f-biharmonic Maps between Riemannian Manifolds

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Abstract

We show that if ψ is an f-biharmonic map from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying a condition, then ψ is an f-harmonic map. We prove that if the f-tension field $\tau_f(\psi)$ of a map ψ of Riemannian manifolds is a Jacobi field and ϕ is a totally geodesic map of Riemannian manifolds, then $\tau_f(\phi \circ \psi)$ is a Jacobi field. We finally investigate the stress f-bienergy tensor, and relate the divergence of the stress f-bienergy of a map ψ of Riemannian manifolds with the Jacobi field of the $\tau_f(\psi)$ of the map.

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1 Introduction

Harmonic maps between Riemannian manifolds were first established by Eells and Sampson in 1964. Chiang, Ratto, Sun and Wolak also studied harmonic and biharmonic maps in [4]-[9]. f-harmonic maps which generalize harmonic maps, were first introduced by Lichnerowicz [25] in 1970, and were studied by Course [12, 13] recently. f-harmonic maps relate to the equations of the motion of a continuous system of spins with inhomogeneous neighbor Heisenberg interaction in mathematical physics. Moreover, F-harmonic maps between Riemannian manifolds were first introduced by Ara [1, 2] in 1999, which could be considered as the special cases of f-harmonic maps.

Let $f : (M_1, g) \to (0, \infty)$ be a smooth function. f-biharmonic maps between Riemannian manifolds are the critical points of f-bienergy

$$E_2^f(\psi) = \frac{1}{2} \int_{M_1} f |\tau_f(\psi|^2 dv,$$

where dv the volume form determined by the metric g. f-biharmonic maps between Riemannian manifolds were first studied by Ouakkas, Nasri and Djaa [26] in 2010, which generalized biharmonic maps by Jiang [20, 21] in 1986.

In section two, we describe the motivation, and review f-harmonic maps and their relationship with F-harmonic maps. In Theorem 3.1, we show that if ψ is an f-biharmonic map from a compact Riemannian manifold into a Riemannian manifold with non-positive curvature satisfying a condition, then ψ is an f-harmonic map. It is well-known from [18] that if ψ is a harmonic map of Riemannian manifolds and ϕ is a totally geodesic map of Riemannian manifolds, then $\phi \circ \psi$ is harmonic. However, if ψ is f-biharmonic and ϕ is totally geodesic, then $\phi \circ \psi$ is not necessarily f-biharmonic. Instead, we prove in Theorem 3.3 that if the f-tension field $\tau_f(\psi)$ of a smooth map ψ of Riemannian manifolds is a Jacobi field and ϕ is totally geodesic, then $\tau_f(\phi \circ \psi)$ is a Jacobi field. It implies Corollary 3.4 [8] that if ψ is a biharmonic map between Riemannian manifolds and ϕ is totally geodesic, then $\phi \circ \psi$ is a biharmonic map. We finally investigate the stress f-bienergy tensors. If ψ is an f-biharmonic of Riemannian manifolds, then it usually does not satisfy the conservation law for the stress f-bienergy tensor $S_2^f(\psi)$. However, we obtain in Theorem 4.2 that if $\psi : (M_1, g) \to (M_2, h)$ be a smooth map between two Riemannian manifolds, then the divergence of the stress f-bienergy tensor $S_2^f(\psi)$ can be related with the Jacobi field of the f-tension field $\tau_f(\psi)$ of the map ψ. It implies Corollary 4.4 [22] that if ψ is a biharmonic map between Riemannian manifolds, then ψ satisfies the conservation law for the stress bi-energy tensor$S_2(\psi)$. We also discuss a few results concerning the vanishing of the stress f-bienergy tensors.

2 Preliminaries

2.1 Motivation

In mathematical physics, the equation of the motion of a continuous system of spins with inhomogeneous neighborhood Heisenberg interaction is

$$\frac{\partial \psi}{\partial t} = f(x)(\psi \times \triangle \psi) + \nabla f \cdot (\psi \times \nabla \psi), \tag{2.1}$$

