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Groupoid of partially invertible elements of W^*-algebras

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BASIC DEFINITIONS

Groupoid over the base set B is a set $\mathcal{G}$ with actions:

(i) **source map s** : $\mathcal{G} \to B$ and **target map t** : $\mathcal{G} \to B$

(ii) **product** $m : \mathcal{G}^{(2)} \to \mathcal{G}$

$$m(g,h) =: gh,$$

defined on **the set of composable pairs**

$$\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} : \ \mathbf{s}(g) = \mathbf{t}(h)\},$$

(iii) iniective **identity section** $\varepsilon : B \to \mathcal{G}$,

(iv) **inverse map** $\iota : \mathcal{G} \to \mathcal{G}$, which satisfy the following conditions:

$$\mathsf{s}(gh) = \mathsf{s}(h), \quad \mathsf{t}(gh) = \mathsf{t}(g), \tag{1}$$

$$k(gh) = (kg)h, \tag{2}$$

$$\varepsilon(\mathsf{t}(g))g = g = g\varepsilon(\mathsf{s}(g)), \tag{3}$$

$$\iota(g)g = \varepsilon(\mathsf{s}(g)), \qquad g\iota(g) = \varepsilon(\mathsf{t}(g)), \tag{4}$$

where $g, k, h \in \mathcal{G}$.
Notation: $\mathcal{G} \rightrightarrows B$.

GROUPOID $\mathcal{G}(\mathfrak{M})$

Left support $l(x) \in \mathcal{L}(\mathfrak{M})$ (**right support** $r(x) \in \mathcal{L}(\mathfrak{M})$) of $x \in \mathfrak{M}$ is the least projection in $\mathfrak{M}$, such that

$$l(x)x = x \qquad (\text{resp. } x\, r(x) = x). \tag{5}$$

If $x \in \mathfrak{M}$ is selfadjoint, then **support** $s(x)$

$$s(x) := l(x) = r(x).$$

Polar decomposition for $x \in \mathfrak{M}$

$$x = u|x|, \tag{6}$$

where $u \in \mathfrak{M}$ is partial isometry and $|x| := \sqrt{x^* x} \in \mathfrak{M}^+$. Then

$$l(x) = s(|x^*|) = uu^*, \qquad r(x) = s(|x|) = u^* u.$$

Let $G(p\mathfrak{M}p)$ - the group of all invertible elements in W^*-subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$.
We define the set $\mathcal{G}(\mathfrak{M})$ of **partially invertible** elements in $\mathfrak{M}$

$$\mathcal{G}(\mathfrak{M}) := \{x \in \mathfrak{M}; \ \ |x| \in G(p\mathfrak{M}p), \text{ where } p = s(|x|)\}$$

Remark: $\mathcal{G}(\mathfrak{M}) \subsetneq \mathfrak{M}$.

Theorem

The set $\mathcal{G}(\mathfrak{M})$ with

1. the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$
$$\mathbf{s}(x) := r(x), \qquad \mathbf{t}(x) := l(x),$$
2. the product defined as the product in $\mathfrak{M}$ on the set
$$\mathcal{G}(\mathfrak{M})^{(2)} := \{(x,y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M});\ \ \mathbf{s}(x) = \mathbf{t}(y)\},$$
3. the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ as the identity,
4. the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ defined by
$$\iota(x) := |x|^{-1} u^*,$$

is the groupoid over $\mathcal{L}(\mathfrak{M})$.

GROUPOID REPRESENTATIONS

The set $\mathcal{G}(\mathbb{E})$ of linear isomorphisms $e^n_m : \mathbb{E}_m \tilde{\to} \mathbb{E}_n$ between the fibres of a vector bundle $\pi : \mathbb{E} \to M$ has the groupoid structure over M.
It is called the **structural groupoid** of the bundle $\pi : \mathbb{E} \to M$.

