifold.

Quantization in the Schrödinger picture ∼

$$\lambda_k \mapsto \hat{\lambda}_k, \quad w_k \mapsto \hat{w}_k = -i\hbar\frac{\partial}{\partial\lambda_k}, \quad \{\lambda_k, w_l\} = \delta_{kl} \mapsto [\hat{\lambda}_k, \hat{w}_l] = i\hbar\delta_{kl}\,\mathbb{I}\,.$$

∼ Representation of the algebra of phase space symmetry group on a Lagrangian manifold.

Example: Classical Heisenberg Model

Isotropic Landau—Lifshits equation:

$$\frac{\partial \boldsymbol{\mu}}{\partial t} = \frac{1}{2c_0}\left[\boldsymbol{\mu}, \frac{\partial^2 \boldsymbol{\mu}}{\partial x^2}\right] + \frac{c_1}{2c_0}\frac{\partial \boldsymbol{\mu}}{\partial x}, \qquad \boldsymbol{\mu} = \left(\mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}\right).$$

As a system on a coadjoint orbit of the loop algebra $\mathfrak{su}(2) \times \mathcal{P}(\lambda, \lambda^{-1})$

$$L(\lambda) = \begin{pmatrix} \mathrm{i}\mu_3(\lambda) & \mu_1(\lambda) - \mathrm{i}\mu_2(\lambda) \\ -\mu_1(\lambda) - \mathrm{i}\mu_2(\lambda) & -\mathrm{i}\mu_3(\lambda) \end{pmatrix}$$

$$N = 2 \quad \begin{aligned} \mu_1(\lambda) &= \mu_1^{(0)} + \mu_1^{(1)}\lambda, \\ \mu_2(\lambda) &= \mu_2^{(0)} + \mu_2^{(1)}\lambda, \\ \mu_3(\lambda) &= \mu_3^{(0)} + \mu_3^{(1)}\lambda + \varkappa\lambda^2, \qquad \varkappa = \mathrm{const}\,. \end{aligned}$$

Spectral curve

$$\mathcal{R}: \quad \lambda^4 w^2 = -\varkappa^2\lambda^4 - h_3\lambda^3 - h_2\lambda^2 - c_1\lambda - c_0.$$

Phase Space Structure

Poisson structure of the phase space:

$$\{\mu_k^{(0)}, \mu_l^{(0)}\} = 0, \quad \{\mu_k^{(0)}, \mu_l^{(1)}\} = \varepsilon_{klj}\mu_j^{(0)}, \quad \{\mu_k^{(1)}, \mu_l^{(1)}\} = \varepsilon_{klj}\mu_j^{(1)}.$$

$\mu_1^{(0)}$, $\mu_2^{(0)}$, $\mu_3^{(0)}$, $\mu_1^{(1)}$, $\mu_2^{(1)}$, $\mu_3^{(1)}$ form the algebra of **Euclidian group** $\mathrm{E}(3) = \mathrm{SO}(3) \ltimes \mathrm{T}_3$, which is the **phase space symmetry group**.

Then denote

$$\mu_1^{(0)} = p_1, \quad \mu_2^{(0)} = p_2, \quad \mu_3^{(0)} = p_3,$$
$$\mu_1^{(1)} = L_1, \quad \mu_2^{(1)} = L_2, \quad \mu_3^{(1)} = L_3.$$

Equations of the orbit

$$c_0 = -(\boldsymbol{p}, \boldsymbol{p})$$
$$c_1 = -2(\boldsymbol{p}, \boldsymbol{L})$$

Hamiltonians

$$h_2 = -(\boldsymbol{L}, \boldsymbol{L}) - 2\varkappa p_3$$
$$h_3 = -2\varkappa L_3$$

Separation of Variables

$$\lambda_1,\, \lambda_2 : \quad \mu_1(\lambda_k) + \mathrm{i}\mu_3(\lambda_k) = 0, \qquad w_k = \mathrm{i}\mu_2(\lambda_k)/\lambda_k^2.$$

$$p_2 = \mathrm{i}\lambda_1\lambda_2 \frac{\lambda_1 w_1 - \lambda_2 w_2}{\lambda_1 - \lambda_2},$$

$$p_1 = \frac{\mathrm{i}}{2}\left(\varkappa\lambda_1\lambda_2 + \frac{c_0}{\varkappa\lambda_1\lambda_2} - \frac{\lambda_1\lambda_2(\lambda_1 w_1 - \lambda_2 w_2)^2}{\varkappa(\lambda_1 - \lambda_2)^2}\right),$$