where $\Omega \subset R^m$ is a smooth domain in the Euclidean space, f is a real-valued function defined on Ω, $\psi(x,t) \in S^2$, $\times$ is the cross product in R^3 and $\triangle$ is the Laplace operator in R^m. Such a model is called the inhomogeneous Heisenberg ferromagnet [10, 11, 14]. Physically, the function f is called the coupling function, and is the continuum of the coupling constant between the neighboring spins. It is known [18] that the tension field of a map ψ into S^2 is $\tau(\psi) = \triangle \psi + |\nabla \psi|^2 \psi$. We can easily see that the right hand side of (2.1) can be expressed as

$$\psi \times (f\tau(\psi) + \nabla f \cdot \nabla \psi) = 0. \tag{2.2}$$

It implies that ψ is a smooth stationary solution of (2.1) if and only if

$$f\tau(\psi) + \nabla f \cdot \nabla \psi = 0, \tag{2.3}$$

i.e., ψ is an f-harmonic map. Consequently, there is a one-to-one correspondence between the set of the stationary solutions of the inhomogeneous Heisenberg spin system (2.1) on the domain

Ω and the set of f-harmonic maps from Ω into S^2. The inhomogeneous Heisenberg spin system (2.1) is also called inhomogeneous Landau-Lifshitz system (cf. [23, 24, 19]).

2.2 f-harmonic maps

Let $f : (M_1, g) \to (0, \infty)$ be a smooth function. f-harmonic maps which generalize harmonic maps, were introduced in [25], and were studied in [12, 13, 19, 24] recently. Let $\psi : (M_1,\ g) \to (M_2,\ h)$ be a smooth map from an m-dimensional Riemannian manifold $(M_1,\ g)$ into an n-dimensional Riemannian manifold $(M_2,\ h)$. A map $\psi : (M_1, g) \to (M_2, h)$ is $f-harmonic$ if and only if ψ is a critical point of the f-energy

$$E_f(\psi) = \frac{1}{2}\int_{M_1} f|d\psi|^2 dv.$$

In terms of the Euler-Lagrange equation, ψ is $f-harmonic$ if and only if the $f-tension\,field$

$$\tau_f(\psi) = f\tau(\psi) + d\psi(grad\, f) = 0, \tag{2.4}$$

where $\tau(\psi) = Trace_g Dd\psi$ is the tension field of ψ. In particular, when $f = 1$, $\tau_f(\psi) = \tau(\psi)$.

Let $F : [0, \infty) \to [0, \infty)$ be a C^2 function such that $F' > 0$ on $(0, \infty)$. F-harmonic maps between Riemannian manifolds were introduced in [1, 2]. For a smooth map $\psi : (M_1,\ g) \to (M_2, h)$ of Riemannian manifolds, the F-energy of ψ is defined by

$$E_F(\psi) = \int_{M_1} F(\frac{|d\psi|^2}{2})dv. \tag{2.5}$$

When $F(t) = t$, $\frac{(2t)^{p/2}}{p}(p \geq 4)$, $(1 + 2t)^\alpha$ ($\alpha > 1$, dim M=2), and e^t, they are the energy, the p-energy, the α-energy of Sacks-Uhlenbeck [27], and the exponential energy, respectively. A map ψ is F-harmonic iff ψ is a critical point of the F-energy functional. In terms of the Euler-Lagrange equation, $\psi : M_1 \to M_2$ is an $F-harmonic$ map iff the F-tension field

$$\tau_F(\psi) = F'(\frac{|d\psi|^2}{2})\tau(\psi) + \psi_*\left\{grad(F'(\frac{|d\psi|^2}{2}))\right\} = 0. \tag{2.6}$$

Prposition 2.1. *If $\psi : (M_1,\ g) \to (M_2,\ h)$ an F-harmonic map without critical points (i.e., $|d\psi_x| \neq 0$ for all $x \in M_1$), then it is an f-harmonic map with $f = F'(\frac{|d\psi|^2}{2})$. In particular, a p-harmonic map without critical points is an f-harmonic map with $f = |d\psi|^{p-2}$.*

Proof. It follows from (2.4) and (2.6) immediately.

