Fermionic screenings and line bundle twisted chiral de Rham complex on toric manifolds.

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1. Introduction

Chiral de Rham (CDR) complex is a certain sheaf of vertex operator algebras which is a string generalization of the usual de Rham complex. It has been introduced for every smooth manifold by Malikov Shechtman and Vaintrob. The CDR on Calabi-Yau (CY) compact manifold is a most important case to consider in string theory. In this situation CDR is closely related to the space of states of the (open) string on CY manifold.

1.1 Chiral de Rham complex in local coordinates.

On the affine space A with local coordinates $a_1, ..., a_d$ it is given by the set of bosonic fields

$$a_{\mu}(z) = \sum_{n \in \mathbb{Z}} a_{\mu}[n] z^{-n},$$
$$a^{*}_{\mu}(z) = \sum_{n \in \mathbb{Z}} a^{*}_{\mu}[n] z^{-n-1}, \qquad (1)$$

and fermionic fields

$$\alpha_\mu(z) = \sum_{n\in\mathbb{Z}} \alpha_\mu[n] z^{-n-\frac{1}{2}},$$
$$\alpha^*_\mu(z) = \sum_{n\in\mathbb{Z}} \alpha^*_\mu[n] z^{-n-\frac{1}{2}},$$
$$\mu = 1, ..., d \qquad (2)$$

with the commutation relations between the modes

$$[a^*_\mu[n], a_\nu[m]] = \delta_{\mu\nu}\delta_{n+m,0},$$
$$\left\{\alpha^*_\mu[n], \alpha_\nu[m]\right\} = \delta_{\mu\nu}\delta_{n+m,0} \qquad (3)$$

The space of sections M_A of the chiral de Rham complex over the affine space A is generated from vacuum state $|0>$ by the non-positive modes of the fields $a_\mu(z)$, $\alpha_\nu(z)$ and by negative modes of $a^*_\mu(z)$, $\alpha^*_\nu(z)$.

1.2. The change of coordinates.

The important property is the behavior of the ($bc\beta\gamma$)fields (1), (2) under the local change of coordinates on A (Malikov, Shechtman, Vaintrob). For each new set of coordinates

$$\begin{aligned} b_\mu &= g_\mu(a_1, ..., a_d) \\ a_\mu &= f_\mu(b_1, ..., b_d) \end{aligned} \tag{4}$$

the isomorphic $bc\beta\gamma$ system of fields is given by

$$\begin{aligned} b_\mu(z) &= g_\mu(a_1(z), ..., a_d(z)) \\ b^*_\mu(z) &= \frac{\partial f_\nu}{\partial b_\mu}(a_1(z), ..., a_d(z))a^*_\nu(z) + \\ &\frac{\partial^2 f_\lambda}{\partial b_\mu \partial b_\nu}\frac{\partial g_\nu}{\partial a_\rho}(a_1(z), ..., a_d(z))\alpha^*_\lambda(z)\alpha_\rho(z) \\ \beta_\mu(z) &= \frac{\partial g_\mu}{\partial a_\nu}(a_1(z), ..., a_d(z))\alpha_\nu(z) \\ \beta^*_\mu(z) &= \frac{\partial f_\nu}{\partial b_\mu}(a_1(z), ..., a_d(z))\alpha^*_\nu(z) \end{aligned} \tag{5}$$

(the normal ordering is implied).

1.3. The geometric interpretation of the fields:

$$
\begin{aligned}
a_\mu(z) &\leftrightarrow \ coordinate\ a_\mu\ on\ A \\
a^*_\mu(z) &\leftrightarrow \ \frac{\partial}{\partial a_\mu} \\
\alpha_\mu(z) &\leftrightarrow \ da_\mu \\
\alpha^*_\mu(z) &\leftrightarrow \ conjugated\ to\ da_\mu,
\end{aligned}
\tag{6}
$$

1.4. $N = 2$ Virasoro superalgebra currents and Calabi-Yau condition.

