# On the 3-wave Equations with Constant Boundary Conditions

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# Outline

# 3-wave resonant interaction equations

## The case of vanishing boundary conditions

- 3-wave resonant interaction model:

$$\begin{aligned} \mathrm{i}\frac{\partial q_1}{\partial t} + \mathrm{i}v_1\frac{\partial q_1}{\partial x} + \varkappa q_2^* q_3 &= 0, \\ \mathrm{i}\frac{\partial q_2}{\partial t} + \mathrm{i}v_2\frac{\partial q_2}{\partial x} + \varkappa q_1^* q_3 &= 0, \\ \mathrm{i}\frac{\partial q_3}{\partial t} + \mathrm{i}v_3\frac{\partial q_3}{\partial x} + \varkappa q_1 q_2 &= 0. \end{aligned}$$

  $\varkappa$ – interaction constant, $v_i$ – the group velocities of the model, $q_i = q_i(x,t)$, $i = 1, 2, 3$.

- The 3-wave equations can be solved through the Inverse Scattering Method.

  Zakharov V. E., Manakov S. V., Zh. Exp. Teor. Fiz., **69** (1975), 1654–1673; (*INF preprint 74-41*, Novosibirsk (1975) (In Russian)).

# ISM as a Generalised Fourier Transform

## Describing the Fundamental Properties of NLEEs

The interpretation of the ISM as a GFT and the expansions over the so called "squared solutions" allows one to study all the fundamental properties of NLEEs:

1. the description of the whole class of NLEE related to a given spectral problem (Lax operator) solvable by the ISM;
2. derivation of the infinite family of integrals of motion;
3. the Hamiltonian properties of the NLEE's.


M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, *Studies in Appl. Math.* **53** (1974), n. 4, 249–315.

Gerdjikov V. S., Kulish P. P., *Physica D* **3**(1981), 549–564.
Gerdjikov V. S., *Inverse Problems* **2** (1986), 51–74.

V.S. Gerdjikov, G. Vilasi, A.B. Yanovski, *Integrable Hamiltonian Hierarchies: Spectral and Geometric Methods*, Lect. Notes Phys. **748**, Springer, Berlin - Heidelberg (2008).

# Describing the Fundamental Properties ...

## The Hamiltonian and ... going to constant boundary conditions

- The (canonical) Hamiltonian of 3WRI eqns. is given by:

$$H_{3-\mathrm{w}} = \frac{1}{2}\int_{-\infty}^{\infty} dx \left( \sum_{k=1}^{3} v_k \left( q_k \frac{\partial q_k^*}{\partial x} - q_k^* \frac{\partial q_k}{\partial x} \right) + \varkappa (q_3 q_1^* q_2^* + q_3^* q_1 q_2) \right)$$

- A special interest deserves the case when some (or all) of the functions $q_k(x,t)$ tend to a constant as $x \to \pm\infty$. Below we choose:

$$q_{1,2}(x,t) \to 0, \qquad q_3(x,t) \to \rho e^{i\phi_\pm}, \qquad x \to \pm\infty.$$

- Here the constants $\theta = \phi_+ - \phi_-$ and $\rho$ are of a physical origin and play a basic role in determining the properties of 3WRI eqns. with CBC and its soliton solutions.

Faddeev L. D., Takhtadjan L. A., *Hamiltonian approach in the theory of solitons*, Springer Verlag, Berlin (1987).

# The Case of Constant Boundary Conditions

## "Bright" vs. "Dark" Solitons

- More specifically, $\rho$ characterizes the end-points of the continuous spectrum of the Lax operator $L(\lambda)$.
- The discrete spectrum, in this case, may consist of real simple eigenvalues $\lambda_k$, $k = 1, \dots, N$ lying in the lacuna $-2\rho < \lambda_k < 2\rho$.
- To them, there correspond the so-called "dark solitons" whose properties and behavior substantially differ from the ones of the bright solitons.
- The 3- and $N$-wave interaction models describe a special class of wave-wave interactions that are not sensitive on the physical nature of the waves and bear an universal character.

Kaup D. J., Reiman A., Bers A, *Rev. Mod. Phys.* **51** (1979), 275–310.

S. V. Manakov, *Teor. Mat. Phys.* **28** (1976), 172–179.

# 3-waves with CBC

## Setting up the problem

- It is normal to expect that the properties 1 – 3, known for the case of vanishing boundary conditions will have their counterparts for the case of constant boundary conditions.
- However there is no easy and direct way to do so. For example, one may relate both cases by taking a limit $\rho \to 0$.
  Of course, in this limit most of the difficulties, related mostly with the end-points of the continuous spectrum disappear.
- The spectral data, the analyticity properties of the Jost solutions and the corresponding Riemann-Hilbert problem are substantially different and more difficult for $\rho > 0$.

Aims:

1. To study the direct scattering problem for the 3WRI eqns with CBC;
2. To study the spectral properties of the associated Lax operator.