Prposition 2.2 [15, 25]. *A map $\psi : (M_1^m,\ g) \to (M_2^n,\ h)$ is $f-harmonic$ if and only if $\psi : (M_1^m, f^{\frac{2}{m-2}} g) \to (M_2^n, h)$ is a harmonic map.*

3 f-biharmonic maps

Let $f : (M_1, g) \to (0, \infty)$ be a smooth function. f-biharmonic maps between Riemannian manifolds were first studied by Ouakkas, Nasri and Djaa [26] in 2010, which generalized biharmonic maps by Jiang [20, 21]. An f-biharmonic map $\psi : (M_1, g) \to (M_2, h)$ between Riemannian manifolds is the critical point of the f-bienergy functional

$$(E_2)_f(\psi) = \frac{1}{2}\int_{M_1} ||\tau_f(\psi)||^2 dv, \tag{3.1}$$

where the f-tension field $\tau_f(\psi) = f\tau(\psi) + d\psi(grad\, f)$. In terms of Euler-Lagrange equation, ψ is f-biharmonic if and only if the $f - bitension\, field$ of ψ

$$(\tau_2)_f(\psi) = (-)\triangle_2^f \tau_f(\psi)(-)fR'(\tau_f(\psi),\, d\psi)d\psi = 0, \tag{3.2}$$

where

$$\begin{aligned} \triangle_2^f \tau_f(\psi) &= D^\psi f D^\psi \tau_f(\psi) - f D^\psi{}_D \tau_f(\psi) \\ &= \sum_{i=1}^{m} (D^\psi{}_{e_i} f D\psi_{e_i} \tau_f(\psi) - f D^\psi_{D_{e_i} e_i} \tau_f(\psi)). \end{aligned}$$

Here, $\{e_i\}_{1 \le i \le m}$ is an orthonormal frame at a point in M_1, and R' is the Riemannian curvature of M_2. There is a + or - sign convention in (3.2), and we take + sign in the context for simplicity. In particular, if $f = 1$, then $(\tau_2)_f(\psi) = \tau_2(\psi)$, the bitension field of ψ.

Theorem 3.1. *If $\psi : (M_1,\, g) \to (M_2,\, h)$ is a f-biharmonic map ($f \neq 1$) from a compact Riemannian manifold M_1 into a Riemannian manifold M_2 with non-positive curvature satisfying*

$$f D_{e_i} D_{e_i} \tau_f(\psi) - D f D \tau_f(\psi) \ge 0, \tag{3.3}$$

then ψ is f-harmonic.

Proof. Since $\psi : M_1 \to M_2$ is f-biharmonic, it follows from (3.2) that

$$(\tau_2)_f(\psi) = D^\psi f D^\psi \tau_f(\psi) - f D^\psi_D \tau_f(\psi) + fR'(\tau_f(\psi), d\psi)d\psi = 0. \tag{3.4}$$

Suppose that the compact supports of $\frac{\partial \psi_t}{\partial t}$ and $\nabla_{e_i} \frac{\partial \psi_t}{\partial t}$ ($\{\psi_t\} \in C^\infty(M_1 \times [0,1], M_2)$ is a one parameter family of maps with $\psi_0 = \psi$) are contained in the interior of M. We compute

$$\begin{aligned} \frac{1}{2} f \triangle ||\tau_f(\psi)||^2 &= f < D_{e_i}\tau_f(\psi), D_{e_i}\tau_f(\psi) > + f < D^* D \tau_f(\psi), \tau_f(\psi) > \\ &= f < D_{e_i}\tau(\psi), D_{e_i}\tau(\psi) > + f < D_{e_i} D_{e_i} \tau_f(\psi) - D_{D_{e_i e_i}} \tau_f(\psi)), \tau_f(\psi) > \\ &= f < D_{e_i}\tau(\psi), D_{e_i}\tau_f(\psi) > + < f D_{e_i} D_{e_i} \tau_f(\psi) \\ &- D f D \tau_f(\psi) + D f D \tau_f(\psi) - f D_{D_{e_i e_i}} \tau_f(\psi),\, \tau_f(\psi) > \\ &= f < D_{e_i}\tau(\psi), D_{e_i}\tau(\psi) > + < f D_{e_i} D_{e_i} \tau_f(\psi) \\ &- D f D \tau_f(\psi) - f(R'(d\psi,\, d\psi)\tau(\psi), \tau(\psi) > \ge 0, \end{aligned} \tag{3.5}$$

($D^* D = DD - D_D$ [20]) by (3.3), (3.4), $f > 0$ and $R' \le 0$. It implies that

$$\frac{1}{2} \triangle ||\tau_f(\psi)||^2 \ge 0.$$

By applying the Bochner's technique, we know that $||\tau_f(\psi)||^2$ is constant and have

$$D_{e_i}\tau_f(\psi)=0, \forall i=1,2,...m.$$

It follows from Eells-Lemaire [15] that $\tau_f(\psi)$=0, i.e., ψ is f-harmonic on M_1. □

In particular, if $f=1$ and $\psi: M_1 \to M_2$ is a biharmonic map from a compact Riemannian M_1 manifold into a Riemannian manifold M_2 with non-positive curvature, then the condition (3.3) is not required and we arrive at the following corollary.

