XIV International conference
Geometry Integrabiliy and Quantization
Varna, 8-13 June 2012, Bulgaria

Reductions and New types of Integrable models in two and more dimensions

V. S. Gerdjikov
Institute for Nuclear Research and Nuclear Energy
Sofia, Bulgaria

Based on:

- V. S. Gerdjikov. $\mathbb{Z}_2$-reductions of spinor models in two dimensions. Submitted to "Physics of Atomic Nuclei" (2012)
- V. S. Gerdjikov. Riemann-Hilbert Problems with canonical normalization and families of commuting operators. Submitted to Pliska Stud. Math. Bulgar. **arXiv:1204.2928v1 [nlin.SI]**.

PLAN

- Reductions of spinor models
- RHP with canonical normalization
- Reductions of polynomial bundles
- New N-wave equations ($k = 2$) in 2 and more dimensions
- Conclusions and open questions

1 Integrable 2-dimensional spinor models

The integrability of the 2-dimensional versions of the Nambu–Jona-Lasinio–Vaks–Larkin (NJLVL), Gross–Neveu model (GN) and Zakharov and Mikhailov – see Zakharov and Mikhailov (1981). NJLVL models are related to $su(N)$ algebras, Gross–Neveu models – to $sp(N)$ and Zakharov–Mikhailov (ZM) models – to $so(N)$.

Lax representations of these models:

$$\Psi_\xi = U(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda), \qquad \Psi_\eta = U(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda),$$

$$U(\xi,\eta,\lambda) = \frac{U_1(\xi,\eta)}{\lambda - a}, \qquad V(\xi,\eta,\lambda) = \frac{V_1(\xi,\eta)}{\lambda + a},$$

where $\eta = t + x$, $\xi = t - x$ and a is a real number.

We also impose the $\mathbb{Z}_2$-reduction:

$$U^\dagger(x,t,\lambda) = -U(x,t,\lambda^*), \qquad V^\dagger(x,t,\lambda) = -V(x,t,\lambda^*).$$

The compatibility condition of the above linear problems reads:

$$U_\eta - V_\xi + [U,V] = 0,$$

which is equivalent to

$$U_{1,\eta} + \frac{1}{2a}[U_1, V_1(\xi,\eta)] = 0, \qquad V_{1,\xi} \quad - \frac{1}{2a}[V_1, U_1(\xi,\eta)] = 0.$$

i) Nambu-Jona-Lasinio-Vaks-Larkin models. Here we choose $\mathfrak{g} \simeq su(N)$. Then $\psi(\xi,\eta)$ and $\phi(\xi,\eta)$ are elements of the group $SU(N)$ and by definition $\hat{\psi}(\xi,\eta) = \psi^\dagger(\xi,\eta)$, $\hat{\phi}(\xi,\eta) = \phi^\dagger(\xi,\eta)$. Next we choose $J = \mathrm{diag}\,(1,0,\dots,0)$ and as a result only the first columns $\phi^{(1)}$, $\psi^{(1)}$ and the first rows $\hat{\phi}^{(1)}$, $\hat{\psi}^{(1)}$ enter into the systems. If we introduce the notations:

$$\phi_\alpha(\xi,\eta) = \phi^{(1)}_{\alpha,1}, \qquad \psi_\alpha(\xi,\eta) = \psi^{(1)}_{\alpha,1},$$

then the explicit form of the system is:

$$\frac{\partial \phi_\alpha}{\partial \eta} = \frac{i}{2a}\psi_\alpha \sum_{\beta=1}^{N} \psi^*_\beta \phi_\beta,$$

$$\frac{\partial \psi_\alpha}{\partial \xi} = \frac{i}{2a}\phi_\alpha \sum_{\beta=1}^{N} \phi^*_\beta \psi_\beta.$$

The functional of the action is:

$$A_{\mathrm{NJLVL}} = \int_{-\infty}^{\infty} dx\, dt \left(i \sum_{\alpha=1}^{N} \left(\phi^*_\alpha \frac{\partial \phi_\alpha}{\partial \eta} + \psi^*_\alpha \frac{\partial \psi_\alpha}{\partial \xi} \right) - \frac{1}{2a} \left| \sum_{\alpha=1}^{N} (\psi^*_\alpha \phi_\alpha) \right|^2 \right).$$

ii) Gross-Nevew models. Here we choose $\mathfrak{g} \simeq sp(2N,\mathbb{R})$; then $\psi(\xi,\eta)$ and $\phi(\xi,\eta)$ are elements of the group $\mathfrak{G} \simeq SP(2N,\mathbb{R})$. Following [?] we use the standard definition of symplectic group elements:

$$\hat{\psi}(\xi,\eta) = \mathfrak{J}\psi^T(\xi,\eta)\hat{\mathfrak{J}}, \qquad \hat{\phi}(\xi,\eta) = \mathfrak{J}\phi^T(\xi,\eta)\hat{\mathfrak{J}}, \qquad \mathfrak{J} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Then the corresponding Lie algebraic elements acquire the following block-matrix structure:

$$U_1(\xi,\eta) = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where A, B, C are arbitrary real $N \times N$ matrices. Next we choose

$$J = \begin{pmatrix} 0 & B_0 \\ 0 & 0 \end{pmatrix}, \qquad B_0 = \mathrm{diag}\,(1, 0, \ldots, 0, 0).$$

