XIVth International Conference on Geometry, Integrability and Quantization

The Dynamics of the Field of Linear Frames and Gauge Gravitation

Some Comments-2

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$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$S = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} F_{\mu\nu} F_{\varkappa\lambda}\, g^{\mu\varkappa} g^{\nu\lambda} = \frac{1}{2}(\bar{E}^2 - \bar{B}^2)$$

$$P = -\frac{1}{4} F_{\mu\nu} \check{F}^{\mu\nu} = -\frac{1}{8} \varepsilon^{\mu\nu\lambda\varkappa} F_{\mu\nu} F_{\lambda\varkappa} = \bar{E} \cdot \bar{B}$$

$$L[F] = \ell(S, P)$$

$g_{\mu\nu}$ - Minkowskian

<u>Maxwell</u>: $L = S$

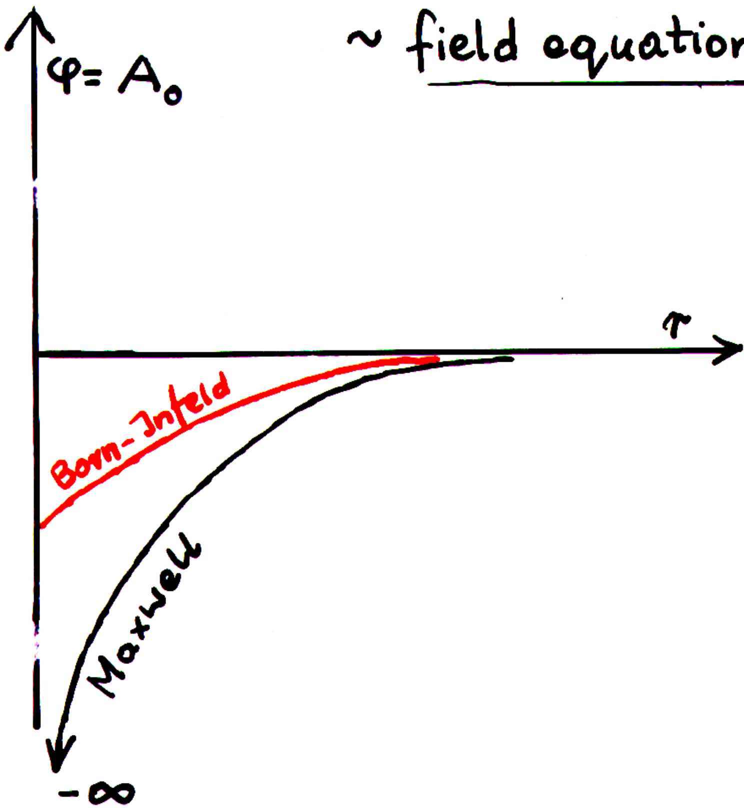
<u>Born-Infeld</u>: $L = b^2 - b^2\sqrt{1 - \frac{2}{b^2} S - \frac{1}{b^4} P^2} =$

$$= b^2\sqrt{|\det[g_{\mu\nu}]|} \boxed{- \sqrt{|\det[b g_{\mu\nu} + F_{\mu\nu}]|}}$$

<u>Born original</u>: $b^2\left(\sqrt{1 + \frac{1}{b^2}(\bar{B}^2 - \bar{E}^2)} - 1\right)$

Motivation: ~ finite electromagnetic self-energy of the electr.
(maximal electrostatic field)

~ field equations $\Rightarrow$ equations of motion (like in GR)



$$\bar{E}(\bar{r}) = \frac{e}{\sqrt{r_0^4 + r^4}} \frac{\bar{r}}{r}$$

$$r_0 = \sqrt{\frac{e}{b}}$$

$$\varphi(r) = \int_r^\infty \frac{e\,dx}{\sqrt{r_0^4 + x^4}}$$

$\varphi, \bar{E}$ – finite

$\bar{D}$ – infinite at $r = 0$

$w = T_{00}$ – infinite at $r = 0$, but $\mathcal{E} = \int w \, d_3\bar{r}$ – finite

Exceptionality of the Born-Infeld model:

~ gauge-invariant

~ energy positively definite

~ finite electromagnetic mass of point sources

~ energy current - non-spacelike

~ no birefringence

~ plane waves on the background of the constant electromagnetic field, solitary waves

(J. Plebański, Z. Białynicka-Birula)

Most interesting among nonlinear models, nevertheless, disappointing(?):

~ no convincing results in: field equations $\Rightarrow$ equations of motion (success of G.R.: Bianchi identities following from the general covariance nonlinearity relevant, but indirectly, as implied by general covariance)

~ the spectra of superheavy atoms do not seem to support B.-I.