$$p_3 = \frac{1}{2}\left(\varkappa\lambda_1\lambda_2 - \frac{c_0}{\varkappa\lambda_1\lambda_2} + \frac{\lambda_1\lambda_2(\lambda_1 w_1 - \lambda_2 w_2)^2}{\varkappa(\lambda_1 - \lambda_2)^2}\right),$$

$$L_2 = -\mathrm{i}\frac{\lambda_1^2 w_1 - \lambda_2^2 w_2}{\lambda_1 - \lambda_2},$$

$$L_1 = \frac{\mathrm{i}}{2}\left(-\varkappa(\lambda + \lambda_2) + \frac{c_1}{\varkappa\lambda_1\lambda_2} + \frac{c_0(\lambda_1 + \lambda_2)}{\varkappa\lambda_1^2\lambda_2^2} + \frac{\lambda_1^2 w_1^2 - \lambda_2^2 w_2^2}{\varkappa(\lambda_1 - \lambda_2)}\right),$$

$$L_3 = \frac{1}{2}\left(-\varkappa(\lambda + \lambda_2) - \frac{c_1}{\varkappa\lambda_1\lambda_2} - \frac{c_0(\lambda_1 + \lambda_2)}{\varkappa\lambda_1^2\lambda_2^2} - \frac{\lambda_1^2 w_1^2 - \lambda_2^2 w_2^2}{\varkappa(\lambda_1 - \lambda_2)}\right).$$

Quantization $\sim$ the algebra $\mathfrak{e}(3)$ representation

The canonical quantization $\lambda_k \mapsto \hat{\lambda}_k, \;\; w_k \mapsto \hat{w}_k = -i\hbar\partial/\partial\lambda_k$

gives a representation of $\mathfrak{e}(3) = \{\hat{L}_3,\; \hat{L}_\pm = \hat{L}_1 \pm \mathrm{i}\hat{L}_2,\; \hat{p}_3,\; \hat{p}_\pm = \hat{p}_1 \pm \mathrm{i}\hat{p}_2\}$

$$[\hat{L}_3, \hat{L}_\pm] = \pm\hat{L}_\pm, \quad [\hat{L}_+, \hat{L}_-] = 2\hat{L}_3, \quad [\hat{p}_3, \hat{p}_\pm] = 0, \quad [\hat{p}_+, \hat{p}_-] = 0,$$

$$[\hat{L}_3, \hat{p}_\pm] = [\hat{p}_3, \hat{L}_\pm] = \pm\hat{p}_\pm, \quad [\hat{L}_+, \hat{p}_-] = [\hat{p}_+, \hat{L}_-] = 2\hat{p}_3.$$

With $z = 2\varkappa\lambda/\hbar,\;\; \tilde{c}_0 = 4\varkappa^2 c_0/\hbar^4,\;\; \tilde{c}_1 = 2\varkappa c_1/\hbar^3$

$$\hat{L}_3 = \frac{z_1^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} + \frac{\tilde{c}_1}{z_1^3} + \frac{\tilde{c}_0}{z_1^4}\right) - \frac{z_2^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_2^2} - \frac{1}{4} + \frac{\tilde{c}_1}{z_2^3} + \frac{\tilde{c}_0}{z_2^4}\right),$$

$$\hat{L}_\pm = \frac{\mathrm{i}z_1^2}{z_1 - z_2}\left(-\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} - \frac{\tilde{c}_1}{z_1^3} - \frac{\tilde{c}_0}{z_1^4} \mp \frac{\partial}{\partial z_1}\right) - \frac{\mathrm{i}z_2^2}{z_1 - z_2}\left(-\frac{\partial^2}{\partial z_2^2} - \frac{1}{4} - \frac{\tilde{c}_1}{z_2^3} - \frac{\tilde{c}_0}{z_2^4} \mp \frac{\partial}{\partial z_2}\right),$$