# Lax representation

## Algebraic Setup

- Consider the pair of Lax operators:

$$
\begin{aligned}
L\psi &\equiv \left( i\frac{\partial}{\partial x} + [J, Q(x,t)] - \lambda J \right) \psi(x,t,\lambda) = 0, \\
M\psi &\equiv \left( i\frac{\partial}{\partial t} + [I, Q(x,t)] - \lambda I \right) \psi(x,t,\lambda) = 0,
\end{aligned}
$$

with

$$
Q = \begin{pmatrix} 0 & q_1 & q_3 \\ q_1^* & 0 & q_2 \\ q_3^* & q_2^* & 0 \end{pmatrix}, \qquad \begin{aligned} J &= \operatorname{diag}(J_1, J_2, J_3), \\ I &= \operatorname{diag}(I_1, I_2, I_3). \end{aligned}
$$

- $Q(x,t)$, $I$ and $J$ are traceless matrices (i.e. the Lax operators $\in sl(3,\mathbb{C})$)
- The eigenvalues of $I$ and $J$ are ordered as follows: $J_1 > J_2 > J_3$, $I_1 > I_2 > I_3$ ($J_1 + J_2 + J_3 = 0$ and $I_1 + I_2 + I_3 = 0$).
- Here $\lambda \in \mathbb{C}$ is a spectral parameter.

# Lax representation

## Zero Curvature Equation

- The compatibility condition:

$$i[J, Q_t] - i[I, Q_x] + [[I, Q], [J, Q]] = 0,$$

- The group velocities take the form:

$$v_1 = \frac{I_1 - I_2}{J_1 - J_2}, \qquad v_2 = \frac{I_2 - I_3}{J_2 - J_3}, \qquad v_3 = \frac{I_1 - I_3}{J_1 - J_3},$$

- The interaction constant $\varkappa$ reads:

$$\varkappa = J_1 I_2 + J_2 I_3 + J_3 I_1 - J_2 I_1 - J_3 I_2 - J_1 I_3.$$

# Imposing Boundary Conditions

- Constant boundary conditions as $|x| \to \infty$:

$$\lim_{x\to\pm\infty} q_1(x,t) = \lim_{x\to\pm\infty} q_2(x,t) = 0, \qquad \lim_{x\to\pm\infty} q_3(x,t) = q_3^{\pm} = \rho e^{i\phi_\pm}.$$

- For the potential matrix $Q(x,t)$ one can write:

$$\lim_{x\to\pm\infty} Q(x,t) = Q_\pm, \qquad Q_\pm = \begin{pmatrix} 0 & 0 & \rho e^{i\phi_\pm} \\ 0 & 0 & 0 \\ \rho e^{-i\phi_\pm} & 0 & 0 \end{pmatrix}$$

- The difference

$$\theta = \phi_+ - \phi_-$$

of the asymptotic phases $\phi_\pm$ plays a crucial rôle in the Hamiltonian formulation of the 3-wave model with constant boundary conditions: its values label the leaf on the phase space $\mathcal{M}$ of the 3WRI model, where one can determine the class of admissible functionals, and to construct a Hamiltonian formulation.

# Imposing Boundary Conditions

## The Asymptotic phases

- The two asymptotic potentials $Q_{\pm}$ are related by

$$Q_{+} = Q(\theta) Q_{-}(t) Q^{-1}(\theta),$$

where $\theta = \phi_{+} - \phi_{-}$ and

$$Q(\theta) = \left( \begin{array}{ccc} e^{i\theta/2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\theta/2} \end{array} \right).$$

# Direct Scattering Problem for $L$

## Jost Solutions

- The direct and the inverse scattering problem for the Lax operator will be done for fixed $t$ and in most of the corresponding formulae $t$ will be omitted.
- The starting point in developing the DSP for 3WRI eqns are the eigenfunctions (the Jost solutions) of the auxiliary spectral problem

$$L(x,t,\lambda)\psi_{\pm}(x,t,\lambda) = 0,$$

- Asymptotic behavior for $x \to \pm\infty$ respectively:

$$\lim_{x\to\pm\infty} \psi_{\pm}(x,t,\lambda) e^{iJ(\lambda)x} = \psi_{\pm,0}(\lambda) P(\lambda),$$

where $P(\lambda)$ is a projector:

$$P(\lambda) = \mathrm{diag}\,(\theta(|\mathrm{Re}\,\lambda| - 2\rho), 1, \theta(|\mathrm{Re}\,\lambda| - 2\rho))$$

and $\theta(z)$ is the step function.

# Jost Solutions

## Asymptotic Lax operators

- $P(\lambda)$ ensures that the continuous spectrum of $L$ has multiplicity 3 for $|\mathrm{Re}\,\lambda| - 2\rho > 0$ and multiplicity 1, for $-2\rho < \mathrm{Re}\,\lambda < 2\rho$.
- The $x$ and $t$-independent matrices $\psi_{\pm,0}(\lambda)$ in diagonalize the asymptotic Lax operators:

$$L_{\pm}(x,t,\lambda) = i\frac{\partial}{\partial x} + [J, Q_{\pm}] - \lambda J.$$