~ quantization difficulties (non-polynomial structure)

~ QED is not afraid of infinities (renormalization). The electron mass is not purely electromagnetic (cancellation of infinities)

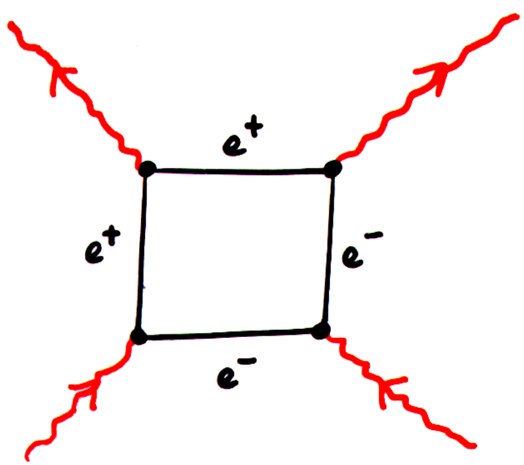
~ B.-I. does not suit well the external charged matter (other than the internal one, described by singularities of $\overline{D}$).

E.g., for the quantum coherent matter:

$$L = b^2\sqrt{|g|} - \sqrt{|bg+F|} + g^{\mu\nu} D_\mu \overline{\psi} D_\nu \psi \sqrt{|g|} - m^2 \overline{\psi}\psi\sqrt{|g|} \;; \quad D_\mu = \partial_\mu + ieA_\mu$$

(Non-rational structure of field equations).

B-I : useful as a classical model of some QEM - effects like the light-light-scattering :



Nonlinearity of the electromagnetic field: an affective description (Ersatz-Modell) of the nonlinear interaction between the linear Maxwell field and matter. (Nonlinearity replaces matter).

Nevertheless, the peculiarity of B-I seems to suggest that it was motivated by some good intuitions. One should discuss it and reformulate in more geometric terms, including the general covariance.

Linear models: $L(y^A; y^A{}_{,\mu})$ - quadratic in $y^A{}_{,\mu}$, y^A with y-independent coefficients; typically:

$$L(y, \partial y) = \ell(y, \partial y)\sqrt{|g|},$$

$$\ell(y, \partial y) = a_{KL}(x) y^K{}_{,\mu} y^L{}_{,\nu} g^{\mu\nu} + b_{KL}(x) y^K y^L .$$

Quasilinear: coefficients at ∂y depend on y:

$$\ell(y, \partial y) = a_{KL}(x, y) y^K{}_{,\mu} y^L{}_{,\nu} g^{\mu\nu} + b(x, y)$$

GR - quasilinear, but generally-covariant theories must be nonlinear.

Simple nonlinearities introduced „by hand": adding to L some terms of degree higher than 2 in ∂y, y. It does not work even in quasilinear GR.

<u>Density philosophy</u>:

Primary dynamical quantities are <u>Lagrangian tensors</u> $L_{\mu\nu}(y, \partial y)$.

$$\boxed{L(y,\partial y) := \sqrt{\left|\det\left[L_{\mu\nu}(y,\partial y)\right]\right|}}$$

GR - artificial in this language:

$$L = \operatorname{sign} R \sqrt{\left|\det\left[|R|^{2/n} g_{\mu\nu}\right]\right|} \underset{\text{if } n=4}{=} \operatorname{sign} R \sqrt{\left|\det\left[\sqrt{|R|}\, g_{\mu\nu}\right]\right|}$$

Locally:

$$L_{\mu\nu} = |R|^{n/2} g_{\mu\nu} \underset{\text{if } n=4}{=} \sqrt{|R|}\, g_{\mu\nu} .$$

The simplest models, opposite to linear theories: $L_{\mu\nu}$ is a low-order polynomial of derivatives. B-I - first-order polynomial.

Scalar field; the B-I counterpart of the scalar theory of light:

$$L = b^2 \sqrt{|\det[g_{\mu\nu}]|} - \sqrt{|\det[b g_{\mu\nu} + \phi_{,\mu}\phi_{,\nu}]|}$$

instead the linear model $g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\sqrt{|\det[g_{\mu\nu}]|}$.