On M_A the $N = 2$ Virasoro superalgebra acts by the currents

$$J(z) = \sum_{\mu} \alpha_{\mu}^{*}(z)\alpha_{\mu}(z) = \sum_{n\in\mathbb{Z}} J[n]z^{-n-1}$$
$$T(z) = \sum_{\mu}(a_{\mu}^{*}\partial_z a_{\mu} + \frac{1}{2}(\partial_z \alpha_{\mu}^{*}\alpha_{\mu} - \alpha_{\mu}^{*}\partial_z\alpha_{\mu})) = \sum_{n\in\mathbb{Z}} L[n]z^{-n-2}$$
$$G^{+}(z) = -\sum_{\mu} \alpha_{\mu}^{*}(z)\partial_z a_{\mu}(z) = \sum_{n\in\mathbb{Z}} G^{+}[n]z^{-n-\frac{3}{2}}$$
$$G^{-}(z) = \sum_{\mu} \alpha_{\mu}(z)a_{\mu}^{*}(z) = \sum_{n\in\mathbb{Z}} G^{-}[n]z^{-n-\frac{3}{2}} \qquad (7)$$

Commutation relations

$$
\begin{aligned}
[L[n], L[m]] &= (n-m)L[n+m] + \frac{d}{4}(n^3 - n)\delta_{n+m,0}, \\
\left\{G^{+}[n], G^{-}[m]\right\} &= 2L[n+m] + (n-m)J[n+m] + \\
& d(n^2 - \frac{1}{4})\delta_{n+m,0}, \\
[J[n], J[m]] &= nd\delta_{n+m,0}, \\
\left[L[n], G^{\pm}[m]\right] &= (\frac{n}{2} - m)G^{\pm}[n+m], \\
\left[J[n], G^{\pm}[m]\right] &= \pm G^{\pm}[n+m], \\
[L[n], J[m]] &= -mJ[n+m]
\end{aligned}
\tag{8}
$$

One can see that

$$G^{-}[0] = \sum_{\mu} \sum_{n \in \mathbb{Z}} \alpha_{\mu}[n] a^{*}_{\mu}[-n] \qquad (9)$$

is the string generalization of the de Rham differential and on zero level with respect to $L[0]$-grading we have the usual de Rham complex

$$\begin{array}{c} \alpha_{\lambda}[0] ... \alpha_{\gamma}[0] a_{\mu}[0] ... a_{\nu}[0] |0> \in M_A \leftrightarrow forms \\ d_{DR} = \sum_{\mu} \alpha_{\mu}[0] a^{*}_{\mu}[0] \leftrightarrow de\ Rham\ differential \end{array} \qquad (10)$$

•$N = 2$ **Virasoro superalgebra is defined globally when the manifold is Calabi-Yau (CY)**

This is most important for the string theory applications. If it is not, only zero modes $G^{+}[0]$, $G^{-}[0]$, $J[0]$, $L[0]$ are defined globally.

For the case of toric manifold as well as CY hypersurface in toric manifold the explicit construction of chiral de Rham complex has been found recently by L.Borisov.

In this talk I would like to represent a generalization of Borisov construction to include chiral de Rham complex twisted by line bundle defined on toric manifold. (In case of d-dimensional CY manifold it may describe the state of states of bound (d,d-2) bound system of D-branes.)

1.5. Approach.

The main object of the construction is a set of fermionic screening currents

$$S_i^*(z) = \sum_n S_i^*[n] z^{-n-1} \tag{11}$$

associated to the points of the polytope Δ^* defining the toric manifold $\mathbb{P}_{\Delta^*}$. Zero modes of these currents are used to build up a differential

$$D_{\Delta^*} = \sum_i S_i^*[0] \tag{12}$$

whose cohomology calculated in some lattice vertex algebra gives the global sections of a chiral de Rham complex on $\mathbb{P}_{\Delta^*}$. The approach can also be applied for the case when we have a pair of dual (reflexive) polytopes Δ^* and Δ defining CY hypersurface in toric manifold $\mathbb{P}_{\Delta^*}$.