Indeed,

$$([J, Q_{\pm}] - \lambda J)\psi_{\pm,0}(\lambda) = -\psi_{\pm,0}(\lambda) J(\lambda),$$

where $J(\lambda) = -\mathrm{diag}\,(J_1(\lambda), J_2(\lambda), J_3(\lambda))$ and

$$J_1(\lambda) = \frac{1}{2}\left[J_2\lambda + (J_1 - J_3)\sqrt{\lambda^2 - 4\rho^2}\right], \qquad J_2(\lambda) = -\lambda J_2,$$

$$J_3(\lambda) = \frac{1}{2}\left[J_2\lambda - (J_1 - J_3)\sqrt{\lambda^2 - 4\rho^2}\right], \qquad k(\lambda) = \sqrt{\lambda^2 - 4\rho^2}.$$

# The Scattering Matrix

## Asymptotic Lax operators

- Using the asymptotic phases we have:

$$\psi_{\pm,0}(\lambda) = \frac{1}{\sqrt{2\lambda(k+\lambda)}} \begin{pmatrix} 2\rho & 0 & -(\lambda+k)e^{i\phi_\pm} \\ 0 & 1 & 0 \\ (\lambda+k)e^{-i\phi_\pm} & 0 & 2\rho \end{pmatrix}$$

- We will deal with the Riemannian surface related to $k(\lambda)$; its first sheet is fixed up by the condition: $\operatorname{sign}\operatorname{Im} k(\lambda) = \operatorname{sign}\operatorname{Im}\lambda$.
- The Jost solutions $\psi_+(x,t,\lambda)$ and $\psi_-(x,t,\lambda)$ are related by the scattering matrix $T(t,\lambda)$:

$$T(t,\lambda) = \psi_+^{-1}(x,t,\lambda)\psi_-(x,t,\lambda), \qquad \det T(\lambda) = 1.$$

- For a sake of convenience, from now on, instead of the spectral parameter $\lambda$ we will be using the so-called "uniformizing variable"

$$\zeta = \frac{1}{2\rho}(\lambda + k(\lambda)).$$

# The Scattering Matrix

## The uniformising variable

- In terms of $\zeta$ we have:

$$\lambda = \rho \left( \zeta + \frac{1}{\zeta} \right), \qquad k(\lambda) = \rho \left( \zeta - \frac{1}{\zeta} \right).$$

- Then, the form of $J$ take the form:

$$J(\zeta) = -\rho \,\mathrm{diag}\, \left( J_3 \zeta + \frac{J_1}{\zeta},\ J_2 \left( \zeta + \frac{1}{\zeta} \right),\ J_1 \zeta + \frac{J_3}{\zeta} \right).$$

# Direct Scattering Problem for $L$

## Integrable Representations for the Jost Solutions

- Consider slightly modified Jost solutions:

$$\eta_{\pm}(x,\zeta) = \psi_{\pm,0}^{-1}(\zeta)\psi_{\pm}(x,\zeta)e^{iJ(\zeta)x}, \qquad \lim_{x\to\pm\infty} \eta_{\pm}(x,\zeta) = \mathbb{1},$$

and satisfying the following equation:

$$\begin{aligned} i\frac{\partial \eta_{\pm}}{\partial x} &+ \psi_{\pm,0}^{-1}[J, Q(x,t) - Q_{\pm}]\psi_{\pm,0}\eta_{\pm} \\ &- \rho(\zeta + \zeta^{-1})[J(\zeta), \eta_{\pm}(x,t,\zeta)] = 0. \end{aligned}$$

- Equivalently, $\eta_{\pm}(x,\zeta)$ can be regarded as solutions to the following Volterra-type integral equations:

$$\begin{aligned} \eta_{\pm}(x,\zeta) = \mathbb{1} + i\int_{\pm\infty}^{x} dy\, e^{-iJ(\zeta)(x-y)} \\ \times \psi_{\pm,0}^{-1}[J, Q(y) - Q_{\pm}]\psi_{\pm,0}\eta_{\pm}(y,\zeta)e^{iJ(\zeta)(x-y)}, \qquad \zeta \in \mathbb{R}, \end{aligned}$$

# Direct Scattering Problem for $L$

## Integrable Representations for the Jost Solutions ... (cont'd)

- In addition the second column of the Jost solution is defined also on the unit circle in the $\zeta$-plane:

$$
\begin{aligned}
(\eta_{\pm}^{(2)})_{k2}(x,\zeta) = \delta_{k2} + i \int_{\pm\infty}^{x} dy\, e^{-i(J_k(\zeta)-J_2(\zeta))(x-y)} \\
\times \left( \psi_{\pm,0}^{-1}[J, Q(y) - Q_{\pm}]\psi_{\pm,0}\eta_{\pm}(y,\zeta) \right), \qquad |\zeta| = 1.
\end{aligned}
$$

# Fundamental Analytic Solutions (FAS) for $L$

## Introducing the regions on analyticity

- In order to construct the fundamental analytic solutions (FAS) of $L$ we first need to determine the regions of the complex $\zeta$-plane in which the imaginary parts of the eigenvalues of $J(\zeta)$ are ordered.
- To do this we first need to find the curves on which

$$\mathrm{Im}\,(J_j(\zeta) - J_k(\zeta)) = 0, \qquad 1 \leq j < k \leq 3; \qquad \mathrm{Im}\, J_1(\zeta) = 0.$$