For the spherically-symmetric stationary field (the point particle) the result identical with that for the usual B-I, although $L_{\mu\nu}$ -quadratic in $\partial\phi$. Barbashov-scalar Born-Infeld theory.

~ If so, perhaps the second-order B-I electrodynamics could also work:

$$\sqrt{\left|\det\left[\alpha g_{\mu\nu} + \beta F_{\mu\nu} + \gamma g^{\varkappa\lambda} F_{\mu\varkappa} F_{\lambda\nu} + \delta g^{\varkappa\varrho} g^{\lambda\sigma} F_{\varkappa\lambda} F_{\varrho\sigma}\, g_{\mu\nu}\right]\right|}$$

additional terms: familiar from the Maxwell energy-momentum tensor.

This modification: necessary in the Born-Infeld theory of gauge fields ruled by semisimple groups:

$$\sqrt{\left|\det\left[\alpha g_{\mu\nu} + \gamma g^{\varkappa\lambda} F^{K}{}_{\mu\varkappa} F^{L}{}_{\lambda\nu} h_{KL}\right]\right|}$$

~ If so, it is perhaps possible to overcome the problem of incompatibility between B-I and „external" matter, e.g., in this way:

$$\sqrt{\left|\det\left[a g_{\mu\nu} + b F_{\mu\nu} + c \overline{D_{\mu}\Psi} D_{\nu}\Psi\right]\right|}$$

+ perhaps the terms quadratic in F.

The resulting field equations are rational, although Lagrangian is not.

Natural models for scalar multiplets.

(M, g) - space-time , coordinates x^μ

(W, η) - target space , coordinates y^A

η - pseudo-Euclidean or hermitian

$\Phi : M \to W$ $\quad$ $y^A = \Phi^A(x^\mu)$

$$\boxed{L = \frac{1}{2}\eta_{AB}(\Phi(x))\frac{\partial \Phi^A}{\partial x^\mu}\frac{\partial \Phi^B}{\partial x^\nu} g^{\mu\nu}\sqrt{|\det[g_{\mu\nu}]|}}$$

– „d'Alembert"
(but η may depend on Φ).

If W - linear and η - constant, algebraic (mass) term possible :

$$\boxed{L = \frac{1}{2}\eta_{AB}\frac{\partial \Phi^A}{\partial x^\mu}\frac{\partial \Phi^B}{\partial x^\nu} g^{\mu\nu}\sqrt{|\det[g_{\mu\nu}]|} - \frac{m^2}{2}\eta_{AB}\Phi^A\Phi^B\sqrt{|\det[g_{\mu\nu}]|}}$$

+ possibly nonlinear terms $f(\eta_{AB}\Phi^A\Phi^B)$, e.g., $c\left(\eta_{AB}\Phi^A\Phi^B - \lambda\right)^2$

Born-Infeld version:

$$\boxed{L = \sqrt{\left|\det\left[\alpha g_{\mu\nu} + \beta \eta_{AB}\frac{\partial \Phi^A}{\partial x^\mu}\frac{\partial \Phi^B}{\partial x^\nu} + \gamma \eta_{AB}\Phi^A \Phi^B g_{\mu\nu}\right]\right|}}$$

fixed $g_{\mu\nu}$ - nongeometric feature; violation of the general covariance.

From now on: generally-covariant B-I schemes:

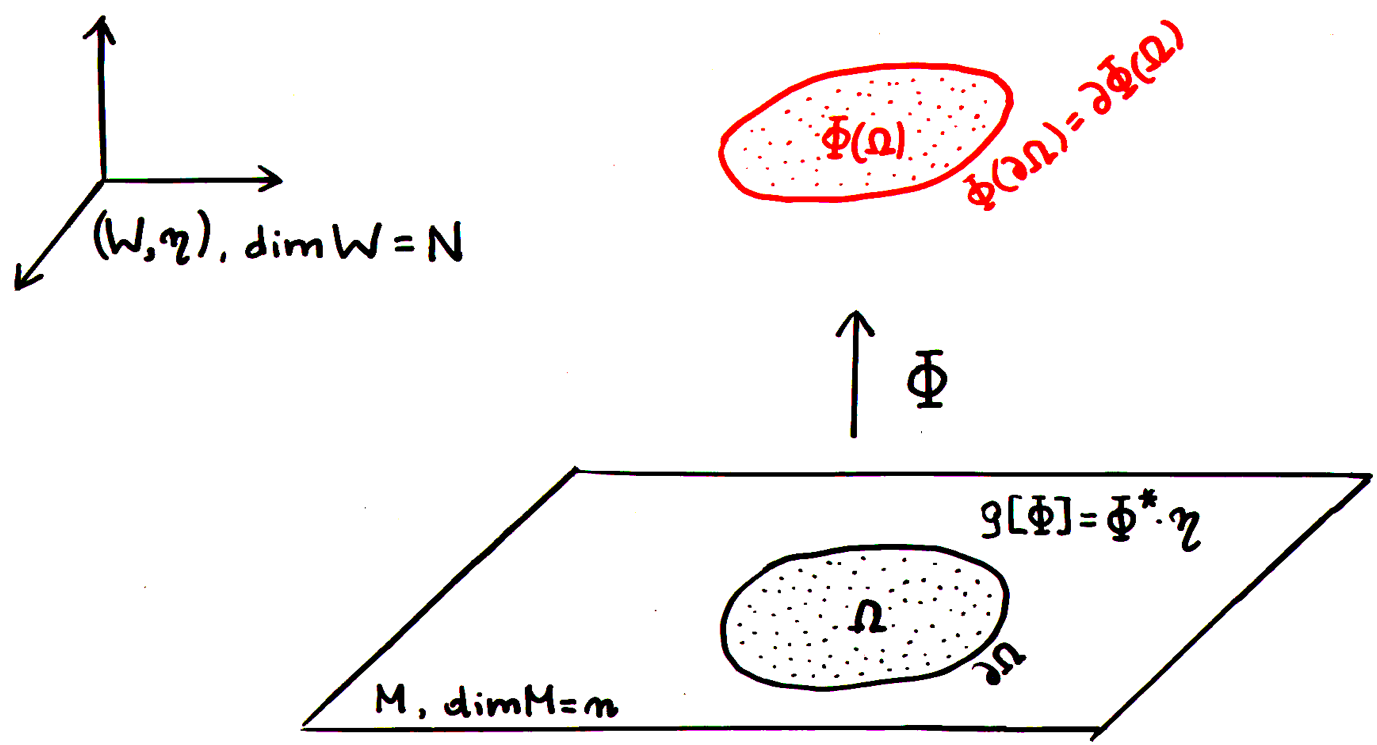
M - amorphous; no fixed metric g

(W, η) - target space endowed with the geometry

$$g[\Phi] := \Phi^* \eta \quad , \qquad \boxed{g[\Phi]_{\mu\nu} := \eta_{AB}\frac{\partial \Phi^A}{\partial x^\mu}\frac{\partial \Phi^B}{\partial x^\nu}} \; .$$

$$\boxed{L[\Phi] = \sqrt{\left|\det\left[g[\Phi]_{\mu\nu}\right]\right|} = \sqrt{\left|\det\left[\eta_{AB}\,\Phi^{A}{}_{,\mu}\,\Phi^{B}{}_{,\nu}\right]\right|}}$$

$L_{\mu\nu}$ - quadratic in derivatives.



Minimizing the n-dimensional η-volume of $\Phi(\Omega)$, keeping the $(n-1)$-dimensional boundary $\Phi(\partial\Omega) = \partial\Phi(\Omega)$ fixed.

Minimal surfaces, soap films. If $N \leq n$ – trivial.

Invariant under $\mathrm{Diff}\, M \times \mathrm{Diff}(W, \eta)$.

Without $\mathrm{Diff}(W, \eta)$-invariance – a wider class:

$$\boxed{L[\Phi] = f(\Phi)\sqrt{|\det[g[\Phi]_{\mu\nu}|} = \sqrt{|k(\Phi)\det[g[\Phi]]_{\mu\nu}|}}$$

f, k – a „potential" term.

If W - complex-linear, and η - hermitian, then:

$$L[\Phi] = \sqrt{|\det[g[\Phi]_{\mu\nu}]|} = \sqrt{|\det[\eta_{AB}\overline{\Phi^{A}{}_{,\mu}}\,\Phi^{B}{}_{,\nu}]|}$$

$g[\Phi]_{\mu\nu}$ – Φ-dependent hermitian metric on M.