Let $\mathcal{G}$ be a groupoid over B. **Representation** of $\mathcal{G} \rightrightarrows B$ on a vector bundle $\pi : \mathbb{E} \to M$ is a groupoid morphism of $\mathcal{G}$ into the structural groupoid $\mathcal{G}(\mathbb{E})$ of this bundle:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\quad \phi \quad} & \mathcal{G}(\mathbb{E}) \\ \mathsf{s} \downarrow\downarrow \mathsf{t} & & \mathsf{s}' \downarrow\downarrow \mathsf{t}' \\ B & \xrightarrow{\quad \varphi \quad} & M \end{array} \tag{7}$$

One has the bundle $\pi : \mathcal{M}_R(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$, where

$$\mathcal{M}_R(\mathfrak{M}) := \{(y,p) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : \quad p\, r(y) = r(y)\}$$

and $\pi := pr_2$. The fibre $\pi^{-1}(p)$ over $p \in \mathcal{L}(\mathfrak{M})$ is isomorphic to the right W^*-ideal (=$\mathfrak{M}$-modul) $p\mathfrak{M}$ of $\mathfrak{M}$ generated by p.

Fact

For $x \in \mathcal{G}(\mathfrak{M})_p^q$ the left action

$$L_x : p\mathfrak{M} \tilde{\to} q\mathfrak{M}$$

is the isomorphism of the right W^*-ideals.

Theorem

1. The structural groupoid $\mathcal{G}(\mathcal{M}_R(\mathfrak{M}))$ of the bundle $\pi : \mathcal{M}_R(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of the right W^*-ideals is isomorphic to $\mathcal{G}(\mathfrak{M})$.

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2. The structural groupoid $\mathcal{G}(\mathcal{M}_L(\mathfrak{M}))$ of the bundle $\pi : \mathcal{M}_L(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of the left W^*-ideals is isomorphic to $\mathcal{G}(\mathfrak{M})$.

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2. The structural groupoid $\mathcal{G}(\mathcal{M}_L(\mathfrak{M}))$ of the bundle $\pi : \mathcal{M}_L(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of the left W^*-ideals is isomorphic to $\mathcal{G}(\mathfrak{M})$.
3. The structural groupoid $\mathcal{G}(\mathcal{A}(\mathfrak{M}))$ of the bundle $\pi : \mathcal{A}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ of W^*-subalgebras is isomorphic to $\mathcal{G}(\mathfrak{M})$, where $\mathcal{A}(\mathfrak{M}) := \{(y, p) \in \mathfrak{M} \times \mathcal{L}(\mathfrak{M}) : \quad y \in p\mathfrak{M}p\}$.

DIFFERENTIAL STRUCTURES

We consider the following locally convex topologies, for which hold:

$$\sigma\text{–topology} \prec s\text{–topology} \prec s^*\text{–topology} \prec \text{uniform topology}$$

where for $\omega \in \mathfrak{M}_*^+$

1. σ-topology is defined by a family of semi-norms $\|x\|_\sigma := |\langle x, \omega\rangle|$,
2. s-topology is defined by a family of semi-norms $\|x\|_\omega := \sqrt{\langle x^* x, \omega\rangle}; \quad x \in \mathfrak{M}$;
3. s^*-topology is defined by a family of semi-norms $\{\|\cdot\|_\omega, \|\cdot\|_\omega^* : \ \omega \in \mathfrak{M}_*^+\}$ where $\|x\|_\omega^* := \sqrt{\langle x x^*, \omega\rangle}; \quad x \in \mathfrak{M}$.

Theorem

For an infinite dimmentional W^*-algebra $\mathfrak{M}$ the groupoid $\mathcal{G}(\mathfrak{M})$ is not topological with respect to above topologies of $\mathfrak{M}$.

Example: Let $p \in \mathcal{L}(\mathfrak{M})$ and $x_n \in \mathcal{G}(\mathfrak{M})$ as

$$x_n = p + \frac{1}{n}(1 - p), \quad n \in \mathbb{N}.$$

Then

$$\mathsf{s}(x_n) = \mathsf{t}(x_n) = 1 \quad \text{and} \quad \mathsf{s}(p) = \mathsf{t}(p) = p.$$

The uniform limit $\lim_{n\to\infty} x_n = p$, so the source and target maps are not continuous.