Corollary 3.2 [20]. *If $\psi:(M_1,\, g)\to(M_2,\, h)$ is a biharmonic map from a compact Riemannian M_1 manifold into a Riemannian manifold M_2 with non-positive curvature, then ψ is harmonic.*

Proof. When $f=1$ and $\psi: M_1\to M_2$ is a biharmonic map from a compact Riemannian M_1 manifold into a Riemannian manifold M_2 with non-positive curvature, (3.2) becomes

$$\tau_2(\psi)=D^*D\tau(\psi)+R'(\tau(\psi), d\psi)d\psi=0.$$

The first identity of (3.5) implies that

$$\begin{aligned}\frac{1}{2}\triangle||\tau(\psi)||^2 &= < D_{e_i}\tau(\psi), D_{e_i}\tau(\psi) > + < D^*D\tau(\psi), \tau(\psi) > \\ &= < D_{e_i}\tau(\psi), D_{e_i}\tau(\psi) > - < R'(d\psi,\, d\psi)\tau(\psi), \tau(\psi) > \geq 0\end{aligned}$$

$(D^*D=DD-D_D)$, since ψ is biharmonic, and M_2 is a Riemannian manifold with non-positive curvature R'. It follows from the similar arguments as Theorem 3.1 that ψ is harmonic. □

It is well-known from [18] that if $\psi:(M_1,\, g)\to(M_2,\, h)$ is a harmonic map of two Riemannian manifolds and $\phi:(M_2,h)\to(M_3,k)$ is totally geodesic of two Riemannian manifolds, then $\phi\circ\psi:(M_1,\, g)\to(M_3,\, k)$ is harmonic. However, if $\psi:(M_1,\, g)\to(M_2,\, h)$ is an f-biharmonic map, and $\phi:(M_2,h)\to(M_3,k)$ is totally geodesic, then $\phi\circ\psi:(M_1,\, g)\to(M_3,\, k)$ is not necessarily an f-biharmonic map. We obtain the following theorem instead.

Theorem 3.3. *If $\tau_f(\psi)$ is a Jacobi field for a smooth map $\psi:(M_1,g)\to(M_2,h)$ of two Riemannian manifolds, and $\phi:(M_2,h)\to(M_3,k)$ is a totally geodesic map of two Riemannian manifolds, then $\tau_f(\phi\circ\psi)$ is a Jacobi field.*

Proof. Let $D, D', \bar{D}, \bar{D}', \bar{D}'', \hat{D}, \hat{D}', \hat{D}''$ be the connections on $TM_1, TM_2, \psi^{-1}TM_2, \phi^{-1}TM_3$, $(\phi\circ\psi)^{-1}TM_3, T^*M_1\otimes\psi^{-1}TM_2, T^*M_2\otimes\phi^{-1}TM_3,\ T^*M_1\otimes(\phi\circ\psi)^{-1}TM_3$, respectively. We first have

$$\bar{D}''_X d(\phi\circ\psi)(Y)=(\hat{D}'_{d\psi(X)}d\phi)d\psi(Y)+d\phi\circ\bar{D}_X d\psi(Y), \tag{3.6}$$

$\forall\, X,\, Y\ \in\Gamma(TM_1)$. We also have

$$R^{M_3}(d\phi(X'), d\phi(Y'))d\phi(Z')=R^{\phi^{-1}TM_3}(X',Y')d\phi(Z'), \tag{3.7}$$

$\forall\, X',\, Y',\, Z'\ \in\Gamma(TM_2)$.