If W - linear, η - constant, (W, η) - pseudo-Euclidean, the natural class of scalars is given by:

$$f[\Phi] = F(\|\Phi\|^2) = F(\eta_{AB}\Phi^A\Phi^B).$$

L - invariant under $\mathrm{Diff}\, M \times O(W, \eta)$. Other representation:

$$L[\Phi] = \sqrt{|\det[T_{\mu\nu}]|} = \sqrt{|\det[\omega\eta_{AB}\Phi^A{}_{,\mu}\Phi^B{}_{,\nu} + \varkappa\lambda_\mu\lambda_\nu]|},$$

$\omega, \varkappa$ - functions of $\|\Phi\|^2$, $\lambda_\mu = \frac{1}{2}(\|\Phi\|^2)_{,\mu} = \eta_{AB}\Phi^A\Phi^B{}_{,\mu}$.

No absolute conceptual gap between Lagrangians:

$$L_{d'A} = \frac{1}{2}\eta_{AB}\Phi^A{}_{,\mu}\Phi^B{}_{,\nu}g^{\mu\nu}\sqrt{|g|}, \quad L_{BI} = \sqrt{\left|\det\left[\eta_{AB}\Phi^A{}_{,\mu}\Phi^B{}_{,\nu}\right]\right|}.$$

(fixed g)

Namely, let us take:

$$\boxed{\mathcal{L}[\Phi, G] = \frac{1}{2}\eta_{AB}\Phi^A{}_{,\mu}\Phi^B{}_{,\nu}G^{\mu\nu}\sqrt{|G|} + C\sqrt{|G|}}\quad, \quad C\text{-constant},$$

Φ, G - dynamical on the equal footing. Generally-covariant. (Polyakov approach).

~ If $n > 2$, $G_{\mu\nu} = \frac{2-n}{2C}\, g[\Phi]_{\mu\nu}$, and Φ satisfies L_{BI}-equations.

If $C = \frac{2-n}{2}$, $G_{\mu\nu} = g[\Phi]_{\mu\nu}$.

~ If $n = 2$, $C = 0$, $G_{\mu\nu} = \lambda\, g[\Phi]_{\mu\nu}$, λ-arbitrary function, and Φ again satisfies L_{BI}-equations.

Minimal surfaces, general covariance, and scalar B-I - models

$$L = \sqrt{\left|\det\left[\eta_{AB}\,\Phi^{A}{}_{,\mu}\,\Phi^{B}{}_{,\nu}\right]\right|}\quad , \quad \eta_{AB} = \text{const.}$$

$$\boxed{g^{\mu\nu}\,\overset{g[\Phi]}{\nabla}_{\mu}\,\overset{g[\Phi]}{\nabla}_{\nu}\,\Phi^{A} = 0}, \qquad A = 1, \ldots, N \qquad \text{„d'Alembert"}$$

$\overset{g[\Phi]}{\nabla}$ - covariant differentiation with respect to the $g[\Phi]$-Levi-Civita

Written down:

$$\boxed{g^{\mu\nu}\,\Phi^{A}{}_{,\mu\nu} + \Phi^{A}{}_{,\nu}\left(\frac{1}{2}g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)g_{\alpha\beta,\mu} = 0}$$

The mean curvature of $\Phi(M)$ in (W, η) vanishes.

General covariance, one needs coordinate conditions, e.g.,

$$\boxed{\left(\tfrac{1}{2}g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)g_{\alpha\beta,\mu} = 0} \qquad (*)$$

thus:

$$\boxed{g^{\mu\nu}\,\Phi^{A}{}_{,\mu\nu} = 0 \quad , \quad A = 1, \ldots, N}\,.$$

The simplest gauge for eliminating superfluous variables:

$\Phi^{\varrho} = x^{\varrho}, \quad \varrho = 1, \ldots, n;$ $\qquad (*)$-holds automatically

$\Phi^{r}, \quad r = n+1, \ldots, N$ - true degrees of freedom.

If $\eta_{\mu r} = 0, \ \mu = 1, \ldots, n; \quad r = n+1, \ldots, N$ - block structure, then:

$$\boxed{g_{\mu\nu} = \eta_{\mu\nu} + \eta_{rs}\,\Phi^{r}{}_{,\mu}\,\Phi^{s}{}_{,\nu}}\,.$$

The true dynamics: $\boxed{g^{\mu\nu}\,\Phi^{r}{}_{,\mu\nu} = 0}, \quad r = n+1, \ldots, N$

Exactly equivalent to the vanishing of the mean curvature.