$\mathcal{L}(\mathfrak{M})$ as a BANACH MANIFOLD

For $p \in \mathcal{L}(\mathfrak{M})$ let us define the set

$$\Pi_p := \{q \in \mathcal{L}(\mathfrak{M}) : \quad \mathfrak{M} = q\mathfrak{M} \oplus (1-p)\mathfrak{M}\}$$

then $q \wedge (1-p) = 0, \qquad q \vee (1-p) = 1$
and $\qquad q = x - y \ \in q\mathfrak{M} \oplus (1-p)\mathfrak{M}$.
Define the maps

$$\sigma_p : \Pi_p \to q\mathfrak{M}p, \qquad \varphi_p : \Pi_p \tilde{\to} (1-p)\mathfrak{M}p$$

by

$$\sigma_p(q) := x, \qquad \varphi_p(q) := y.$$

The map φ_p is the bijection of Π_p onto the Banach space $(1-p)\mathfrak{M}p$.

In order to find the transitions maps

$$\varphi_p \circ \varphi_{p'}^{-1} : \varphi_{p'}(\Pi_p \cap \Pi_{p'}) \to \varphi_p(\Pi_p \cap \Pi_{p'})$$

in the case $\Pi_p \cap \Pi_{p'} \neq \emptyset$, let us take for $q \in \Pi_p \cap \Pi_{p'}$ the following splittings

$$\begin{aligned}\mathfrak{M} = q\mathfrak{M} \oplus (1-p)\mathfrak{M} = p\mathfrak{M} \oplus (1-p)\mathfrak{M} \\ \mathfrak{M} = q\mathfrak{M} \oplus (1-p')\mathfrak{M} = p'\mathfrak{M} \oplus (1-p')\mathfrak{M}.\end{aligned} \tag{8}$$

The splittings (9) lead to the corresponding decompositions of p and p'

$$\begin{aligned} p &= x - y \qquad & p &= a + b \\ p' &= x' - y' \qquad & 1 - p &= c + d \end{aligned} \tag{9}$$

where $x \in q\mathfrak{M}p$, $y \in (1-p)\mathfrak{M}p$, $x' \in q\mathfrak{M}p'$, $y' \in (1-p')\mathfrak{M}p'$, $a \in p'\mathfrak{M}p$, $b \in (1-p')\mathfrak{M}p$, $c \in p'\mathfrak{M}(1-p)$ and $d \in (1-p')\mathfrak{M}(1-p)$.

Combining equations from (10) we obtain

$$q = \iota(x') + y'\iota(x') \tag{10}$$

$$q = (a + cy)\iota(x) + (b + dy)\iota(x). \tag{11}$$

Comparing (11) and (12) we find that

$$\iota(x') = (a + cy)\iota(x) \tag{12}$$

$$y'\iota(x') = (b + dy)\iota(x). \tag{13}$$

After substitution (13) into (14) and noting that $\mathbf{t}(a + cy) \leqslant p'$ we finally get the formula

$$y' = (\varphi_{p'} \circ \varphi_p^{-1})(y) = (b + dy)\iota(a + cy).$$

Theorem

The family of maps

$$(\Pi_p, \varphi_p) \quad p \in \mathcal{L}(\mathfrak{M})$$

defines a smooth atlas on a $\mathcal{L}(\mathfrak{M})$. This atlas is modeled by the family of Banach spaces $(1-p)\mathfrak{M}p$, for $p \in \mathcal{L}(\mathfrak{M})$.

Fact: If $p' \in \mathcal{O}_p$ then $(1-p)\mathfrak{M}p \cong (1-p')\mathfrak{M}p'$.