- Writing down $\zeta = |\zeta| e^{i\phi_0}$ we find:

$$\mathrm{Im}\,(J_1(\zeta) - J_2(\zeta)) = \rho \left( (J_2 - J_3)|\zeta| + \frac{J_1 - J_2}{|\zeta|} \right) \sin\phi_0,$$

$$\mathrm{Im}\,(J_2(\zeta) - J_3(\zeta)) = \rho \left( (J_1 - J_2)|\zeta| + \frac{J_2 - J_3}{|\zeta|} \right) \sin\phi_0,$$

$$\mathrm{Im}\,(J_1(\zeta) - J_3(\zeta)) = \rho (J_1 - J_3) \left( |\zeta| + \frac{1}{|\zeta|} \right) \sin\phi_0.$$

# Fundamental Analytic Solutions (FAS) for $L$

## Introducing the regions on analyticity ... (cont'd)

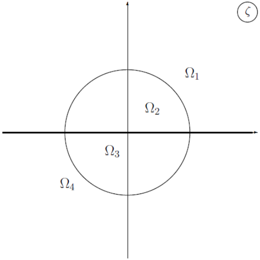
$$\operatorname{Im} J_2(\zeta) = -J_2\rho \left( |\zeta| - \frac{1}{|\zeta|} \right) \sin \phi_0 .$$

- Since $J_1 - J_2 > 0$, $J_1 - J_3 > 0$ and $J_2 - J_3 > 0$ it is easy to see that the solutions of the fist set of eqs. is the real axis in the complex $\zeta$-plane;
- In addition $\operatorname{Im} J_2(\zeta) = 0$ for $|\zeta|^2 = 1$.

# Fundamental Analytic Solutions (FAS) for $L$

## Ordering in the regions on analyticity

The complex $\zeta$-plane is split into four regions $\Omega_k$, $k = 1, \dots, 4$ formed by the intersections of the upper and lower complex half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$ with the unit circle $\mathcal{S}$.




The ordering of $\mathrm{Im}', J_k(\lambda)$ in each of them depends on the sign of $J_2$ and are as follows:

$$
\begin{aligned}
\Omega_1: &\quad \mathrm{Im}\, J_1(\lambda) > 0 > \mathrm{Im}\, J_2(\lambda) > \mathrm{Im}\, J_3(\lambda),\\
\Omega_2: &\quad \mathrm{Im}\, J_1(\lambda) > \mathrm{Im}\, J_2(\lambda) > 0 > \mathrm{Im}\, J_3(\lambda),\\
\Omega_3: &\quad \mathrm{Im}\, J_3(\lambda) > 0 > \mathrm{Im}\, J_2(\lambda) > \mathrm{Im}\, J_1(\lambda),\\
\Omega_4: &\quad \mathrm{Im}\, J_3(\lambda) > \mathrm{Im}\, J_2(\lambda) > 0 > \mathrm{Im}\, J_1(\lambda),
\end{aligned}
$$

for $J_2 > 0$

# Fundamental Analytic Solutions (FAS) for $L$

## Ordering in the regions on analyticity ... (cont'd)

and

$$\begin{aligned}
\Omega_1 :& \qquad \mathrm{Im}\, J_1(\lambda) > \mathrm{Im}\, J_2(\lambda) > 0 > \mathrm{Im}\, J_3(\lambda), \\
\Omega_2 :& \qquad \mathrm{Im}\, J_1(\lambda) > 0 > \mathrm{Im}\, J_2(\lambda) > \mathrm{Im}\, J_3(\lambda), \\
\Omega_3 :& \qquad \mathrm{Im}\, J_3(\lambda) > \mathrm{Im}\, J_2(\lambda) > 0 > \mathrm{Im}\, J_1(\lambda), \\
\Omega_4 :& \qquad \mathrm{Im}\, J_3(\lambda) > 0 > \mathrm{Im}\, J_2(\lambda) > \mathrm{Im}\, J_1(\lambda),
\end{aligned}$$

for $J_2 < 0$.

# Constructing the Fundamental Analytic Solutions

## Integral representations for the FAS in $\Omega_1$

- introduce the FAS as the solution of the following set of integral equations:

$$\left\{\xi^+_{(1)}(x,\zeta)\right\}_{kl} = \delta_{kl} + i\int_\infty^x dy\, e^{-i(J_k(\zeta)-J_l(\zeta))(x-y)} \times \left\{\psi^{-1}_{+,0}[J, Q(y)-Q_+]\psi_{+,0}\xi^+_{(1)}(y,\zeta)\right\}_{kl}, \qquad k<l.$$

$$\left\{\xi^+_{(1)}(x,\zeta)\right\}_{kl} = i\int_{-\infty}^x dy\, e^{-i(J_k(\zeta)-J_l(\zeta))(x-y)} \times \left\{\psi^{-1}_{+,0}[J, Q(y)-Q_-]\psi_{+,0}\xi^+_{(1)}(y,\zeta\right\}_{kl}, \qquad k\geq l.$$

- The proof of the fact that $\xi^+_{(1)}(x,\zeta)$ is an analytic function of $\zeta$ for any $\zeta\in\Omega_1$ is based on the fact, that due the ordering all exponential factors for $\zeta\in\mathbb{C}_+$ are decaying.
- This ensures the convergence of all integrals.