1.6. Generalization.

We consider toric manifold $\mathbb{P}_{\Delta^*}$ and generalize differential allowing also non-zero modes for screening currents:

$$D_{\Delta^*} \to \tilde{D}_{\Delta^*} = \sum_i S_i^*[N_i] \tag{13}$$

Proposition.
The numbers N_i of screening current modes from $\tilde{D}_{\Delta^*}$ define the support function of toric divisor of a line bundle on $\mathbb{P}_{\Delta^*}$.

By this means, the chiral de Rham complex on $\mathbb{P}_{\Delta^*}$ appears to be twisted by the line bundle.

2. Chiral de Rham complex on $\mathbb{P}^d$.

$$\mathbb{P}_{\Delta^*} = \mathbb{P}^d \qquad (14)$$

Let me review the construction of chiral de Rham complex and its cohomology for the case of projective space $\mathbb{P}^d$.

2.1. The fan of $\mathbb{P}^d$.

Let

$$\Lambda = \mathbb{Z}e_1 \oplus ... \oplus \mathbb{Z}e_d \qquad (15)$$

be the lattice in $\mathbb{R}^d$. The vertices of the polytope $\Delta^* \subset \Lambda$ are given by the vectors $e_1, ..., e_d$ and the vector e_0

$$e_0 = -e_1 - ... - e_d \qquad (16)$$

Then one considers the fan

$$\Sigma \subset \Lambda \qquad (17)$$

coding the toric dates of $\mathbb{P}^d$. The fan Σ is a union of finite number of cones.

In our case d-dimensional cones are

$$C_i = Span(e_0, ..., \hat{e_i}, ..., e_d) \subset \Sigma,$$
$$i = 0, 1, 2, ..., d, \qquad (18)$$

where the vector e_i is missing. The standard toric construction associates to the cone C_i the affine space

$$A_i \approx \mathbb{C}^d \qquad (19)$$

The affine spaces A_i, $i = 0, 1, ..., d$ cover the toric manifold $\mathbb{P}^d$

$$\mathbb{P}^d = \cup A_i \qquad (20)$$

The intersection of an arbitrary number of cones C_i is also a cone in Σ:

$$C_i \cap C_j ... \cap C_k = C_{ij...k} \subset \Sigma \qquad (21)$$

For positive codimensions cones C_{ij}, C_{ijk},... the toric construction associates the spaces

$$A_{ij} = A_i \cap A_j \approx \mathbb{C}^{d-1} \times \mathbb{C}^*,$$
$$A_{ijk} = A_i \cap A_j \cap A_k \approx \mathbb{C}^{d-2} \times (\mathbb{C}^*)^2 ... \qquad (22)$$

The collection of spaces A_i, A_{ij},...,$A_{012...d}$ can be used to calculate $\check{C}$ech cohomology of anything on $\mathbb{P}^d$.

2.2. Chiral de Rham complex over A_i in logarithmic coordinates.

Let

$$
\begin{aligned}
X_\mu(z) &= x_\mu + X_\mu[0]\ln(z) - \sum_{n\neq 0}\frac{X_\mu[n]}{n}z^{-n} \\
X^*_\mu(z) &= x^*_\mu + X^*_\mu[0]\ln(z) - \sum_{n\neq 0}\frac{X^*_\mu[n]}{n}z^{-n} \\
\psi_\mu(z) &= \sum_{n\in\mathbb{Z}}\psi_\mu[n]z^{-n-\frac{1}{2}} \\
\psi^*_\mu(z) &= \sum_{n\in\mathbb{Z}}\psi^*_\mu[n]z^{-n-\frac{1}{2}}, \\
&\mu = 1, ..., d
\end{aligned}
\tag{23}
$$

be free bosonic and free fermionic fields

$$
\begin{aligned}
[x_\mu, X^*_\nu[0]] = \delta_{\mu\nu},\ \left[x^*_\mu, X_\nu[0]\right] &= \delta_{\mu\nu}, \\
[X_\mu[n], X^*_\nu[m]] = n\delta_{\mu\nu}\delta_{n+m,0},\ n,m &\neq 0, \\
\{\psi_\mu[n], \psi^*_\nu[m]\} &= \delta_{\mu\nu}\delta_{n+m,0}
\end{aligned}
\tag{24}
$$