Quantization ∼ the algebra $\mathfrak{e}(3)$ representation

$$\hat{p}_3 = -\frac{\hbar}{2\varkappa}\left[\frac{z_1 z_2}{(z_1-z_2)^2}\left(z_1^2\frac{\partial^2}{\partial z_1^2} + z_2^2\frac{\partial^2}{\partial z_2^2} - 2z_1 z_2\frac{\partial^2}{\partial z_1 \partial z_2}\right) - \frac{z_1 z_2}{4} + \frac{\tilde{c}_0}{z_1 z_2} - \right.$$
$$\left. - \frac{2z_1^2 z_2^2}{(z_1-z_2)^3}\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}\right)\right],$$

$$\hat{p}_\pm = \frac{\mathrm{i}\hbar}{2\varkappa}\left[\frac{z_1 z_2}{(z_1-z_2)^2}\left(z_1^2\frac{\partial^2}{\partial z_1^2} + z_2^2\frac{\partial^2}{\partial z_2^2} - 2z_1 z_2\frac{\partial^2}{\partial z_1 \partial z_2}\right) + \frac{z_1 z_2}{4} + \frac{\tilde{c}_0}{z_1 z_2} - \right.$$
$$\left. - \frac{2z_1^2 z_2^2}{(z_1-z_2)^3}\left(\frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2}\right) \pm \frac{z_1 z_2}{z_1 - z_2}\left(z_1\frac{\partial}{\partial z_1} - z_2\frac{\partial}{\partial z_2}\right)\right].$$

Hamiltonians:

$$\hat{h}_2 = \frac{\hbar^2 z_1^2 z_2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} + \frac{\tilde{c}_0}{z_1^4} + \frac{\tilde{c}_1}{z_1^3}\right) - \frac{\hbar^2 z_1 z_2^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_2^2} - \frac{1}{4} + \frac{\tilde{c}_0}{z_2^4} + \frac{\tilde{c}_1}{z_2^3}\right),$$

$$\hat{h}_3 = -\frac{2\varkappa\hbar z_1^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} + \frac{\tilde{c}_0}{z_1^4} + \frac{\tilde{c}_1}{z_1^3}\right) + \frac{2\varkappa\hbar z_2^2}{z_1 - z_2}\left(\frac{\partial^2}{\partial z_2^2} - \frac{1}{4} + \frac{\tilde{c}_0}{z_2^4} + \frac{\tilde{c}_1}{z_2^3}\right).$$

Harmonic Analysis: the case of $c_0 = 0$, $c_1 = 0$

Consider the orbit $\mathcal{O}_0$ with $c_0 = 0$, $c_1 = 0$:

$$(\boldsymbol{p}, \boldsymbol{p}) = 0, \qquad (\boldsymbol{p}, \boldsymbol{L}) = 0.$$

The spectral curve $\mathcal{R}$: $\lambda^2 w^2 = -\varkappa^2 \lambda^2 - h_3 \lambda - h_2 \lambda.$
On the orbit $\mathcal{O}_0$ the algebra $\mathfrak{sl}(2) = \{\hat{L}_+, \hat{L}_-, \hat{L}_3\}$ acts.

Representation space

$$\hat{L}_3 f(z_1, z_2) = m f(z_1, z_2), \qquad f(z_1, z_2) = W(z_1) W(z_2)$$

$$W'' + \left(-\frac{1}{4} - \frac{m}{z} - \frac{C}{z^2}\right) W = 0, \quad C = \mu^2 - 1/4$$

— the Whittaker equation with solutions $W_{-m,\mu}(z)$.

$$\hat{L}^2 f(z_1, z_2) = J(J+1) f(z_1, z_2) \quad \Rightarrow \quad C = J(J+1).$$

Representation of the algebra $\mathfrak{sl}(2)$

For $\mu = -J - 1/2$ Whittaker functions $W_{-m,\mu}$ are connected with associated Laguerre polynomials L_n^α.

Basis functions in the representation space

$$f_{Jm}(z_1, z_2) \sim (z_1 z_2)^{-J} e^{-(z_1+z_2)/2} L_{J-m}^{-2J-1}(z_1 + z_2).$$

Every function $f_{JJ}(z_1, z_2) = (z_1 z_2)^{-J} e^{-(z_1+z_2)/2}$, $J = 0, 1, \ldots,$ gives rise to the $\mathfrak{sl}(2)$ **Verma module** $\{f_{Jm} = \hat{L}_-^{J-m} f_{JJ},\ m = J, J-1, \ldots\}$:

$$\hat{L}_3 f_{Jm} = m f_{Jm}, \quad \hat{L}_- f_{Jm} = f_{J,m-1}, \quad \hat{L}_+ f_{Jm} = (J-m)(J+m+1) f_{J,m+1}.$$

The representation is **not standard**

$$\hat{L}_\pm \tilde{f}_{Jm} = \sqrt{(J \mp m)(J \pm m + 1)}\, \tilde{f}_{J,m\pm1}, \quad \hat{L}_3 \tilde{f}_{Jm} = m \tilde{f}_{Jm}.$$