It is well-known from [18] that the tension field of the composition $\phi\ \circ\psi$ is given by

$$\tau(\phi\circ\psi)=d\phi(\tau(\psi))+Tr_gDd\phi(d\psi,d\psi)=d\phi(\tau(\psi)),$$

since ϕ is totally geodesic. Then the f-tension field of the composition of $\phi\ \circ\psi$ is

$$\tau_f(\psi\circ\phi)=d\phi(\tau_f(\psi))+fTr_gDd\phi(d\psi,d\psi)=d\psi(\tau_f(\psi)),$$

since ϕ is totally geodesic. Note that $\{e_i\}_{i=1}^m$ is an orthonormal frame at a point in M_1, and let $\bar{D}^*\bar{D}=\bar{D}_{e_k}\bar{D}_{e_k}-\bar{D}_{D_{e_k}e_k}$ and $\bar{D}''^*\bar{D}''=\bar{D}''_{e_k}\bar{D}''_{e_k}-\bar{D}''_{D_{e_k}e_k}$. Thus we arrive at

$$\begin{aligned}\bar{D}''^*\bar{D}''\tau_f(\phi\circ\psi) &= \bar{D}''^*\bar{D}''(d\phi\circ\tau_f(\psi))\\ &= \bar{D}''_{e_k}\bar{D}''_{e_k}(d\phi\circ\tau_f(\psi))-\bar{D}''_{D_{e_k}e_k}(d\phi\circ\tau_f(\psi)).\end{aligned}\tag{3.8}$$

We derive from (3.6) that

$$\begin{aligned}\bar{D}''_{e_k}(d\phi\circ\tau_f(\psi)) &= (\hat{D}'_{\hat{D}_{e_j}d\psi(e_k)}d\phi)(\tau_f(\psi))+d\phi\circ\bar{D}_{e_k}(\tau_f(\psi))\\ &= d\phi\circ\bar{D}_{e_k}\tau_f(\psi),\end{aligned}$$

since ϕ is totally geodesic. Therefore, we have

$$\bar{D}''_{e_k}\bar{D}''_{e_k}(d\phi\circ\tau_f(\psi))=\bar{D}''_{e_k}(d\phi\circ\bar{D}_{e_k}\tau_f(\psi))=d\phi\circ\bar{D}_{e_k}\bar{D}_{e_k}\tau_f(\psi),\tag{3.9}$$

and

$$\bar{D}''_{D_{e_k}e_k}(d\phi\circ\tau(\psi))=d\phi\circ\bar{D}_{D_{e_k}e_k}\tau_f(\psi).\tag{3.10}$$

Substituting (3.9), (3.10) into (3.8), we deduce

$$\bar{D}''^*\bar{D}''\tau_f(\phi\circ\psi)=d\phi\circ\bar{D}^*\bar{D}\tau_f(\psi).\tag{3.11}$$

It follows from (3.7) that

$$\begin{aligned}R^{M_3} &\quad (d(\phi\circ\psi)(e_i),\tau_f(\phi\circ\psi))d(\phi\circ\psi)(e_i)\\ &= R^{\phi^{-1}TM_3}(d\psi(e_i),\tau_f(\psi))d\phi(d\psi(e_i))\\ &= d\phi\circ R^{M_2}(d\psi(e_i),\tau_f(\psi))d\psi(e_i).\end{aligned}\tag{3.12}$$

By (3.11) and (3.12) we obtain

$$\begin{aligned}\bar{D}''^*\bar{D}''\tau_f(\phi\circ\psi) &+ R^{M_3}(d(\phi\circ\psi)(e_i),\tau_f(\phi\circ\psi))d(\phi\circ\psi)(e_i)\\ &= d\phi\circ[\bar{D}^*\bar{D}\tau_f(\psi)+R^{M_2}(d\psi(e_i),\tau_f(\psi))d\psi(e_i)].\end{aligned}\tag{3.13}$$

Consequently, if $\tau_f(\psi)$ is a Jacobi field, then $\tau_f(\phi\circ\psi)$ is a Jacobi field. □

Corollary 3.4 [8]. *If $\psi:(M_1,g)\to(M_2,h)$ is a biharmonic map between two Riemannian manifolds and $\phi:(M_2,h)\to(M_3,k)$ is totally geodesic, then $\phi\circ\psi:(M_1,g)\to(M_3,k)$ is a biharmonic map.*

Proof. If $f = 1$ and $\psi : (M_1, g) \to (M_2, h)$ is a biharmonic map of two Riemannian manifolds, then $\tau_f(\psi) = \tau(\psi)$ is a Jacobi field and (3.13) becomes

$$\begin{aligned} \bar{D}''^{*}\bar{D}''\tau(\phi\circ\psi) &+ R^{M_3}(d(\phi\circ\psi)(e_i), \tau(\phi\circ\psi))d(\phi\circ\psi)(e_i) \\ &= d\phi\circ[\bar{D}^{*}\bar{D}\tau(\psi) + R^{M_2}(d\psi(e_i), \tau(\psi))d\psi(e_i)], \end{aligned}$$

i.e., $\tau_2(\phi\circ\psi) = d\phi\circ(\tau_2(\psi))$, where $\tau_2(\psi)$ is the bi-tension field of ψ. Hence, the result follows immediately.