# Constructing the Fundamental Analytic Solutions

## Integral representations for the FAS in $\Omega_4$

- Similarly, the FAS in the region $\Omega_4$ is the solution of the set of integral equations:

$$\left\{\xi^{-}_{(4)}(x,\zeta)\right\}_{kl} = \delta_{kl} + i\int_{-\infty}^{x} dy\, e^{-i(J_k(\zeta)-J_l(\zeta))(x-y)} \times \left\{\psi^{-1}_{+,0}[J, Q(y) - Q_-]\psi_{+,0}\xi^{-}_{(4)}(y,\zeta)\right\}_{kl}, \qquad k \leq l.$$

$$\left\{\xi^{-}_{(4)}(x,\zeta)\right\}_{kl} = i\int_{\infty}^{x} dy\, e^{-i(J_k(\zeta)-J_l(\zeta))(x-y)} \times \left\{\psi^{-1}_{+,0}[J, Q(y) - Q_+]\psi_{+,0}\xi^{-}_{(4)}(y,\zeta\right\}_{kl}. \qquad k > l.$$

- The proof of the analyticity of $\xi^{-}_{(4)}(x,\zeta)$ for any $\zeta \in \Omega_4$ is similar to the one for $\xi^{+}_{(1)}(x,\zeta)$.

# Constructing the Fundamental Analytic Solutions

## Integral representations for the FAS in $\Omega_2$ and $\Omega_3$

- It remains to outline the construction of $\xi^-_{(2)}(x,\lambda)$ and $\xi^+_{(3)}(x,\lambda)$ for the regions $\Omega_2$ and $\Omega_3$. To this end we make use of the involution of the Lax operator $L$ that is a consequence of $Q = -Q^\dagger$. Then we conclude:

$$\chi^-_{(3)}(x,\zeta) = (\chi^{+,\dagger}_{(1)})^{-1}(x,1/\zeta^*), \qquad \chi^+_{(2)}(x,\zeta) \; = (\chi^{-,\dagger}_{(4)})^{-1}(x,1/\zeta^*).$$

- The next step is to analyze the interrelations between the Jost solutions $\psi_\pm(x,\zeta)$ and the FAS $\chi^\pm(x,\zeta)$. Skipping the details we note that:

$$\begin{aligned}
\chi^+_{(\alpha)}(x,\zeta) &= \psi_-(x,\zeta)S^+_{(\alpha)}(\zeta), & \chi^+_{(\alpha)}(x,\zeta) &= \psi_+(x,\zeta)T^-_{(\alpha)}(\zeta)D^+_{(\alpha)}(\zeta), \\
\chi^-_{(\beta)}(x,\zeta) &= \psi_-(x,\zeta)S^-_{(\beta)}(\zeta), & \chi^-_{(\beta)}(x,\zeta) &= \psi_+(x,\zeta)T^+_{(\beta)}(\zeta)D^-_{(\beta)}(\zeta),
\end{aligned}$$

where $\zeta \in \mathbb{R} \cup \mathcal{S}$, $\alpha = 1,3$ and $\beta = 2,4$ match the indices of the regions of analyticity $\Omega_k$.

# Gauss Decomposition for $T$

- Gauss decomposition of the scattering matrix:

$$T(\zeta) = T^{\mp}(\zeta)D^{\pm}(\zeta)(S^{\pm}(\zeta))^{-1},$$

where:

$$T^{+}(\zeta) = \begin{pmatrix} 1 & T_1^{+}(\zeta) & T_3^{+}(\zeta) \\ 0 & 1 & T_2^{+}(\zeta) \\ 0 & 0 & 1 \end{pmatrix}, \quad T^{-}(t,\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ T_1^{-}(\zeta) & 1 & 0 \\ T_3^{-}(\zeta) & T_2(\zeta) & 1 \end{pmatrix},$$

$$S^{+}(\zeta) = \begin{pmatrix} 1 & S_1^{+}(\zeta) & S_3^{+}(\zeta) \\ 0 & 1 & S_2^{+}(\zeta) \\ 0 & 0 & 1 \end{pmatrix}, \quad S^{-}(t,\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ S_1^{-}(\zeta) & 1 & 0 \\ S_3^{-}(\zeta) & S_2(\zeta) & 1 \end{pmatrix},$$

$$D^{+}(\zeta) = \mathrm{diag}\left( m_1^{+}(\zeta), \frac{m_2^{+}(\zeta)}{m_1^{+}(\zeta)}, \frac{1}{m_2^{+}(\zeta)} \right),$$

$$D^{-}(\zeta) = \mathrm{diag}\left( \frac{1}{m_2^{-}(\zeta)}, \frac{m_2^{-}(\zeta)}{m_1^{-}(\zeta)}, m_1^{-}(\zeta) \right).$$