For the lattice

$$
\Gamma = \Lambda \oplus \Lambda^* \tag{25}
$$

where Λ^* is dual to Λ we introduce the direct sum of Fock modules

$$\Phi_\Gamma = \oplus_{(p,p^*)\in\Gamma} F_{(p,p^*)} \qquad (26)$$

where $F_{(p,p^*)}$ is the Fock module generated from the vacuum $|p,p^* >$

$$\begin{aligned} X^*_\mu[0]|p,p^* > &= p^*_\mu|p,p^* > \\ X_\mu[0]|p,p^* > &= p_\mu|p,p^* > \end{aligned} \qquad (27)$$

by the negative modes $X_\mu[n]$, $X^*_\mu[n]$, $\psi^*_\mu[n]$, $n < 0$ and by non-positive modes $\psi_\mu[n]$, $n \leq 0$.

2.3. Fermionic screenings.

For every vector e_i, $i = 0, 1, ..., d$ generating 1-dimensional cone from Σ, one defines the fermionic screening current and screening charge

$$S_i^*(z) = e_i \cdot \psi^* \exp(e_i \cdot X^*)(z)$$
$$\oint dz S_i^*(z) = S_i^*[0] \qquad (28)$$

We form the BRST operator for every maximal dimension cone C_i

$$D_i^* = S_0^*[0] + ... + \widehat{S^*}_i[0] + ... + S_d^*[0],$$
$$i = 0, ..., d \quad (29)$$

where $S_i^*[0]$ is missing. Then, one considers the space

$$\Phi_{C_i \otimes \Lambda^*} = \oplus_{(p,p^*) \in C_i \otimes \Lambda^*} F_{(p,p^*)} \qquad (30)$$

Proposition (L.Borisov).

The space of sections M_i of the chiral de Rham complex over the affine space A_i is given by the cohomology of $\Phi_{C_i \otimes \Lambda^*}$ with respect to the operator D_i^*.

M_i is generated by the creation modes of the $bc\beta\gamma$ system of fields

$$\begin{aligned} a_{i\mu}(z) = \exp[w^*_{i\mu} \cdot X](z) \\ a^*_{i\mu}(z) = (e_\mu \cdot \partial X^* - \\ w^*_{i\mu} \cdot \psi e_\mu \cdot \psi^*) \exp[-w^*_{i\mu} \cdot X](z) \\ \alpha_{i\mu}(z) = w^*_{i\mu} \cdot \psi \exp[w^*_{i\mu} \cdot X](z), \\ \alpha^*_{i\mu}(z) = e_\mu \cdot \psi^* \exp[-w^*_{i\mu} \cdot X](z), \\ \mu = 0, ...\hat{i}, ...d \end{aligned} \quad (31)$$

from the vacuum state

$$|0> = |(p = 0, p^* = 0)> \quad (32)$$

where

$$w^*_{i\mu}(e_\nu) = \delta_{\mu\nu}, \;\; \mu, \nu = 0, ..., \hat{i}, ..., d \quad (33)$$

2.4. Cohomology of chiral de Rham complex on $\mathbb{P}^d$.

Proposition(L.Borisov).

The cohomology of the chiral de Rham complex on $\mathbb{P}^d$ can be calculated as a $\check{C}$ech cohomology of the covering by A_i, i=0,...,d

$$0 \to \oplus_i M_i \to \oplus_{k<j} M_{kj} \to \cdots$$
$$\cdots \to M_{012...d} \to 0 \qquad (34)$$

and coincide with the cohomology with respect to

$$D_{\Delta^*} = S_0^*[0] + ... + S_d^*[0] \qquad (35)$$

The modules $M_{kj...l}$ are the sections of CDR over the $A_{kj...l}$. They are given by the cohomology of $\Phi_{C_{kj...l} \otimes \Lambda^*}$ with respect to the operator

$$D^*_{kj...l} = S_0^*[0] + ... + \hat{S^*}_k[0] + ...$$
$$+ \hat{S^*}_j[0] + ... + \hat{S^*}_l[0] + ... + S_d^*[0] \qquad (36)$$

and they are generated by $bc\beta\gamma$ systems also.