4 Stress f-bienergy tensors

Let $\psi : (M_1, g) \to (M_2, h)$ be a smooth map between two Riemannian manifolds. The stress energy tensor [3] is defined by

$$S(\psi) = e(\psi)g - \psi^{*}h,$$

where $e(\psi) = \frac{|d\psi|^2}{2}$. Thus we have $divS(\psi) = -<\tau(\psi), d\psi>$. Hence, if ψ is harmonic, then ψ satisfies the conservation law for S (i.e., $div\, S(\psi) = 0$). In [26], the stress f-energy tensor of the smooth map $\psi : M_1 \to M_2$ was similarly defined as

$$S^f(\psi) = fe(\psi)g - f\psi^{*}h,$$

and they obtained

$$div\, S^f(\psi) = -<\tau_f(\psi), d\psi> + e(\psi)df.$$

In this case, an f-harmonic map usually does not satisfy the conservation law for S^f. In particular, setting $f = F'(\frac{d\psi|^2}{2})$, then $S^f(\psi) = F'(\frac{d\psi|^2}{2})e(\psi)g - F'(\frac{d\psi|^2}{2})\psi^{*}h$. It is different than following [3] to define $S^F(\psi) = F(\frac{|d\psi|^2}{2})g - F'(\frac{d\psi|^2}{2})\psi^{*}h$, and we have

$$div\, S^F(\psi) = -<\tau_F(\psi), d\psi>.$$

It implies that if $\psi : M_1 \to M_2$ is an F-harmonic map between Riemannian manifolds, then it satisfies the conservation law for S^F (cf. [1]).

The stress bienergy tensors and the conservation laws of biharmonic maps between Riemannian manifolds were first studied in [22] in 1987. Following Jiang's notion, we define the stress f-bienergy tensor of a smooth map as follows.

Definition 4.1. Let $\psi : (M_1, g) \to (M_2, h)$ be a smooth map between two Riemannian manifolds. The stress f-bienergy tensor of ψ is defined by

$$\begin{aligned} S_2^f(X, Y) &= \frac{1}{2}|\tau_f(\psi)|^2 <X, Y> + <d\psi, D(\tau_f(\psi)><X, Y> \\ &- <d\psi(X), D_Y\tau_f(\psi)> - <d\psi(Y), D_X\tau_f(\psi)>, \end{aligned} \tag{4.1}$$

$\forall X, Y \in \Gamma(TM_1)$.

Note that if $\psi : (M_1, g) \to (M_2, h)$ is an f-biharmonic map between two Riemannian manifolds, then ψ does not necessarily satisfy the conservation law for the stress f-bienrgy tensor S_2^f. Instead, we obtain the following theorem.

Theorem 4.2. *If $\psi : (M_1, g) \to (M_2, h)$ be a smooth map between two Riemannian manifolds, then we have*

$$div\, S_2^f(Y) = (-) < J_{\tau_f(\psi)}(Y),\ d\psi(Y) >,\ \ \forall Y \in \Gamma(TM_1), \tag{4.2}$$

where $J_{\tau_f(\psi)}$ is the Jacobi field of $\tau_f(\psi)$.

Proof. For the map $\psi : M_1 \to M_2$ between two Riemannian manifolds, set $S_2^f = K_1 + K_2$, where K_1 and K_2 are (0, 2)-tensors defined by

$$\begin{aligned} K_1(X,Y) &= \frac{1}{2}|\tau_f(\psi)|^2 < X,Y > + < d\psi, D\tau_f(\psi) >< X,Y >, \\ K_2(X,Y) &= - < d\psi(X), D_Y\tau_f(\psi) > - < d\psi,\, D_X\tau_f(\psi) > . \end{aligned}$$