Effective Lagrangian for Φ^r, $r = n+1, \ldots, N$:

$$\boxed{L_{\text{eff}} = \sqrt{\left|\det\left[\eta_{\mu\nu} + \eta_{rs}\Phi^r{}_{,\mu}\Phi^s{}_{,\nu}\right]\right|}}$$

„Traditional" B-I form with the fixed spatio-temporal metric $\eta_{\mu\nu}$

Without the block structure:

$$\boxed{L_{\text{eff}} = \sqrt{\left|\det\left[\eta_{\mu\nu} + 2\eta_{r(\mu}\Phi^r{}_{,\nu)} + \eta_{rs}\Phi^r{}_{,\mu}\Phi^s{}_{,\nu}\right]\right|}\,.}$$

„Vacuum" solutions – affine injections:

$$\Phi^r = C^r{}_\mu x^\mu + C^r, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \eta_{rs} C^r{}_\mu C^s{}_\nu,$$

$C^r{}_\mu$, C^r – constants. $C^r{}_\mu = 0$ if well-behaving at infinity:

$$\Phi^r = C^r, \qquad g_{\mu\nu} = \eta_{\mu\nu}.$$

Small corrections φ^r to the vacuum background, Jacobi fields:

$$\Phi^r = C^r + \varphi^r \quad , \qquad \eta^{\mu\nu}\varphi^r{}_{,\mu\nu} = 0 \quad .$$

Ruled by the quadratic background Lagrangian:

$$\frac{1}{2}\eta_{rs}\varphi^r{}_{,\mu}\varphi^s{}_{,\nu}\eta^{\mu\nu}\sqrt{|\det[\eta_{\mu\nu}]|} \; .$$

Special example: scalar Born-Infeld electrodynamics:

$$N = 5 \, , \quad n = 4 \, , \quad [\eta_{AB}] = \mathrm{diag}\left(\eta, 1, -1, -1, -1\right) \, , \quad \eta > 0 .$$

$$L_{eff} = \sqrt{|\det[\eta_{\mu\nu} + \eta \, \Phi_{,\mu}\Phi_{,\nu}]|}$$

Spherically-symmetric, stationary solutions:

$$\boxed{\Phi(r) = \pm \frac{\sqrt{C}}{\sqrt{\eta}} \int_0^r \frac{dx}{\sqrt{C + x^4}}} \; ; \quad C > 0 \text{ - integration constant}$$

Conclusion: traditional Born-Infeld schemes are fixed-gauge-descriptions of generally-covariant models with higher-dimensional target spaces.

Special examples:

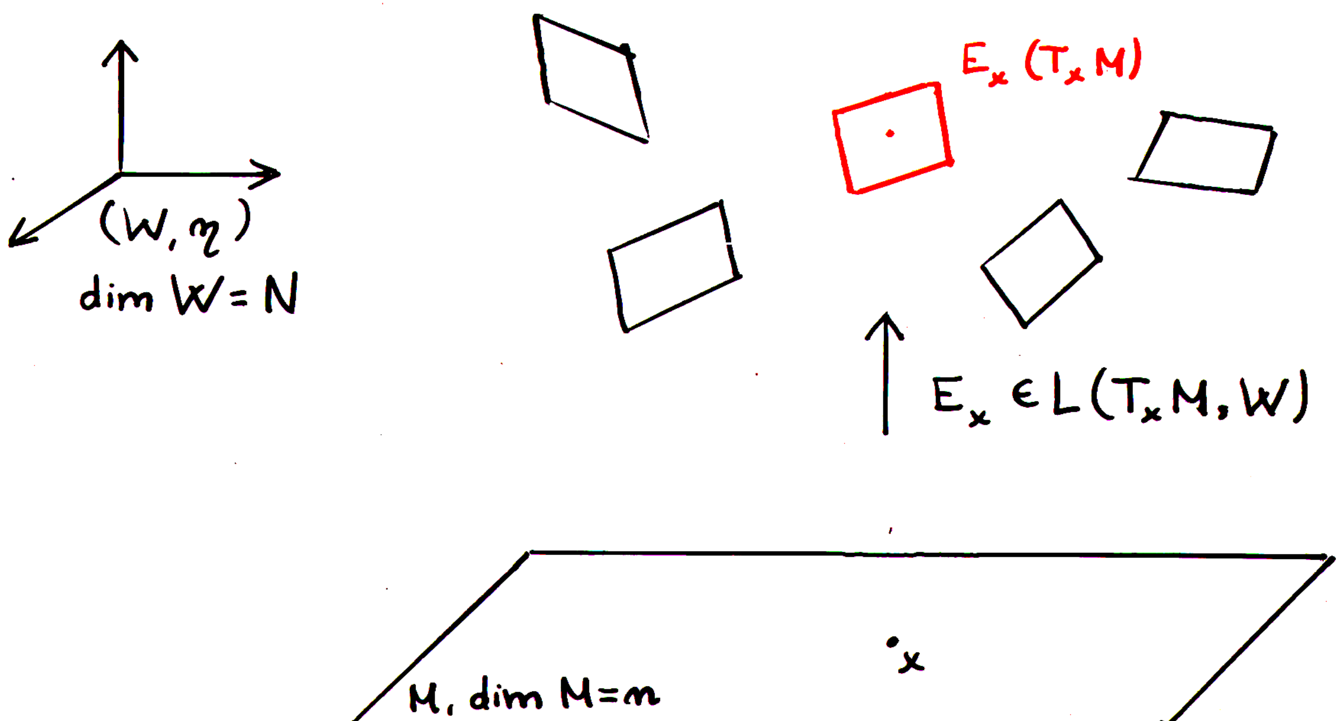
- $N=3$, $n=2$ - soap films, rubber films
- N-arbitrary, $n=1$ - geodesics
- $N=4$, η - Minkowskian, $n=1$, $L_{eff} \simeq \sqrt{1-v^2/c^2}$ - relativistic material point
- $N=4$, η - Minkowskian, $n=2$ - strings
- N-arbitrary, η - Riemannian, $n=1$, $f=\sqrt{2(E-V)}$ - Jacobi-Maupertuis variational principle with the potential V.