# Gauss Decomposition for $T$

## Explicit form of the Gauss Factors

- Here $m_k^{\pm}(\zeta)$ are the principal upper/lower minors of order $k$ of the scattering matrix;
- the explicit expressions for the matrix elements of $T^{\pm}$ and $S^{\pm}$ in terms of $T_{ij}(\zeta)$ are given by:

$$
\begin{aligned}
S_1^+(\zeta) &= -\frac{T_{12}(\zeta)}{T_{11}(\zeta)}, \quad S_2^+(\zeta) = \frac{T_{13}(\zeta)T_{21}(\zeta) - T_{11}(\zeta)T_{23}(\zeta)}{T_{11}(\zeta)T_{22}(\zeta) - T_{12}(\zeta)T_{21}(\zeta)}, \\
S_3^+(\zeta) &= \frac{T_{12}(\zeta)T_{23}(\zeta) - T_{13}(\zeta)T_{22}(\zeta)}{T_{11}(\zeta)T_{22}(\zeta) - T_{12}(\zeta)T_{21}(\zeta)}, \\
T_1^-(\zeta) &= \frac{T_{21}(\zeta)}{T_{11}(\zeta)}, \quad T_2^-(\zeta) = \frac{T_{32}(\zeta)T_{11}(\zeta) - T_{31}(\zeta)T_{12}(\zeta)}{T_{11}(\zeta)T_{22}(\zeta) - T_{12}(\zeta)T_{21}(\zeta)}, \\
T_3^-(\zeta) &= \frac{T_{31}(\zeta)}{T_{11}(\zeta)}.
\end{aligned}
$$

- Due to the the special choice of the matrix $Q(x,t)$, it follows that $S^-(\zeta) = (S^+(1/\zeta^*))^\dagger$ and $T^+(\zeta) = (T^-(1/\zeta^*))^\dagger$.

# On the Inverse Scattering Problem for $L$

## Casting to a Riemann-Hilbert Problem

- One of the most effective method for solving the ISP for a given $L$ is to reduce it to a RHP.
- On the complex $\zeta$-plane it can be formulated as follows:

$$\xi^{+}_{(\alpha)}(x,\zeta) = \xi^{-}_{(\beta)}(x,\zeta) G_{\alpha,\beta}(x,t,\zeta), \qquad \lim_{k\to\infty} \xi^{+}(x,\zeta) = \mathbb{1},$$
$$G(\zeta) = e^{-iJ(\zeta)x - iF(\zeta)t} (S^{-})^{-1} S^{+} e^{iJ(\zeta)x + iF(\zeta)t}.$$

  This relation holds true for $k \in \mathbb{R}$ in the complex $k$-plane.
- The RHP for Lax operators with vanishing boundary conditions look similarly. However, in our case, it is more complicated due to the fact, that we are dealing with an RHP formulated on the Riemannian surface related to the root $k(\lambda) = \sqrt{\lambda^2 - 4\rho^2}$.
- The sewing function $G(x,\zeta)$ gives the minimal set of scattering data, sufficient to reconstruct the scattering matrix $T(\zeta)$.

# Fixing up the Dispersion Law

- Start with a bit more general $M$-operator:

$$M\psi \equiv \left( i\frac{\partial}{\partial t} + [I, Q(x,t)] - \zeta I \right) \psi(x,t,\zeta) = \psi(x,t,\zeta)F(\zeta).$$

- The compatibility condition $[L, M] = 0$ holds true for any $x$- and $t$-independent matrix $F(\zeta)$.
- $F(\zeta)$ is an eigenfunction for the asymptotic $M$-operator, when $x \to \pm\infty$:

$$([I, Q_\pm] - \zeta I)\psi_{\pm,0} = \psi_{\pm,0}F(\zeta).$$

- It is easy to check that $\psi_{\pm,0}$ diagonalize also $[I, Q_\pm] - \zeta I$ and therefore $F(\zeta)$ is a diagonal matrix:

$$F(\zeta) = \mathrm{diag}\,(f_1(\zeta), f_2(\zeta), f_3(\zeta)),$$

# Time Evolution of the Scattering Data

- In terms of $\zeta$ we have:

$$F(\zeta) = \rho \,\mathrm{diag}\left( l_1\zeta + \frac{l_3}{\zeta},\; l_2\left(\zeta - \frac{1}{\zeta}\right),\; l_3\zeta + \frac{l_1}{\zeta}\right).$$

- Time evolution of the associated scattering matrix $T(\zeta)$:

$$i\frac{dT}{dt} - [F(\zeta), T(t,\zeta)] = 0.$$

As a consequence the Gauss factors of $T(t,\zeta)$ satisfy

$$i\frac{dT^{\pm}}{dt} - [F(\zeta), T^{\pm}(t,\zeta)] = 0, \quad i\frac{dS^{\pm}}{dt} - [F(\zeta), S^{\pm}(t,\zeta)] = 0.$$

$$i\frac{dD^{\pm}}{dt} = 0.$$

# Time Evolution of the Gauss Factors

- The principle minors $m_1^{\pm}(\zeta)$ and $m_2^{\pm}(\zeta)$ are time-independent and can be considered as generating functionals of the integrals of motion
- For the off-diagonal ones we get:

$$T_{ij}(t,\zeta) = T_{ij}(0,\zeta)e^{-i(f_i(\zeta)-f_j(\zeta))t}.$$

- The function $F(\zeta)$ is known as the dispersion law for the 3-wave equations with constant boundary conditions.