As a consequence again only the first columns $\phi^{(1)}$, $\psi^{(1)}$ and the first rows $\hat{\phi}^{(1)}$, $\hat{\psi}^{(1)}$ enter into the systems. If we introduce the N-component complex vectors:

$$\phi_\alpha(\xi,\eta) = \frac{1}{2}(\phi^{(1)}_{\alpha,1} + i\phi^{(1)}_{N+\alpha,1}), \qquad \psi_\alpha(\xi,\eta) = \frac{1}{2}(\psi^{(1)}_{\alpha,1} + i\psi^{(1)}_{N+\alpha,1})$$

then the explicit form of the system is:

$$\frac{\partial\phi_\alpha}{\partial\eta} = \frac{i}{a}\psi_\alpha\sum_{\beta=1}^{N}(\psi_\beta\phi_\beta^* - \psi_\beta^*\phi_\beta),$$

$$\frac{\partial\psi_\alpha}{\partial\xi} = -\frac{i}{a}\phi_\alpha\sum_{\beta=1}^{N}(\phi_\beta\psi_\beta^* - \phi_\beta^*\psi_\beta).$$

The functional of the action is:

$$A_{\rm GN} = \int_{-\infty}^{\infty} dx\, dt\, \left(i\sum_{\alpha=1}^{N}\left(\phi_\alpha^*\frac{\partial\phi_\alpha}{\partial\eta} + \psi_\alpha^*\frac{\partial\psi_\alpha}{\partial\xi}\right) - \frac{1}{2a}\left(\sum_{\alpha=1}^{N}(\psi_\alpha^*\phi_\alpha - \phi_\alpha^*\psi_\alpha)\right)^2\right).$$

iii) Zakharov–Mikhailov models. Now we choose $\mathfrak{g} \simeq so(N,\mathbb{R})$; then $\psi(\xi,\eta)$ and $\phi(\xi,\eta)$ are elements of the group $\mathfrak{G} \simeq SO(N,\mathbb{R})$. Following [**?**] we use the standard definition of orthogonal group elements:

$$\hat{\psi}(\xi,\eta) = \psi^T(\xi,\eta), \qquad \hat{\phi}(\xi,\eta) = \phi^T(\xi,\eta).$$

Now we choose

$$J = E_{1,N} - E_{N,1},$$

where the $N \times N$ matrices E_{kp} are defined by $(E_{kp})_{nm} = \delta_{kn}\delta_{pm}$. As a consequence now the first and the last columns $\phi^{(1)}, \phi^{(N)}$, $\psi^{(1)}, \psi^{(N)}$ and the first and the last rows $\hat{\phi}^{(1)}, \hat{\phi}^{(N)}$, $\hat{\psi}^{(1)}, \hat{\psi}^{(N)}$ enter into the systems. If we introduce the N-component complex vectors:

$$\phi_\alpha(\xi,\eta) = \frac{1}{2}(\phi^{(1)}_{\alpha,1} + i\phi^{(N)}_{\alpha,N}), \qquad \psi_\alpha(\xi,\eta) = \frac{1}{2}(\psi^{(1)}_{\alpha,1} + i\psi^{(N)}_{\alpha,N})$$

then the explicit form of the system becomes:

$$i\frac{\partial\psi_\alpha}{\partial\xi} = \frac{i}{a}\sum_{\beta=1}^{N}(\phi^*_\alpha\phi_\beta\psi_\beta - \phi_\alpha\phi^*_\beta)\psi_\beta),$$

$$i\frac{\partial\phi_\alpha}{\partial\eta} = \frac{i}{a}\sum_{\beta=1}^{N}(\psi^*_\alpha\psi_\beta\phi_\beta - \psi_\alpha\psi^*_\beta)\phi_\beta,$$

The functional of the action is:

$$A_{\mathrm{ZM}} = \int_{-\infty}^{\infty} dx\, dt\, \left(i \sum_{\alpha=1}^{N} \left(\phi_\alpha^* \frac{\partial \phi_\alpha}{\partial \eta} + \psi_\alpha^* \frac{\partial \psi_\alpha}{\partial \xi} \right) - \frac{1}{2a} \left(\sum_{\alpha,\beta=1}^{N} (\phi_\alpha^* \phi_\beta - \phi_\beta^* \phi_\alpha)(\psi_\alpha^* \psi_\beta - \psi_\beta^* \psi_\alpha) \right) \right) .$$

2 $\mathbb{Z}_2$-Reductions of the spinor models

Apply the idea of the reduction group – Mikhailov (1980) and obtain new types of spinor models generalizing the previous ones.

Start with the Lax representation:

$$\Psi_\xi = U_{\rm R}(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda), \qquad \Psi_\eta = V_{\rm R}(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda),$$

$$U_{\rm R}(\xi,\eta,\lambda) = \frac{U_1(\xi,\eta)}{\lambda - a} + \frac{CU_1(\xi,\eta)C^{-1}}{\epsilon\lambda^{-1} - a}, \qquad V_{\rm R}(\xi,\eta,\lambda) = \frac{V_1(\xi,\eta)}{\lambda + a} + \frac{CV_1(\xi,\eta)C^{-1}}{\epsilon\lambda^{-1} + a},$$

where $\epsilon = \pm 1$, $a \neq 1$ is a real number and C is an involutive automorphism of $\mathfrak{g}$. It satisfy also:

$$U_{\rm R}(\xi,\eta,\lambda) = CU_{\rm R}(\xi,\eta,\epsilon\lambda^{-1})C^{-1}, \qquad V_{\rm R}(\xi,\eta,\lambda) = CV_{\rm R}(\xi,\eta,\epsilon\lambda^{-1})C^{-1},$$