Let $\{e_i\}$ be the geodesic coordinates at a point $a \in M_1$, and write $Y = Y^i e_i$ at the point a. We first compute

$$\begin{aligned} div\, K_1(Y) &= \sum_i (D_{e_i}K_1)(e_i, Y) = \sum_i (e_i(K_1(e_i, Y) - K_1(e_i,\, D_{e_i}Y)) \\ &= \sum_i (e_i(\frac{1}{2}|\tau_f(\psi)|^2 Y^i + \sum_k < d\psi(e_k),\, D_{e_k}\tau_f(\psi) > Y^i) \\ &- \frac{1}{2}|\tau_f(\psi)|^2 Y^i e_i - \sum_k < d\psi(e_k),\, D_{e_k}\tau_f(\psi) > Y^i e_i)) \\ &= < D_Y\tau_f(\psi), \tau_f(\psi) > + \sum_i < d\psi(Y, e_i), D_{e_i}\tau_f(\psi) > \\ &+ \sum_i < d\psi(e_i),\, D_Y D_{e_i}\tau_f(\psi) > \\ &= < D_Y\tau_f(\psi), \tau_f(\psi) > + trace < Dd\psi(Y, .),\, D.\tau_f(\psi) > \\ &+ trace < d\psi(.), D^2\tau_f(\psi)(Y, .) > . \end{aligned} \tag{4.3}$$

We then compute

$$\begin{aligned} div\, K_2(Y) &= \sum_i (D_{e_i}K_2)(e_i, Y) = \sum_i (e_i(K_2(e_i, Y) - K_2(e_i, D_{e_i}Y)) \\ &= - < D_Y\tau_f(\psi),\, \tau_f(\psi) > - \sum_i < Dd\psi(Y, e_i),\, D_{e_i}\tau_f(\psi) > \\ &- \sum_i < d\psi(e_i),\, D_{e_i}D_Y\tau_f(\psi) - D_{D_{e_i}Y}\tau_f(\psi) > + < d\psi(Y), \triangle\tau_f(\psi) > \\ &= - < D_Y\tau_f(\psi), \tau_f(\psi) > - trace < Dd\psi(Y, .),\, D_.\tau_f(\psi) > \\ &- trace < d\psi(.), D^2\tau_f(\psi)(., Y) > + < d\psi(Y), \triangle\tau_f(\psi) > . \end{aligned} \tag{4.4}$$

Adding (4.3) and (4.4), we arrive at

$$\begin{aligned} divS_2^f(Y) &= (-) < d\psi(Y),\ \triangle\tau_f(\psi) + \sum_i < d\psi(e_i), R'(Y,e_i)\tau_f(\psi) > \\ &= (-) < J_{\tau_f(\psi)}(Y),\ d\psi(Y) >, \end{aligned} \tag{4.5}$$

where $J_{\tau_f(\psi)}$ is the Jacobi field of $\tau_f(\psi)$. □

Corollary 4.3. *If $\tau_f(\psi)$ is a Jacobi field for a map $\psi : M_1 \to M_2$, then it satisfies the conservation law (i.e., $div\, S_2^f = 0$) for the stress f-bienergy tensor S_2^f.*

Corollary 4.4. [22]. *If $\psi : (M_1,\, g) \to (M_2,\, h)$ is biharmonic between two Riemannian manifolds, then it satisfies the conservation law for stress bienergy tensor S_2*

Proof. If $f = 1$ and $\psi : (M_1,\, g) \to (M_2,\, h)$ is biharmonic, then (4.5) yields to

$$\begin{aligned} div\, S_2(Y) &= (-) < d\psi,\ \triangle\tau(\psi) + \sum_i (d\psi(e_i), R'(Y,X_i)\tau(\psi) > \\ &= (-) < J_{\tau(\psi)}(Y),\ d\psi(Y) > \\ &= (-) < \tau_2(\psi),\ d\psi(Y) >, \end{aligned}$$

where $\tau_2(\psi)$ is the bi-tension field of ψ (i.e., $\tau(\psi)$ is a Jacobi field). Hence, we can conclude the result. □

Proposition 4.5. *Let $\psi : (M_1,\, g) \to (M_2, h)$ be a submersion such that $\tau_f(\psi)$ is basic, i.e., $\tau_f(\psi) = W \circ \psi$ for $W \in \Gamma(TM_2)$. Suppose that W is Killing and $|W|^2 = c^2$ is non-zero constant. If M_1 is non-compact, then $\tau_f(\psi)$ is a non-trivial Jacobi field.*