# The Class of Admissible Potentials

- The crucial fact that determines the spectral properties of the operator $L(\zeta)$ is the choice of the class of functions where from we shall choose the potential $Q(x)$.
- For a sake of simplicity, we assume that $Q(x,t)$ satisfies

  **Condition C.1** $Q(x,t)$ is smooth for all $x$ and $t$ and is such that

$$\lim_{x\to\pm\infty} |x|^p(Q(x,t) - Q_\pm) = 0 \quad \text{for all} \quad p = 0, 1, \dots .$$

- The FAS $\chi^{\pm}(x,\zeta)$ of $L(\zeta)$ allows one to construct the resolvent of the operator $L$ and then to investigate its spectral properties.

# The resolvent of $L(\zeta)$

- By a resolvent of $L(\zeta)$ we understand an integral operator $R(\zeta)$ with kernel $R(x,y,\zeta)$ which satisfies

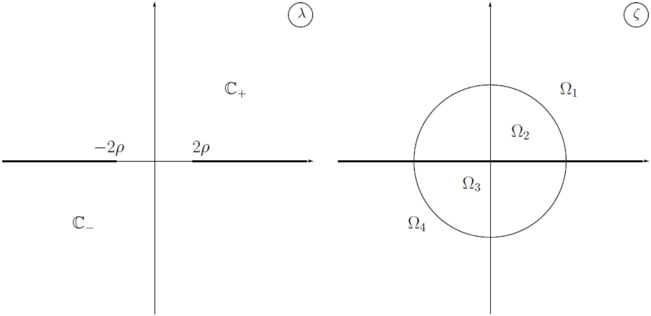
$$L(\zeta)(R(\zeta)f)(x) = f(x),$$

where $f(x)$ is an 3-component vector complex-valued function with bounded norm, i.e.

$$\int_{-\infty}^{\infty} dy |f^T(y) f(y)| < \infty.$$

- From the general theory of linear operators we know that the point $\zeta$ in the complex $\zeta$-plane is a regular point if $R(\zeta)$ is a bounded integral operator.
- In each connected subset of regular points $R(\zeta)$ is analytic in $\zeta$.
- The points $\zeta$ which are not regular constitute the spectrum of $L(\zeta)$.

# The Spectrum of $L(\zeta)$




Roughly speaking the spectrum of $L(\zeta)$ consist of two types of points:

- i) the continuous spectrum of $L(\zeta)$ consists of all points $\zeta$ for which $R(\zeta)$ is an unbounded integral operator;
- ii) the discrete spectrum of $L(\zeta)$ consists of all points $\zeta$ for which $R(\zeta)$ develops pole singularities.

# Properties of the Resolvent
## Explicit form using FAS

If we write down $R(\zeta)$ in the form:

$$R(\zeta)f(x) = \int_{-\infty}^{\infty} R(x,y,\zeta)f(y),$$

the kernel $R(x,y,\zeta)$ of the resolvent is given by:

$$R_{(\alpha)}(x,y,\zeta) = R^{\pm}_{(\alpha)}(x,y,\zeta), \qquad \text{for } \zeta \in \Omega_{(\alpha)}.$$

where

$$R^{\pm}_{(\alpha)}(x,y,\zeta) = -i\chi^{\pm}_{(\alpha)}(x,\zeta)\Theta^{\pm}_{(\alpha)}(x-y)(\chi^{\pm}_{(\alpha)})^{-1}(y,\zeta), \qquad \zeta \in \Omega_{(\alpha)},$$

# Properties of the Resolvent

## Explicit form using FAS

Here

$$\begin{aligned}
\Theta_1^+(x-y) &= \operatorname{diag}\left(-\theta(y-x), \theta(x-y), \theta(x-y)\right),\\
\Theta_2^-(x-y) &= \operatorname{diag}\left(-\theta(y-x), -\theta(y-x), \theta(x-y)\right),\\
\Theta_3^+(x-y) &= \operatorname{diag}\left(\theta(x-y), -\theta(y-x), -\theta(y-x)\right),\\
\Theta_4^-(x-y) &= \operatorname{diag}\left(-\theta(x-y), \theta(x-y), -\theta(y-x)\right),
\end{aligned}$$

for $J_2 > 0$, and

$$\begin{aligned}
\Theta_1^+(x-y) &= \operatorname{diag}\left(-\theta(y-x), -\theta(y-x), \theta(x-y)\right),\\
\Theta_2^-(x-y) &= \operatorname{diag}\left(-\theta(y-x), \theta(x-y), \theta(x-y)\right),\\
\Theta_3^+(x-y) &= \operatorname{diag}\left(\theta(x-y), \theta(x-y), -\theta(y-x)\right),\\
\Theta_4^-(x-y) &= \operatorname{diag}\left(-\theta(x-y), -\theta(y-x), -\theta(y-x)\right),
\end{aligned}$$

for $J_2 < 0$.