The new Lax representation is:

$$\frac{\partial U_{\rm R}}{\partial \eta} - \frac{\partial V_{\rm R}}{\partial \xi} + [U_{\rm R}, V_{\rm R}] = 0,$$

which is equivalent to

$$U_{1,\eta} + [U_1, V_{\rm R}(\xi,\eta,a)] = 0, \qquad V_{1,\xi} + [V_1, U_{\rm R}(\xi,\eta,-a)] = 0.$$

In the same way as above we get:

i) $\mathbb{Z}_2$-NJLVL models. Here $\mathfrak{G} \simeq SU(N)$ and the system takes the form:

$$i\frac{\partial\vec{\phi}}{\partial\eta} + \frac{1}{2a}\vec{\psi}(\vec{\psi}^{\,\dagger}\vec{\phi}) + \frac{1}{\epsilon a^{-1}+a}C\vec{\psi}(\vec{\psi}^{\,\dagger}\hat{C}\vec{\phi})(\xi,\eta) = 0,$$
$$i\frac{\partial\vec{\psi}}{\partial\xi} + \frac{1}{2a}\vec{\phi}(\vec{\phi}^{\,\dagger}\vec{\psi}) + \frac{1}{\epsilon a^{-1}+a}C\vec{\phi}(\vec{\phi}^{\,\dagger}\hat{C}\vec{\psi})(\xi,\eta) = 0.$$

where $\vec{\psi} = (\psi_{\alpha,1},\ldots,\psi_{\alpha,N})^T$ and $\vec{\phi} = (\phi_{\alpha,1},\ldots,\phi_{\alpha,N})^T$.

For the automorphism C of the $SU(N)$ group we may have

$$\text{a)} \qquad C_N = \operatorname{diag}(\epsilon_1,\epsilon_2,\ldots,\epsilon_N), \qquad \epsilon_j = \pm 1, \qquad \text{b)} \qquad C'_N = \begin{pmatrix} 1 & 0 \\ 0 & C_{N-1} \end{pmatrix}.$$

where C_{N-1} belongs to the Weyl group of $SU(N-1)$ and is such that $C_{N-1}^2 = \mathbb{1}$. These two special choices of C are such that $\lim_{\xi\to\pm\infty} U_R(\xi,\eta) = \lim_{\xi\to\pm\infty} CU_R(\xi,\eta)\hat{C}$.

ii) $\mathbb{Z}_2$-GN models. Here $\mathfrak{G} \simeq SP(2N,\mathbb{R})$. Two typical choices of C are given by:

$$\text{a)} \qquad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix}, \qquad\qquad \text{b)} \qquad C' = \begin{pmatrix} 0 & C_2 \\ C_2 & 0 \end{pmatrix},$$

where $C_1^2 = C_2^2 = \mathbb{1}$.

$$\frac{\partial \vec{\phi}}{\partial \eta} = -\frac{i}{a}\vec{\psi}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right) - \frac{2i}{a+\epsilon a^{-1}} C_1\vec{\psi}\left((\vec{\psi}^{\,\dagger}C_1\vec{\phi}) - (\vec{\phi}^{\,\dagger}C_1\vec{\psi})\right),$$
$$\frac{\partial \vec{\psi}}{\partial \xi} = \frac{i}{a}\vec{\phi}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right) + \frac{2i}{a+\epsilon a^{-1}} C_1\vec{\phi}\left((\vec{\psi}^{\,\dagger}C_1\vec{\phi}) - (\vec{\phi}^{\,\dagger}C_1\vec{\psi})\right).$$

The corresponding action can be written as follows:

$$A_{\mathbb{Z}_2,\mathrm{GNa}} = \int_{-\infty}^{\infty} dx\, dt\, \left(i\left(\vec{\phi}^{\,\dagger}\frac{\partial\vec{\phi}}{\partial\eta} + \vec{\psi}^{\,\dagger}\frac{\partial\vec{\psi}}{\partial\xi}\right) - \frac{1}{2a}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right)^2 \right.$$
$$\left. - \frac{1}{\epsilon a^{-1}+a}\left((\vec{\psi}^{\,\dagger}C_1\vec{\phi}) - (\vec{\phi}^{\,\dagger}C_1\vec{\psi})\right)^2 \right).$$

The second $\mathbb{Z}_2$-reduced GN-system is:

$$\frac{\partial\vec{\phi}}{\partial\eta} = -\frac{i}{a}\vec{\psi}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right) + \frac{2i}{a+\epsilon a^{-1}}C_2\vec{\psi}^{\,*}\left((\vec{\psi}^T C_2\vec{\phi}) + (\vec{\psi}^{\,\dagger}C_2\vec{\phi}^{\,*})\right),$$

$$\frac{\partial\vec{\psi}}{\partial\xi} = \frac{i}{a}\vec{\phi}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right) + \frac{2i}{a+\epsilon a^{-1}}C_2\vec{\phi}^{\,*}\left((\vec{\phi}^T C_2\vec{\psi}) + (\vec{\phi}^{\,\dagger}C_2\vec{\psi}^{\,*})\right).$$