Proof. Since $\tau_f(\psi)$ is basic,

$$\begin{aligned} S_2^f(X,Y) &= [\frac{c^2}{2} + < d\psi, D\tau_f(\psi) >](X,Y) - < d\psi(X),\ D_Y\tau_f(\psi) > \\ &- \ < d\psi(Y),\ D_X\tau_f(\psi) >, \end{aligned} \tag{4.6}$$

where $X, Y \in \Gamma(TM_1)$. Let a be a point in M_1 with the orthonormal frame $\{e_i\}_{i=1}^m$ such that $\{e_j\}_{j=1}^n$ are in $T_a^H M_1 = (T_a^V M_1)^\perp$ and $\{e_k\}_{k=n+1}^m$ are in $T_a^V M_1 = ker\, d\psi(a)$. Because W is Killing, we have

$$\begin{aligned} < d\psi,\ D\tau_f(\psi) > (a) &= \sum_j < d\psi_a(e_j),\ D_{e_j}\tau_f(\psi) > + \sum_k < d\psi_a(e_k),\ D_{e_k}\tau_f(\psi) > \\ &= \sum_j < d\psi_a(e_j),\ D^{M_2}_{d\psi_a(e_j)} W >= 0. \end{aligned} \tag{4.7}$$

Therefore,

$$\begin{aligned} S_2^f(a)(X,Y) &= \frac{c^2}{2}(X,Y) + < d\psi_a(X), D^{M_2}_{d\psi_a(Y)} W > \\ &- \ < d\psi_a(Y),\ D^{M_2}_{d\psi_a(X)} W >= \frac{c^2}{2}(X,Y). \end{aligned}$$

If M_1 is not compact, $S_2^f = \frac{c^2}{2} g$ is divergence free and $\tau_f(\psi)$ is a non-trivial Jacobi field due to $c \neq 0$. □

Proposition 4.6. If $\psi : (M_1^2, g) \to (M_2, h)$ is a map from a surface with $S_2^f = 0$, then ψ is f-harmonic.

Proof. Since $S_2^f = 0$, it implies

$$\begin{aligned} 0 = trace S_2^f &= |\tau_f(\psi)|^2 + 2 < D\tau_f(\psi), d\psi > -2 < D\tau_f(\psi), d\psi > \\ &= |\tau_f(\psi)|^2. \end{aligned}$$

Proposition 4.7. *If* $\psi : (M_1^m, g) \to (M_2, h)\ (m \neq 2)$ *with* $S_2^f = 0$, *then*

$$\begin{aligned} \tfrac{1}{m-2} & |\tau_f(\psi)|^2(X,Y) + < D_X\tau_f(\psi), d\psi(Y) > \\ + & < D_Y\tau_f(\psi), d\psi(X) >= 0, \end{aligned} \tag{4.8}$$

$\forall\, X,\, Y \in \Gamma T(M_1)$.

Proof. Suppose that $S_2^f = 0$, it implies trace $S_2^f = 0$. Therefore,

$$< D\tau_f(\psi), d\psi >= -\frac{m}{2(m-2)}|\tau_f(\psi)|^2 (m \neq 2). \tag{4.9}$$

Substituting it into the definition of S_2^f, we arrive at

$$\begin{aligned} 0 &= S_2^f(X,Y) = -\frac{1}{m-2}|\tau_f(\psi)|^2(X,Y) \\ &- < D_X\tau_f(\psi >, d\psi(Y)) - < D_Y\tau_f(\psi), d\psi(X) > . \end{aligned} \tag{4.10}$$

Corollary 4.8. *If* $\psi : (M_1, g) \to (M_2, h)\ (m > 2)$ *with* $S_1^f = 0$ *and* $rank\,\psi \leq m-1$, *then* ψ *is f-harmonic.*

Proof. Since $rank\,\psi(a) \leq m-1$, for a point $a \in M_1$ there exists a unit vector $X_a \in Ker\, d\psi_a$. Letting $X = Y = X_a$, (4.8) gives to $\tau_f(\psi) = 0$.

Corollary 4.9. *If* $\psi : (M_1, g) \to (M_2, h)$ *is a submersion (*$m > n$*) with* $S_2^f = 0$, *then* ψ *is f-harmonic.*

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