•**Example:** M_{ij} is generated by the creation modes of the fields

$$a_{i\mu}(z),\ a^*_{i\mu}(z),\ \mu \neq i,j,$$
$$a_{i\mu}(z),\ a^{-1}_{i\mu}(z),\ a^*_{i\mu}(z),\ \mu = j,$$
$$\alpha_{i\mu}(z),\ \alpha^*_{i\mu}(z),\ \mu \neq i \qquad (37)$$

Thus M_{ij} is a localization of M_i with respect to the multiplicative system generated by the monomial fields

$$a^{m_1}_{i1}(z)...\widehat{a}_{ii}(z)...a^{m_d}_{id}(z),$$
$$where$$
$$(m_1 w^*_{i1} + ...m_d w^*_{id})(C_{ij}) = 0,$$
$$m_1 w^*_{i1} + ...m_d w^*_{id} \in \Lambda^* \qquad (38)$$

The $\check{C}$ech differentials in the complex (34) are given by the localization maps:

$$M_i \to M_{ij} \leftarrow M_j$$
$$..... \qquad (39)$$

3. Line bundle twisted chiral de Rham complex on $\mathbb{P}^d$.

Let us twist the fermionic screening charges $S_i^*[0]$:

$$S_i^*[0] \to S_i^*[N_i] = \oint dz z^{N_i} S_i^*(z), \;\; N_i \in \mathbb{Z} \quad (40)$$

and consider the BRST operator

$$\tilde{D}_i^* = S_0^*[N_0] + ... + \hat{S}^*{}_i[N_i] + ... + S_d^*[N_d] (41)$$

•What is the cohomology of the space $\Phi_{C_i \otimes \Lambda^*}$ with respect to this new BRST operator?

By the direct calculation one can see that the fields $a_{i\mu}(z)$, $\alpha_{i\mu}(z)$, $\alpha^*_{i\mu}(z)$ still commute with the new differential $\tilde{D}_i^*$ but instead of $a^*_{i\mu}(z)$ one has to take

$$\nabla_{i\mu}(z) = a^*_{i\mu}(z) + N_\mu z^{-1} a^{-1}_{i\mu}(z) \qquad (42)$$

The last term in this expression is a $U(1)$ gauge potential on A_i and hence the modes of the fields $\nabla_{i\mu}(z)$ can be regarded as a string version of the covariant derivatives.

Proposition.

In the twisted case the space of sections $\tilde{M}_i$ of the chiral de Rham complex over the affine space A_i is given by the cohomology of $\Phi_{C_I\otimes\Lambda^*}$ with respect to the operator $\tilde{D}_i^*$. $\tilde{M}_i$ is generated by the non-positive modes of the fields $a_{i\mu}(z)$, $\alpha_{i\mu}(z)$ and negative modes of the fields $\nabla_{i\mu}(z)$, $\alpha^*_{i\mu}(z)$ from the vacuum state

$$|\Omega_i >= |(0, -\sum_{\mu\neq i} N_\mu w^*_{i\mu}) > \tag{43}$$

The vacuum $|\Omega_i >$ defines the trivializing izomorphism

$$g_i(z) : \tilde{M}_i \approx M_i \tag{44}$$

3.1. Line bundle and transition functions.

On the intersection $A_i \cap A_j$ one can find the relations between the fields

$$\begin{aligned} a_{i\mu}(z) = a_{j\mu}(z) a_{ji}^{-1}(z),\ \mu \neq i, j, \\ a_{ij}(z) = a_{ji}^{-1}(z), \\ ...... \end{aligned} \tag{45}$$

Using these relations we find that

$$\begin{aligned} g_{ij}|\Omega_j >= |\Omega_i >, \\ g_{ij} = (a_{ji}[0])^{N_0+N_1+N_2+...+N_d} \end{aligned} \tag{46}$$

The functions g_{ij} are the transition functions of a line bundle $O(N)$ on $\mathbb{P}^d$, where

$$N = N_0 + N_1 + N_2 + ... + N_d \tag{47}$$

One can extend the map between the vacua to the map between the modules

$$g_{ij}(z) : \tilde{M}_j \to \tilde{M}_i. \tag{48}$$

if one takes into account the transformation of gauge potential due to different trivializations.