These equations can be obtained from the action:

$$A_{\mathbb{Z}_2,\mathrm{GNb}} = \int_{-\infty}^{\infty} dx\; dt\; \left(i\left(\vec{\phi}^{\,\dagger}\frac{\partial\vec{\phi}}{\partial\eta} + \vec{\psi}^{\,\dagger}\frac{\partial\vec{\psi}}{\partial\xi}\right) - \frac{1}{2a}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right)^2 \right.$$
$$\left. -\frac{1}{\epsilon a^{-1}+a}\left((\vec{\phi}^{\,\dagger}C_2\vec{\psi}^{\,*}) + (\vec{\phi}^T C_2\vec{\psi})\right)^2\right).$$

iii) $\mathbb{Z}_2$-ZM models. Here $\mathfrak{G} \simeq SO(N,\mathbb{R})$. Again we used N-component vectors to cast the $\mathbb{Z}_2$-reduced ZM systems in the form:

$$\frac{\partial\vec{\psi}}{\partial\xi} = \frac{i}{a}\left(\vec{\phi}^{\,*}(\vec{\phi}^T,\vec{\psi}) - \vec{\phi}(\vec{\phi}^{\,\dagger},\vec{\psi})\right) + \frac{2i}{a+\epsilon a^{-1}} C\left(\vec{\phi}^{\,*}(\vec{\phi}^T C\vec{\psi}) - \vec{\phi}(\vec{\phi}^{\,\dagger} C\vec{\psi})\right),$$
$$\frac{\partial\vec{\phi}}{\partial\eta} = \frac{i}{a}\left(\vec{\psi}^{\,*}(\vec{\psi}^T,\vec{\phi}) - \vec{\psi}(\vec{\psi}^{\,\dagger},\vec{\phi})\right) + \frac{2i}{a+\epsilon a^{-1}} C\left(\vec{\psi}^{\,*}(\vec{\psi}^T \hat{C}\vec{\phi}) - \vec{\psi}(\vec{\psi}^{\,\dagger}\hat{C}\vec{\phi})\right),$$

where the involutive automorphism C can be chosen as one of the type:

$$\text{a)} \qquad C = \operatorname{diag}(\epsilon_1,\epsilon_2,\ldots,\epsilon_2,\epsilon_1), \qquad \epsilon_j = \pm 1, \qquad \text{b)} \qquad C' = \begin{pmatrix} 1 & 0 \\ 0 & C_3 \end{pmatrix},$$

with $C_3^2 = \mathbb{1}$. For these choices of C we have $\lim_{\xi\to\pm\infty} U_R(\xi,\eta) = \lim_{\xi\to\pm\infty} C U_R(\xi,\eta)\hat{C}$.

The action for the reduced ZM models is provided by:

$$A_{\mathbb{Z}_2,\mathrm{ZM}} = \int_{-\infty}^{\infty} dx\, dt\ \left(i\left(\vec{\phi}^{\,\dagger} \frac{\partial \vec{\phi}}{\partial \eta} + \vec{\psi}^{\,\dagger} \frac{\partial \vec{\psi}}{\partial \xi} \right) \right.$$
$$+\frac{1}{a}\Big((\vec{\psi}^{\,\dagger}, \vec{\phi}^{\,*})(\vec{\phi}^{T}, \vec{\psi}) - (\vec{\phi}^{\,\dagger}, \vec{\psi})(\vec{\psi}^{\,\dagger}, \vec{\phi}) \Big)$$
$$\left. + \frac{2}{\epsilon a^{-1} + a} \Big((\vec{\psi}^{\,\dagger} C \vec{\phi}^{\,*})(\vec{\phi}^{T} C \vec{\psi}) - (\vec{\phi}^{\,\dagger} C \vec{\psi})(\vec{\psi}^{\,\dagger} C \vec{\phi}) \Big) \right).$$

3 RHP with canonical normalization

$$\xi^{+}(\vec{x}, t, \lambda) = \xi^{-}(\vec{x}, t, \lambda) G(\vec{x}, t, \lambda), \qquad \lambda^k \in \mathbb{R}, \qquad \lim_{\lambda \to \infty} \xi^{+}(\vec{x}, t, \lambda) = \mathbb{1},$$

$\xi^{\pm}(\vec{x}, t, \lambda) \in \mathfrak{G}$

Consider particular type of dependence $G(\vec{x}, t, \lambda)$:

$$i\frac{\partial G}{\partial x_s} - \lambda^k [J_s, G(\vec{x}, t, \lambda)] = 0, \qquad i\frac{\partial G}{\partial t} - \lambda^k [K, G(\vec{x}, t, \lambda)] = 0.$$

where $J_s \in \mathfrak{h} \subset \mathfrak{g}$.

The canonical normalization of the RHP:

$$\xi^{\pm}(\vec{x},t,\lambda) = \exp Q(\vec{x},t,\lambda), \qquad Q(\vec{x},t,\lambda) = \sum_{k=1}^{\infty} Q_k(\vec{x},t)\lambda^{-k}.$$

where all $Q_k(\vec{x},t) \in \mathfrak{g}$. However,

$$\mathcal{J}_s(\vec{x},t,\lambda) = \xi^{\pm}(\vec{x},t,\lambda) J_s \hat{\xi}^{\pm}(\vec{x},t,\lambda), \qquad \mathcal{K}(\vec{x},t,\lambda) = \xi^{\pm}(\vec{x},t,\lambda) K \hat{\xi}^{\pm}(\vec{x},t,\lambda),$$

belong to the algebra $\mathfrak{g}$ for any J and K from $\mathfrak{g}$. If in addition K also belongs to the Cartan subalgebra $\mathfrak{h}$, then

$$[\mathcal{J}_s(\vec{x},t,\lambda), \mathcal{K}(\vec{x},t,\lambda)] = 0.$$

Zakharov-Shabat theorem

Theorem 1. *Let $\xi^{\pm}(x,t,\lambda)$ be solutions to the RHP whose sewing function depends on the auxiliary variables $\vec{x}$ and t as above. Then $\xi^{\pm}(x,t,\lambda)$*

are fundamental solutions of the following set of differential operators:

$$L_s\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x_s} + U_s(\vec{x},t,\lambda)\xi^{\pm}(\vec{x},t,\lambda) - \lambda^k[J_s,\xi^{\pm}(\vec{x},t,\lambda)] = 0,$$