'Unitarization' of $\mathfrak{sl}(2)$ Representation

The standard (canonical) representation is constructed by means of the **intertwining operator** $\hat{A}$

$$\tilde{f}_{Jm} \equiv \hat{A} f_{Jm} = \sqrt{\frac{\Gamma(J+m+1)}{\Gamma(J-m+1)}} f_{Jm} = \mathrm{i}^{J-m}\sqrt{\Gamma(J+m+1)\Gamma(J-m+1)} \times \times (z_1 z_2)^{-J} e^{-(z_1+z_2)/2} L_{J-m}^{-2J-1}(z_1+z_2).$$

The inner product

$$\langle \tilde{f}_{Jm}, \tilde{f}_{Jn} \rangle = \int_0^\infty \int_0^\infty \frac{\tilde{f}^*_{Jm}(z_1, z_2)\tilde{f}_{Jn}(z_1, z_2)}{\sum_{i=0}^{J-n} \frac{\Gamma(-J+i)}{i!} \frac{\Gamma(-n-i)}{(J-n-i)!}} \frac{dz_1\, dz_2}{z_1^{1-J} z_2^{1-J}} = \delta_{nm}.$$

Here the summation theorem and the integral (**divergent**) are used:

$$\int_0^\infty e^{-x} x^\alpha L_n^\alpha(x) L_m^\alpha(x)\, dx = \frac{\Gamma(\alpha+n+1)}{n!} \delta_{nm}, \qquad \alpha = -2J-1.$$

Conclusion and discussion

- A combination of algebraic geometry methods with methods of representation theory for Lie algebras gives a new approach to harmonic analysis on a Lagrangian manifold.
  The case is: an algebra representation is realized by differential operators of high order, and can not be risen to a group.
- There are a lot of integrable systems, for example Gaudin's model, where the proposed scheme provides a basis in the phase space. In particular, it gives an appropriate basis for Bethe anzatz procedure.

Sklyanin E., Separation of variables in the Gaudin model *J. Sov. Math.* **47** (1989), 2473–2488.
Feigin B., Frenkel E., Reshetikhin N., Gaudin Model, Bethe Anzatz and Critical Level *Comm. Math. Phys.*, **166** (1994), 27–62.

The end

Quantization: $\lambda_k \mapsto \hat{\lambda}_k, \quad w_k \mapsto \hat{w}_k = -i\hbar\partial/\partial\lambda_k$

The algebra $\mathfrak{e}(3)$ representation:

$$\hat{p}_2 = \frac{\lambda_1\lambda_2}{\lambda_1 - \lambda_2}\left(\lambda_1\frac{\partial}{\partial\lambda_1} - \lambda_2\frac{\partial}{\partial\lambda_2}\right),$$

$$\hat{p}_1 = \frac{\mathrm{i}}{2\hbar\varkappa}\left(\varkappa^2\lambda_1\lambda_2 + \frac{c_0 + \hat{p}_2^2}{\lambda_1\lambda_2}\right),$$

$$\hat{p}_3 = \frac{1}{2\hbar\varkappa}\left(\varkappa^2\lambda_1\lambda_2 - \frac{c_0 + \hat{p}_2^2}{\lambda_1\lambda_2}\right),$$

$$\hat{L}_2 = \frac{-1}{\lambda_1 - \lambda_2}\left(\lambda_1^2\frac{\partial}{\partial\lambda_1} - \lambda_2^2\frac{\partial}{\partial\lambda_2}\right),$$

$$\hat{L}_1 = \frac{i}{2\hbar\varkappa}\left(-\varkappa^2(\lambda_1 + \lambda_2) + \frac{c_1}{\lambda_1\lambda_2} + \frac{c_0(\lambda_1 + \lambda_2)}{\lambda_1^2\lambda_2^2} + \hat{D}\right),$$

$$\hat{L}_3 = \frac{1}{2\hbar\varkappa}\left(-\varkappa^2(\lambda_1 + \lambda_2) - \frac{c_1}{\lambda_1\lambda_2} - \frac{c_0(\lambda_1 + \lambda_2)}{\lambda_1^2\lambda_2^2} - \hat{D}\right).$$

$$\hat{D} = \frac{2\hat{L}_2\hat{p}_2}{\lambda_1\lambda_2} + \frac{\hat{p}_2^2(\lambda_1 + \lambda_2)}{\lambda_1^2\lambda_2^2} = \frac{-\hbar^2}{\lambda_1 - \lambda_2}\left(\lambda_1^2\frac{\partial^2}{\partial\lambda_1^2} - \lambda_2^2\frac{\partial^2}{\partial\lambda_2^2}\right)$$