It makes chiral de Rham differential

$$G_i^-[0] = \sum_{\mu \neq i} \sum_{n \in \mathbb{Z}} \alpha_{i\mu}[-n] \nabla_{i\mu}[n] \tag{49}$$

acting on $\tilde{M}_i$ globally defined

$$G_i^-[0] = G_j^-[0] \tag{50}$$

3.2. Cohomology of line bundle twisted chiral de Rham complex.

In the twisted case the cohomology of the chiral de Rham complex can similar be calculated as a $\check{C}$ech cohomology of the covering by A_i, i=0,...,d

$$0 \to \oplus_i \tilde{M}_i \to \oplus_{k<j} \tilde{M}_{kj} \to \cdots \\ \cdots \to \tilde{M}_{012\ldots d} \to 0 \qquad (51)$$

where the modules $\tilde{M}_{kj\ldots l}$ are the sections of CDR over the $A_{kj\ldots l}$. They are given by the cohomology of $\Phi_{C_{kj\ldots l} \otimes \Lambda^*}$ with respect to the operator

$$D^*_{kj\ldots l} = S^*_0[N_0] + ... + \hat{S^*}_k[N_k] + ... \\ + \hat{S^*}_j[N_j] + ... + \hat{S^*}_l[N_l] + ... + S^*_d[N_d] \quad (52)$$

3.3. Trivializing vaccua and toric divisor support function.

The module $\tilde{M}_{ij}$ for example, is generated from the vacuum vector

$$|\Omega_{ij}>=|(0,-\sum_{\mu\neq i,j}N_\mu w^*_{i\mu})> \tag{53}$$

by the creation operators of the fields

$$\begin{aligned} a_{i\mu}(z),\ \nabla_{i\mu}(z),\ \mu\neq i,j,\\ a_{i\mu}(z),\ a^{-1}_{i\mu}(z),\ a^*_{i\mu}(z),\ \mu=j\\ \alpha_{i\mu}(z),\ \alpha^*_{i\mu}(z),\ \mu\neq i \end{aligned} \tag{54}$$

$\tilde{M}_{ij}$ can also be generated from the vacuum

$$\begin{aligned} |\Omega_{ji}>=|(0,-\sum_{\mu\neq i,j}N_\mu w^*_{j\mu})>=\\ (a_{ij}[0])^{N-N_i-N_j}|\Omega_{ij}> \end{aligned} \tag{55}$$

by the creation operators of the fields

$$\begin{aligned} a_{j\mu}(z),\ \nabla_{j\mu}(z),\ \mu\neq i,j,\\ a_{j\mu}(z),\ a^{-1}_{j\mu}(z),\ a^*_{j\mu}(z),\ \mu=i\\ \alpha_{j\mu}(z),\ \alpha^*_{j\mu}(z),\ \mu\neq j \end{aligned} \tag{56}$$

The relation (55) is a particular case of compatibility conditions trivializing vaccua to be satisfied. They are the following. For each maximal dimension cone C_i the trivializing vacuum $|\Omega_i>$ defines a linear function $\phi_i \in \Lambda^*$ on this cone:

$$|\Omega_i> = |(0, -\phi_i)>,$$
$$\phi_i = \sum_{\mu \neq i} N_\mu w^*_{i\mu} \in \Lambda^*. \qquad (57)$$

It is easy to see that the collection of ϕ_i satisfies an obvious compatibility condition. Namely, they are coincide on the intersections $C_{ij} = C_i \cap C_j$ and define the new function ϕ_{ij} and so on.

•**The modes N_i of the screening currents define a toric divisor support function ϕ on Σ of the bundle $O(N)$ on $\mathbb{P}^d$.**

4. Conclusion.

The construction of line-bundle twisted CDR can be generalized to any toric manifold as well as CY hypersurface in toric manifold. In the last case the $N = 2$ Virasoro superalgebra acts on the cohomology of the line bundle twisted CDR.