$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(\vec{x},t,\lambda)\xi^{\pm}(\vec{x},t,\lambda) - \lambda^k[K,\xi^{\pm}(\vec{x},t,\lambda)] = 0.$$

Proof. Introduce the functions:

$$g_s^{\pm}(\vec{x},t,\lambda) = i\frac{\partial \xi^{\pm}}{\partial x_s}\hat{\xi}^{\pm}(\vec{x},t,\lambda) + \lambda^k\xi^{\pm}(\vec{x},t,\lambda)J_s\hat{\xi}^{\pm}(\vec{x},t,\lambda),$$

$$g^{\pm}(\vec{x},t,\lambda) = i\frac{\partial \xi^{\pm}}{\partial t}\hat{\xi}^{\pm}(\vec{x},t,\lambda) + \lambda^k\xi^{\pm}(\vec{x},t,\lambda)K\hat{\xi}^{\pm}(\vec{x},t,\lambda),$$

and prove that

$$g_s^{+}(\vec{x},t,\lambda) = g_s^{-}(\vec{x},t,\lambda), \qquad g^{+}(\vec{x},t,\lambda) = g^{-}(\vec{x},t,\lambda),$$

which means that these functions are analytic functions of λ in the whole complex λ-plane. Next we find that:

$$\lim_{\lambda\to\infty} g_s^{+}(\vec{x},t,\lambda) = \lambda^k J_s, \qquad \lim_{\lambda\to\infty} g^{+}(\vec{x},t,\lambda) = \lambda^k K.$$

and make use of Liouville theorem to get

$$g_s^+(\vec{x},t,\lambda) = g_s^-(\vec{x},t,\lambda) = \lambda^k J_s - \sum_{l=1}^{k} U_{s;l}(\vec{x},t)\lambda^{k-l},$$

$$g^+(\vec{x},t,\lambda) = g^-(\vec{x},t,\lambda) = \lambda^k K - \sum_{l=1}^{k} V_l(\vec{x},t)\lambda^{k-l}.$$

We shall see below that the coefficients $U_{s;l}(\vec{x},t)$ and $V_l(\vec{x},t)$ can be expressed in terms of the asymptotic coefficients Q_s. □

Lemma 1. *The set of operators L_s and M commute, i.e. the following set of equations hold:*

$$i\frac{\partial U_s}{\partial x_j} - i\frac{\partial U_j}{\partial x_s} + [U_s(\vec{x},t,\lambda) - \lambda^k J_s, U_j(\vec{x},t,\lambda) - \lambda^k J_j] = 0,$$
$$i\frac{\partial U_s}{\partial t} - i\frac{\partial V}{\partial x_s} + [U_s(\vec{x},t,\lambda) - \lambda^k J_s, V(\vec{x},t,\lambda) - \lambda^k K] = 0.$$

where

$$U_s(\vec{x},t,\lambda) = \sum_{l=1}^{k} U_{s;l}(\vec{x},t)\lambda^{k-l}, \qquad V(\vec{x},t,\lambda) = \sum_{l=0}^{k} V_l(\vec{x},t)\lambda^{k-l}.$$

Proof. The set of the operators L_s and M have a common FAS, i.e. they all must commute.

□

4 Jets of order k

Consider the jets of order k of $\mathcal{J}(x,\lambda)$ and $\mathcal{K}(x,\lambda)$:

$$\mathcal{J}_s(\vec{x},t,\lambda) \equiv \left(\lambda^k \xi^{\pm}(\vec{x},t,\lambda) J_l \hat{\xi}^{\pm}(\vec{x},t,\lambda)\right)_+ = \lambda^k J_s - U_s(\vec{x},t,\lambda),$$
$$\mathcal{K}(\vec{x},t,\lambda) \equiv \left(\lambda^k \xi^{\pm}(\vec{x},t,\lambda) K \hat{\xi}^{\pm}(\vec{x},t,\lambda)\right)_+ = \lambda^k K - V(\vec{x},t,\lambda).$$