Is it possible to remove a fixed geometry also from the linear target space? For scalar field multiplets - NO!

Let us consider multiplets of covectors.

Scalars Φ^A gave rise to local injections $E^K{}_\mu = \Phi^K{}_{,\mu}$.

One can start from non-holonomic fields of such injections, i.e., from W-valued differential one-forms E.



$$\boxed{g(E)_{\mu\nu} := \eta_{KL} E^K{}_\mu E^L{}_\nu} \quad ; \quad g(E)_x = E_x^* \cdot \eta$$

algebraic in E

$$E^{\mu}{}_{K} := \eta_{KL} E^{L}{}_{\nu} g^{\nu\mu} \quad , \quad g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$$

$$E^{\mu}{}_{K} E^{K}{}_{\nu} = \delta^{\mu}{}_{\nu}$$ - <u>left inverse</u>

$$F^{K} = dE^{K} \quad , \quad F^{K}{}_{\mu\nu} = \partial_{\mu} E^{K}{}_{\nu} - \partial_{\nu} E^{K}{}_{\mu}$$

$$\boxed{S^{\lambda}{}_{\mu\nu} := E^{\lambda}{}_{K} F^{K}{}_{\mu\nu}}.$$

The simplest Lagrangian:

$$\boxed{L = \eta_{KL} F^{K}{}_{\mu\nu} F^{L}{}_{\varkappa\lambda} g^{\mu\varkappa} g^{\nu\lambda} \sqrt{|g|} + c\sqrt{|g|}}$$

<u>Quadratic in derivatives</u>, invariant under <u>$\mathrm{Diff}\, M \times O(W,\eta)$</u>

Non-quadratic in E itself $(g_{\alpha\beta}, \sqrt{|g|})$.

<u>Quasilinear</u> field equations.

Another possibility:

$$\boxed{A g_{\alpha\lambda} g^{\beta\mu} g^{\lambda\nu} S^{\alpha}{}_{\beta\sigma} S^{\lambda}{}_{\mu\nu} \sqrt{|g|} + B g^{\mu\nu} S^{\alpha}{}_{\beta\mu} S^{\beta}{}_{\alpha\nu} \sqrt{|g|} + C g^{\mu\nu} S^{\alpha}{}_{\alpha\mu} S^{\beta}{}_{\beta\nu} \sqrt{|g|}}$$

The same invariance under Diff $M \times O(W,\eta)$, although capitals hidden.