$\mathcal{G}(\mathfrak{M})$ as a BANACH MANIFOLD

For projections $\tilde{p}, p \in \mathcal{L}(\mathfrak{M})$ we define the set

$$\Omega_{\tilde{p}p} := \mathbf{t}^{-1}(\Pi_{\tilde{p}}) \cap \mathbf{s}^{-1}(\Pi_p)$$

and a map

$$\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

in the following way

$$\psi_{\tilde{p}p}(x) := \left(\varphi_{\tilde{p}}(\mathbf{t}(x)), \iota(\sigma_{\tilde{p}}(\mathbf{t}(x)))x\sigma_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x))\right),$$

where $p = \sigma_p(q) - \varphi_p(q) \in \; q\mathfrak{M} \oplus (1-p)\mathfrak{M}$.

Theorem

The family of maps

$$(\Omega_{\tilde{p}p}, \psi_{\tilde{p}p}) \quad \tilde{p}, p \in \mathcal{L}(\mathfrak{M})$$

defines a smooth atlas on the groupoid $\mathcal{G}(\mathfrak{M})$. The complex Banach manifold structure of $\mathcal{G}(\mathfrak{M})$ has type $\mathfrak{G}$, where $\mathfrak{G}$ is the set of Banach spaces

$$(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

indexed by the pair of equivalent projections of $\mathcal{L}(\mathfrak{M})$.

$\mathcal{U}(\mathfrak{M})$ as a BANACH MANIFOLD

The groupoid $\mathcal{U}(\mathfrak{M})$ is the set of fixed points of the automorphism $J : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ defined by

$$J(x) := \iota(x^*).$$

Expressing $J : \Omega_{\tilde{p}p} \to \Omega_{\tilde{p}p}$ in the coordinates

$$\psi_{\tilde{p}p}(x) = (\tilde{y}, z, y) \ \in \ (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus ip\mathfrak{M}^h p \oplus (1-p)\mathfrak{M}p$$

we find that

$$\left(\psi_{\tilde{p}p} \circ J \circ \psi_{\tilde{p}p}^{-1}\right)(\tilde{y}, z, y) =$$

$$= \left(\tilde{y}, \iota(\sigma_{\tilde{p}}(\varphi_{\tilde{p}}^{-1}(\tilde{y}))^* \sigma_{\tilde{p}}(\varphi_{\tilde{p}}^{-1}(\tilde{y})))\iota(z^*)\sigma_p(\varphi_p^{-1}(y))^* \sigma_p(\varphi_p^{-1}(y)), y\right),$$

where $z \in \mathcal{G}(\mathfrak{M})_p^{\tilde{p}} \subset \tilde{p}\mathfrak{M}p$. Since $J^2(x) = x$ for $x \in \mathcal{U}(\mathfrak{M})$ one has

$$(DJ(x))^2 = \mathbf{1}$$

for $DJ(x) : T_x\mathcal{G}(\mathfrak{M}) \to T_x\mathcal{G}(\mathfrak{M})$.

Thus one obtains a splitting of the tangent space

$$T_x\mathcal{G}(\mathfrak{M}) = T_x^+\mathcal{G}(\mathfrak{M}) \oplus T_x^-\mathcal{G}(\mathfrak{M}) \tag{14}$$

defined by the Banach space projections

$$P_\pm(x) := \frac{1}{2}\left(\mathbf{1} \pm DJ(x)\right). \tag{15}$$

The Frechét derivative $D\iota(z)$ of the inversion map $\iota : \mathcal{G}(\mathfrak{M})_p^{\tilde{p}} \ni \ z \mapsto \ \iota(z) \in \mathcal{G}(\mathfrak{M})_{\tilde{p}}^p$ at the point z is given by

$$D\iota(z) \triangle z = -\iota(z) \triangle z \ \iota(z),$$

where $\triangle z \in \tilde{p}\mathfrak{M}p$.

Theorem

The groupoid $\mathcal{U}(\mathfrak{M})$ of partial isometries has a natural structure of the real Banach manifold of the type $\mathfrak{G}$, where the family $\mathfrak{G}$ consist of the real Banach spaces

$$(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}^h p \oplus (1-p)\mathfrak{M}p$$

parameterized by the pairs $(\tilde{p}, p) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ of equivalent projections.

Corollary

The groupoid $\mathcal{G}(\mathfrak{M})$ is the complexification of $\mathcal{U}(\mathfrak{M})$.

THANK YOU