Express $U_s(x) \in \mathfrak{g}$ in terms of $Q_s(x)$:

$$\mathcal{J}_s(\vec{x}, t, \lambda) = J_s + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}\,_Q^k J_s, \qquad \mathcal{K}(\vec{x}, t, \lambda) = K + \sum_{k=1}^{\infty} \frac{1}{k!} \mathrm{ad}\,_Q^k K,$$

and therefore for $U_{s;l}$ we get:

$$U_{s;1}(\vec{x}, t) = -\mathrm{ad}\,_{Q_1} J_s, \qquad U_{s;2}(\vec{x}, t) = -\mathrm{ad}\,_{Q_2} J_s - \frac{1}{2} \mathrm{ad}\,_{Q_1}^2 J_s$$

$$U_{s;3}(\vec{x}, t) = -\mathrm{ad}\,_{Q_3} J_s - \frac{1}{2} \left(\mathrm{ad}\,_{Q_2} \mathrm{ad}\,_{Q_1} + \mathrm{ad}\,_{Q_1} \mathrm{ad}\,_{Q_2} \right) J_s - \frac{1}{6} \mathrm{ad}\,_{Q_1}^3 J_s$$

$$\vdots$$

$$U_{s;k}(\vec{x}, t) = -\mathrm{ad}\,_{Q_k} J_s - \frac{1}{2} \sum_{s+p=k} \mathrm{ad}\,_{Q_s} \mathrm{ad}\,_{Q_p} J_s$$

$$- \frac{1}{6} \sum_{s+p+r=k} \mathrm{ad}\,_{Q_s} \mathrm{ad}\,_{Q_p} \mathrm{ad}\,_{Q_r} J_s - \cdots - \frac{1}{k!} \mathrm{ad}\,_{Q_1}^k J_s,$$

and similar expressions for $V_l(\vec{x}, t)$ with J_s replaced by K.

5 Reductions of polynomial bundles

$$\begin{array}{lll} \text{a)} & A\xi^{+,\dagger}(x,t,\epsilon\lambda^*)\hat{A} = \hat{\xi}^{-}(x,t,\lambda), & AQ^{\dagger}(x,t,\epsilon\lambda^*)\hat{A} = -Q(x,t,\lambda), \\ \text{b)} & B\xi^{+,*}(x,t,\epsilon\lambda^*)\hat{B} = \xi^{-}(x,t,\lambda), & BQ^{*}(x,t,\epsilon\lambda^*)\hat{B} = Q(x,t,\lambda), \\ \text{c)} & C\xi^{+,T}(x,t,-\lambda)\hat{C} = \hat{\xi}^{-}(x,t,\lambda), & CQ^{\dagger}(x,t,-\lambda)\hat{C} = -Q(x,t,\lambda), \end{array}$$

where $\epsilon^2 = 1$ and A, B and C are elements of the group $\mathfrak{G}$ such that $A^2 = B^2 = C^2 = \mathbb{1}$. As for the $\mathbb{Z}_N$-reductions we may have:

$$D\xi^{\pm}(x,t,\omega\lambda)\hat{D} = \xi^{\pm}(x,t,\lambda), \qquad DQ(x,t,\omega\lambda)\hat{D} = Q(x,t,\lambda),$$

where $\omega^N = 1$ and $D^N = \mathbb{1}$.

6 On N-wave equations ($k = 1$) in 2 and more dimensions

Lax representation involves two Lax operators linear in λ:

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + [J, Q(x,t)]\xi^{\pm}(\vec{x}, t, \lambda) - \lambda[J, \xi^{\pm}(\vec{x}, t, \lambda)] = 0,$$
$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + [K, Q(x,t)]\xi^{\pm}(\vec{x}, t, \lambda) - \lambda[K, \xi^{\pm}(\vec{x}, t, \lambda)] = 0.$$

The corresponding equations take the form:

$$i\left[J, \frac{\partial Q}{\partial t}\right] - i\left[K, \frac{\partial Q}{\partial x}\right] - [[J, Q], [K, Q(x,t)]] = 0$$

$$Q(x,t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad \begin{aligned} J &= \operatorname{diag}(a_1, a_2, a_3), \\ K &= \operatorname{diag}(b_1, b_2, b_3), \end{aligned}$$

Then the 3-wave equations take the form:

$$\frac{\partial u_1}{\partial t} - \frac{a_1 - a_2}{b_1 - b_2}\frac{\partial u_1}{\partial x} + \kappa\epsilon_1\epsilon_2 u_2^* u_3 = 0,$$
$$\frac{\partial u_2}{\partial t} - \frac{a_2 - a_3}{b_2 - b_3}\frac{\partial u_2}{\partial x} + \kappa\epsilon_1 u_1^* u_3 = 0,$$
$$\frac{\partial u_3}{\partial t} - \frac{a_1 - a_3}{b_1 - b_3}\frac{\partial u_3}{\partial x} + \kappa\epsilon_2 u_1^* u_2^* = 0,$$

where

$$\kappa = a_1(b_2 - b_3) - a_2(b_1 - b_3) + a_3(b_1 - b_2).$$

For 3-dimensional space-time we consider Q as above, but now let u_j and v_j be functions of $x_1 = x$, $x_2 = y$ and t. Let also $J_1 = J$ and $J_2 = I = \operatorname{diag}(c_1, c_2.c_3)$. Now the corresponding solution of the RHP $\xi^{\pm}(x,y,t,\lambda)$ will be FAS not only of L and M above, but also of

$$P\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial y} + [I, Q(x,t)]\xi^{+}(\vec{x},t,\lambda) - \lambda[I, \xi^{+}(\vec{x},t,\lambda)] = 0,$$

and all these three operators will mutually commute, i.e. along with $[L,M]=0$ we will have also $[L,P]=0$ and $[P,M]=0$. As a result $Q(x,y,t)$ will satisfy two more 3-wave NLEE