# Properties of the Resolvent

## Properties of the Kernel of $R$

**Theorem:**

*Let $Q(x)$ satisfy the conditions (C.1) and is such that the minors $m_k^{\pm}(\zeta)$ have a finite number of simple zeroes $\zeta_j^{\pm}$. Then*

1. $R^{\pm}(x,y,\zeta)$ *is an analytic function of* $\zeta$ *for* $\zeta \in \mathbb{C}_{\pm}$ *having pole singularities at* $\zeta_j^{\pm}$;
2. $R^{\pm}(x,y,\zeta)$ *is a kernel of a bounded integral operator for* $\zeta \in \mathbb{R} \cup \mathcal{E}$;
3. $R(x,y,\zeta)$ *is uniformly bounded function for* $\zeta \in \mathbb{R} \cup \mathcal{E}$ *and provides a kernel of an unbounded integral operator;*
4. $R^{\pm}(x,y,\zeta)$ *satisfy the equation:*

$$L(\zeta)R^{\pm}(x,y,\zeta) = \mathbb{1}\delta(x-y).$$

# Properties of the Resolvent ... (cont'd)

- The continuous spectrum of $L$ in the complex $k$-plane coincides with the contour of the RHP $\mathbb{R} \cup \mathcal{S}$ with multiplicity 3 on $\mathbb{R}$ and multiplicity 1 on $\mathcal{S}$.
- On the complex $\lambda$-plane the continuous spectrum of $L$ is on the real axis; it has multiplicity 3 on the semi-axis $\mathrm{Re}\,\lambda < -2\rho$ and $\mathrm{Re}\,\lambda > 2\rho$ and multiplicity 1 in the 'lacuna' $-2\rho < \mathrm{Re}\,\lambda < 2\rho$.

# The Integrals of Motion and the Scattering Data

- The diagonal factors $D^{\pm}(\zeta)$ are time independent and can be used to generate the infinite set of integrals of motion.
- For the 3WRI eqns., these matrices are expressed through the principal upper/lower minors $m^{\pm}(\zeta)$ of the scattering matrix $T(\zeta)$:

$$\ln D^{\pm}_{k,1} = -\frac{i}{4}(J_k - J_{k+1})\mathcal{P}_k + (J_1 - J_3)\mathcal{P}_3,$$

- The momenta $\mathcal{P}_k$, $k = 1, 2, 3$ are given by:

$$\mathcal{P}_1 = \int_{-\infty}^{\infty} dx\, |q_1(x)|^2, \quad \mathcal{P}_2 = \int_{-\infty}^{\infty} dx\, |q_2(x)|^2,$$
$$\mathcal{P}_3 = \int_{-\infty}^{\infty} dx\, (|q_3(x)|^2 - \rho^2).$$

- The fact that $\ln m_1^{\pm}$ generates integrals of motion can be considered as natural analog of the Manley–Rowe relations.

# The Integrals of Motion and the Scattering Data ... (cont'd)

- In our case it means an existence of two additional first integrals:

$$I_1 = (J_1 - J_2)\mathcal{P}_1 + (J_1 - J_3)\mathcal{P}_3 = \mathrm{const},$$
$$I_2 = (J_2 - J_3)\mathcal{P}_2 + (J_1 - J_3)\mathcal{P}_3 = \mathrm{const},$$

- They can be interpreted as relations between the densities $|q_\alpha|^2$ of the waves of type $\alpha$.
- The total momentum for the 3-waves is also a conserved quantity:

$$\mathcal{P} = (J_1 - J_2)\mathcal{P}_1 + (J_2 - J_3)\mathcal{P}_2 + (J_1 - J_3)\mathcal{P}_3 = \mathrm{const}.$$

- The integral of motion $D_2$ is proportional to the Hamiltonian of the 3-wave equations.
- The functional $H_{3-\mathrm{w}}$ remains the same as for the case of vanishing boundary conditions.

# Summary

- We studied the direct scattering problem for the Lax operator and its spectral properties.
  This includes: the construction of Lax representation and the Jost solutions of the Lax operator $L$.
- Furthermore, we outlined the construction of the fundamental analytic solutions (FAS) of $L$ and formulated a Riemann-Hilbert problem for the FAS on a relevant Riemannian surface.
- We also outlined the construction of the resolvent of $L(\zeta)$ in terms of the FAS and the spectral properties of $L$.
- Finally, we briefly discuss the effects of the boundary conditions on the conserved quantities of the 3-wave equations: we showed that the total momentum for non-vanishing boundary conditions needs regularization, while the Hamiltonian remains the same.

# Outlook

- Similar analysis can be done for a 3-wave resonant interaction model with more general boundary conditions: $\lim_{x\to-\infty} q_k(x,t) = q_k^-$ ($k = 1$ or $2$) and $\lim_{x\to+\infty} q_3(x,t) = q_3^-$. This may require the matrices $Q(\theta)$ to have also off-diagonal entries.
- It is an open problem to to derive the soliton solutions in the case of constant boundary conditions (the so-called "dark solitons"), the dark-dark and dark-bright soliton solutions, etc.
- Another challenge is to extend this analysis also for systems, describing resonant interactions of $N$ waves or to $N$-wave type systems related to simple Lie algebras.
- Another open problem is to study the behavior of the scattering data at the end-points of the continuous spectrum in the complex $\lambda$-plane.

# Thank you!

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