Born - Infeld schemes:

$$\boxed{\sqrt{\left|\det\left[C g_{\mu\nu} + \eta_{KL} F^{K}{}_{\mu\varkappa} F^{L}{}_{\lambda\nu} g^{\varkappa\lambda}\right]\right|}}\ , \text{ etc.},$$

invariant under Diff $M \times O(W,\eta)$

$$\boxed{\sqrt{\left|\det\left[S^{\alpha}{}_{\beta\mu} S^{\beta}{}_{\alpha\nu}\right]\right|}}$$

Killing structure of $S^{\alpha}{}_{\beta\mu} S^{\beta}{}_{\alpha\nu}$.

$O(W,\eta)$ still coded in this Lagrangian (although capital indices and η are hidden).

The special case : $\boxed{N=n}$; E - field of linear frames (if $\det[E^{K}{}_{\mu}]\neq 0$).
Then the left inverse is just the η-independent contravariant inverse frame. More generally :

$$\boxed{\begin{gathered} L=\sqrt{\left|\det\left[A\, S^{\alpha}{}_{\beta\mu}S^{\beta}{}_{\alpha\nu}+B\, S^{\alpha}{}_{\alpha\mu}S^{\beta}{}_{\beta\nu}+C\, S^{\alpha}{}_{\alpha\beta}\, S^{\beta}{}_{\mu\nu}\right]\right|} \\ L_{\mu\nu}=A\, S^{\alpha}{}_{\beta\mu}S^{\beta}{}_{\alpha\nu}+B\, S^{\alpha}{}_{\alpha\mu}\, S^{\beta}{}_{\beta\nu}+C\, S^{\alpha}{}_{\alpha\beta}\, S^{\beta}{}_{\mu\nu} \end{gathered}}$$

$GL(W)\simeq GL(n,\mathbb{R})$ - invariant.

S - torsion of the E - teleparallelism connection

– special solutions – Lie-group-spaces (E_K's span a Lie algebra under the Lie bracket operation), or appropriately deformed Lie-group-spaces

– correspondence with GR (with cosmological constant)

With matter, e.g.,

$$\boxed{L = \sqrt{\left|\det\left[S^{\alpha}{}_{\beta\mu} S^{\beta}{}_{\alpha\nu} + \lambda \overline{\Psi}_{,\mu} \Psi_{,\nu}\right]\right|}}$$

Special reference solutions: E – (deformed) Lie group space,
Ψ – combined of matrix elements of a given irreducible representation of the mentioned group.

Heisenberg ideas about fundamental spinors and B-I - models

$\psi : M \to \mathbb{C}^4$

$(\mathbb{C}^4, G)$, $\quad G = \mathrm{diag}(+ + - -)$ – hermitian form of the neutral signature

$$\boxed{L = f\sqrt{\left|\det\left[G_{\bar{r}s}\, \overline{\psi}^{\bar{r}}{}_{,\mu}\, \psi^{s}{}_{,\nu}\right]\right|}},$$

f – „potential" depending on $G_{\bar{r}s}\overline{\psi}^{\bar{r}}\psi^{s}$ (e.g. const $- G_{\bar{r}s}\overline{\psi}^{\bar{r}}\psi^{s}$)

$g[\psi]_{\mu\nu} = G_{\bar{r}s}\, \overline{\psi}^{\bar{r}}{}_{,\mu}\, \psi^{s}{}_{,\nu}$ – assumed normal-hyperbolic.

Generally covariant and U(2,2)-globally-invariant

Small vibrations about affine background-ruled by Klein-Gordon Lagrangian.

Local $U(2,2)$-invariance - gauge field $A^{r}{}_{s\mu}$ with values in the Lie algebra $U(2,2)$!

$$g[\Psi, A]_{\mu\nu} = G_{\bar{r}s} D_\mu \overline{\Psi}^{\bar{r}} D_\nu \Psi^s$$

$$D_\mu \Psi = \partial_\mu \Psi + g A_\mu \Psi + \frac{q-g}{4} \operatorname{Tr} A_\mu \Psi$$

$$L = \left(f + \varkappa \operatorname{Tr}(F_{\mu\nu} F_{\varkappa\lambda}) g^{\mu\varkappa} g^{\nu\lambda}\right) \sqrt{\left|\det\left[g[\Psi, A]_{\mu\nu}\right]\right|}$$, where:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu].$$

Another possibility:

$$L = \sqrt{\left|\det\left[k\, g[\Psi, A]_{\mu\nu} + b \operatorname{Tr}(F_{\mu\varkappa} F_{\lambda\nu}) g^{\varkappa\lambda}\right]\right|},$$

k - function of $G_{\bar{r}s} \overline{\Psi}^{\bar{r}} \Psi^s$