$$
\begin{aligned}
2\frac{\partial u_1}{\partial t} - \frac{a_1-a_2}{b_1-b_2}\frac{\partial u_1}{\partial x} - \frac{a_1-a_2}{c_1-c_2}\frac{\partial u_1}{\partial y} + (\kappa_1+\kappa_2)\epsilon_1\epsilon_2 u_2^* u_3 = 0,\\
2\frac{\partial u_2}{\partial t} - \frac{a_1-a_3}{b_1-b_3}\frac{\partial u_2}{\partial x} - \frac{a_1-a_3}{c_1-c_3}\frac{\partial u_2}{\partial y} + (\kappa_1+\kappa_2)\epsilon_1 u_1^* u_3 = 0,\\
2\frac{\partial u_3}{\partial t} - \frac{a_2-a_3}{b_2-b_3}\frac{\partial u_3}{\partial x} - \frac{a_2-a_3}{c_2-c_3}\frac{\partial u_3}{\partial y} + (\kappa_1+\kappa_2)\epsilon_2 u_1^* u_2^* = 0.
\end{aligned}
$$

$$
\begin{aligned}
\kappa_1 &= a_1(b_2-b_3) - a_2(b_1-b_3) + a_3(b_1-b_2),\\
\kappa_2 &= a_1(c_2-c_3) - a_2(c_1-c_3) + a_3(c_1-c_2).
\end{aligned}
$$

For N-wave equations related to Lie algebras $\mathfrak{g}$ of higher rank r we can add up to r auxiliary variables:

$$
r\frac{\partial Q}{\partial t} - \sum_{s=1}^{r}(\mathrm{ad}_{J_s}^{-1}\mathrm{ad}_J)\frac{\partial Q}{\partial x_s} - i\sum_{s=1}^{r}\mathrm{ad}_{J_s}^{-1}\,[[J,Q],[J_s,Q(\vec{x},t)]] = 0
$$

where Q is an $n \times n$ off-diagonal matrix depending on $r+1$ variables. We remind that if $J = \mathrm{diag}\,(a_1, \ldots, a_n)$ then

$$(\mathrm{ad}\,_J Q)_{jk} \equiv ([J, Q])_{jk} = (a_j - a_k) Q_{jk}, \qquad (\mathrm{ad}\,_J^{-1} Q)_{jk} = \frac{1}{a_j - a_k} Q_{jk},$$

and similarly for the other J_s.

7 New N-wave equations ($k=2$) in 2 and more dimensions

Let $\mathfrak{g} = sl(3)$ and

$$Q_1(\vec{x}, t) = \begin{pmatrix} 0 & u_1 & u_3 \\ -v_1 & 0 & u_2 \\ -v_3 & -v_2 & 0 \end{pmatrix}, \qquad Q_2(\vec{x}, t) = \begin{pmatrix} q_{11} & w_1 & w_3 \\ -z_1 & q_{22} & w_2 \\ -z_3 & -z_2 & q_{33} \end{pmatrix},$$

Fix up $k = 2$. Then the Lax pair becomes

$$L\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial x} + U(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]J, \xi^{\pm}(x,t,\lambda)] = 0,$$
$$M\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial t} + V(x,t,\lambda)\xi^{\pm}(x,t,\lambda) - \lambda^2]K, \xi^{\pm}(x,t,\lambda)] = 0,$$

where

$$U \equiv U_2 + \lambda U_1 = \left([J, Q_2(x)] - \frac{1}{2}[[J, Q_1], Q_1(x)] \right) + \lambda [J, Q_1],$$
$$V \equiv V_2 + \lambda V_1 = \left([K, Q_2(x)] - \frac{1}{2}[[K, Q_1], Q_1(x)] \right) + \lambda [K, Q_1].$$

Impose a $\mathbb{Z}_2$-reduction of type a) with $A = \operatorname{diag}(1, \epsilon, 1)$, $\epsilon^2 = 1$. Thus Q_1 and Q_2 get reduced into:

$$Q_1 = \begin{pmatrix} 0 & u_1 & 0 \\ \epsilon u_1^* & 0 & u_2 \\ 0 & \epsilon u_2^* & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ w_3^* & 0 & 0 \end{pmatrix},$$

New type of integrable 3-wave equations:

$$i(a_1 - a_2)\frac{\partial u_1}{\partial t} - i(b_1 - b_2)\frac{\partial u_1}{\partial x} + \epsilon\kappa u_2^* u_3 + \epsilon\frac{\kappa(a_1 - a_2)}{(a_1 - a_3)}u_1|u_2|^2 = 0,$$

$$i(a_2 - a_3)\frac{\partial u_2}{\partial t} - i(b_2 - b_3)\frac{\partial u_2}{\partial x} + \epsilon\kappa u_1^* u_3 - \epsilon\frac{\kappa(a_2 - a_3)}{(a_1 - a_3)}|u_1|^2 u_2 = 0,$$

$$i(a_1 - a_3)\frac{\partial u_3}{\partial t} - i(b_1 - b_3)\frac{\partial u_3}{\partial x} - \frac{i\kappa}{a_1 - a_3}\frac{\partial(u_1 u_2)}{\partial x}$$
$$+ \epsilon\kappa\left(\frac{a_1 - a_2}{a_1 - a_3}|u_1|^2 + \frac{a_2 - a_3}{a_1 - a_3}|u_2|^2\right)u_1 u_2 + \epsilon\kappa u_3(|u_1|^2 - |u_2|^2) = 0,$$

where:

$$u_3 = w_3 + \frac{2a_2 - a_1 - a_3}{2(a_1 - a_3)}u_1 u_2.$$

The diagonal terms in the Lax representation are λ-independent.

Two of them read:

$$i(a_1 - a_2)\frac{\partial |u_1|^2}{\partial t} - i(b_1 - b_2)\frac{\partial |u_1|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,$$
$$i(a_2 - a_3)\frac{\partial |u_2|^2}{\partial t} - i(b_2 - b_3)\frac{\partial |u_2|^2}{\partial x} - \epsilon\kappa(u_1 u_2 u_3^* - u_1^* u_2^* u_3) = 0,$$

These relations are satisfied identically as a consequence of the NLEE.

Let the sewing function G of the RHP depends on 3 variables: t, $x_1 = x$ and $x_2 = y$ with $J_1 = J$ and $J_2 = I = \mathrm{diag}\,(c_1, c_2, c_3)$. For $k = 2$ we obtain: L, M and

$$P\xi^{\pm} \equiv i\frac{\partial \xi^{\pm}}{\partial y} + W(x,y,t,\lambda)\xi^{\pm}(x,y,t,\lambda) - \lambda^2]I, \xi^{\pm}(x,y,t,\lambda)] = 0,$$
$$\begin{aligned} W &\equiv W_2 + \lambda W_1 \\ &= \left([I, Q_2(x,y,t)] - \frac{1}{2}[[I,Q_1], Q_1(x,y,t)]\right) + \lambda [I, Q_1(x,y,t)], \end{aligned}$$

commuting identically with respect to λ. It is obvious that $[L, P] = 0$ if

$$i(a_1 - a_2)\frac{\partial u_1}{\partial t} - i(c_1 - c_2)\frac{\partial u_1}{\partial y} + \epsilon\kappa_2 u_2^* u_3 + \epsilon\frac{\kappa_2(a_1 - a_2)}{(a_1 - a_3)}u_1|u_2|^2 = 0,$$

$$i(a_2 - a_3)\frac{\partial u_2}{\partial t} - i(c_2 - c_3)\frac{\partial u_2}{\partial y} + \epsilon\kappa_2 u_1^* u_3 - \epsilon\frac{\kappa_2(a_2 - a_3)}{(a_1 - a_3)}|u_1|^2 u_2 = 0,$$

$$i(a_1 - a_3)\frac{\partial u_3}{\partial t} - i(c_1 - c_3)\frac{\partial u_3}{\partial y} - \frac{i\kappa_2}{a_1 - a_3}\frac{\partial(u_1 u_2)}{\partial y}$$

$$+ \epsilon\kappa_2\left(\frac{a_1 - a_2}{a_1 - a_3}|u_1|^2 + \frac{a_2 - a_3}{a_1 - a_3}|u_2|^2\right)u_1 u_2 + \epsilon\kappa_2 u_3(|u_1|^2 - |u_2|^2) = 0,$$

$$
\begin{aligned}
&2i\frac{\partial u_1}{\partial t} - i(\vec{v}_{(1)}\cdot\nabla)u_1 + \epsilon(\kappa_1+\kappa_2)\left(\frac{u_2^* u_3}{a_1-a_2} + \frac{u_1|u_2|^2}{(a_1-a_3)}\right) = 0,\\
&2i\frac{\partial u_2}{\partial t} - i(\vec{v}_{(2)}\cdot\nabla)u_2 + \epsilon(\kappa_1+\kappa_2)\left(\frac{u_1^* u_3}{a_1-a_3} - \frac{u_2|u_1|^2}{(a_1-a_3)}\right) = 0,\\
&2i\frac{\partial u_3}{\partial t} - i(\vec{v}_{(3)}\cdot\nabla)u_3 - i\frac{(\vec{\kappa}\cdot\nabla)(u_1u_2)}{(a_1-a_3)^2} + \frac{\epsilon(\kappa_1+\kappa_2)}{a_1-a_3}(|u_1|^2-|u_2|^2)u_3\\
&\quad + \frac{\epsilon(\kappa_1+\kappa_2)}{(a_1-a_3)^2}\left((a_1-a_2)|u_1|^2 + (a_2-a_3)|u_2|^2\right)u_1u_2 = 0.
\end{aligned}
$$

Here $\nabla = (\partial_x, \partial_y)^T$, the characteristic velocities $\vec{v}_{(j)}$, $j = 1, 2, 3$ and $\vec{\kappa}$ are two-component vectors given by:

$$
\vec{v}_{(1)} = \frac{1}{a_1-a_2}\begin{pmatrix} b_1-b_2 \\ c_1-c_2 \end{pmatrix}, \qquad \vec{v}_{(2)} = \frac{1}{a_2-a_3}\begin{pmatrix} b_2-b_3 \\ c_2-c_3 \end{pmatrix},
$$

$$
\vec{v}_{(3)} = \frac{1}{a_1-a_3}\begin{pmatrix} b_1-b_3 \\ c_1-c_3 \end{pmatrix}, \qquad \vec{\kappa} = \begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix},
$$

and $\kappa_1 = \kappa$.

8 Conclusions and open questions

- We constructed new integrable spinor models and new integrable 3- and N-wave equations.
- These new NLEE must be Hamiltonian. The Poisson brackets for polynomial bundles – see Kulish, Reyman and Semenov-Tyan-Shanskii (1981-1983);
- The method allows one also to apply Zakharov-Shabat dressing method for constructing their explicit (N-soliton) solutions – T. Valchev.
- This approach improves Gel'fand-Dickey approach.