The Uncertainty Principle in Einstein Gravity

Gaetano Vilasi
Università degli Studi di Salerno, Italy
Istituto Nazionale di Fisica Nucleare, Italy

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Outline

- **QM and GR**

  The problem of reconciling quantum mechanics (QM) with general relativity (GR) is a task of modern theoretical physics which has not yet found a consistent and satisfactory solution. The difficulty arises because general relativity deals with events which define the world-lines of particles, while QM does not allow the definition of trajectory; indeed, the determination of the position of a quantum particle involves a measurement which introduces an uncertainty into its momentum (Wigner, 1957; Saleker and Wigner, 1958; Feynman and Hibbs, 1965).

- **Weak Equivalence Principle?**

  These conceptual difficulties have their origin, as argued in Candelas and Sciama (1983) and Donoghue et al. (1984, 1985), in the violation, at the quantum level, of the *weak principle of equivalence* on which GR is based. Such a problem becomes more involved in the formulation of a quantum theory of gravity owing to the non-renormalizability of general relativity when one quantizes it as a local quantum field theory (QFT) (Birrel and Davies, 1982).

- **Planck length**

  Nevertheless, one of the most interesting consequences of this unification is that in quantum gravity there exists a minimal observable distance on the order of the Planck distance, $l_P = \sqrt{G\hbar/c^3} \simeq 10^{-33} cm$, where G is the Newton constant. The existence of such a fundamental length is a dynamical phenomenon due to the fact that, at Planck scales, there are fluctuations of the background metric, *i.e.*, a limit of the order of the Planck length appears when quantum fluctuations of the gravitational field are taken into account. Other "Planck quantities" are: $T_P = l_p/c, \quad m_p = \hbar/l_p c$.

$$l_P = \sqrt{G\hbar/c^3} \simeq 10^{-33} cm \qquad T_P = \sqrt{G\hbar/c^5} \simeq 0.54 \cdot 10^{-43} s$$

$$m_p = \sqrt{\hbar c/G} \simeq 2.2 \cdot 10^{-5} g \qquad E_p = \sqrt{\hbar c^5/G} \simeq 1.2 \cdot 10^{19} GeV$$

- In the absence of a theory of quantum gravity, one tries to analyze quantum aspects of gravity retaining the gravitational field as a classical background, described by general relativity, and interacting with a matter field (Lambiase et al. 2000). This semiclassical approximation leads to QFT and QM in curved space-time and may be considered as a preliminary step toward a complete quantum theory of gravity. In other words, we take into account a theory where geometry is classically defined while the source of the Einstein equations is an effective stress-energy tensor where contributions of matter quantum fields, gravity self-interactions, and quantum matter - gravity interactions appear (Birrel and Davies, 1982).

- A theory containing a fundamental length on the order of l_P (which can be also related to the extension of particles) is string theory. It provides a consistent theory of quantum gravity and avoids the above-mentioned difficulties. In fact, unlike point particle theories, the existence of a fundamental length plays the role of a natural cutoff. In such a way the ultraviolet divergences are avoided without appealing to renormalization and regularization schemes (Green et al., 1987).

- By studying string collisions at Planckian energies and through a renormalization group-type analysis (Veneziano, 1986; Amati et al., 1987, 1988, 1989, 1990; Gross and Mende, 1987, 1988; Konishi et al., 1990; Guida and Konishi, 1991; Yonega, 1989), the emergence of a minimal observable distance yields the generalized uncertainty principle

$$\Delta x \simeq \frac{\hbar}{\Delta p} + l_p^2 \frac{\Delta p}{\hbar}$$

At energies much below the Planck energy, the extra term in the previous equation is irrelevant, and the Heisenberg relation is recovered, while as we approach the Planck energy this term becomes relevant and is related to the minimal observable length.

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- In the early days of quantum theory, Heisenberg showed that the uncertainty principle follows as a direct consequence of the quantization of electromagnetic radiation (photons). Consider a wave scattering from an electron into a microscope and thereby giving a measurement of the position of the electron. According both to optics and the intuition, with an electromagnetic wave of wavelength λ we cannot obtain better precision than

$$\Delta x_H \simeq \lambda$$

Such a wave is quantized in the form of photons, each with a momentum

$$p = \frac{h}{\lambda}$$

In order to interact with the electron an entire photon in the wave must scatter and thereby impart to the electron a significant part of its momentum, which produces an uncertainty in the electron momentum of about $\Delta p \simeq p$. Thus we obtain the standard Heisenberg position-momentum uncertainty relation

$$\Delta x_H \cdot \Delta p \simeq \lambda(\frac{h}{\lambda}) \simeq \hbar$$

Until now, no gravitational interaction between the photon and the electron has ben considered.

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- The photon is assumed to be located in an experimental region of characteristic size L inside of which it interacts with the photon, and to behave as a classical particle with an effective mass equal to E/c^2 (Adler Santiago 2008). Because of the gravity,

$$\vec{a} = -G\frac{E}{c^2r^3}\vec{r}$$

where r is the electron-photon distance. During the interaction (which occurs in characteristic time L/c), because of the gravity the electron will acquire a velocity $\Delta v \simeq G\frac{E}{c^2r^2}\left(\frac{L}{c}\right)$ and move a distance given by

$$\Delta x_G \simeq G\frac{E}{c^2r^2}\left(\frac{L}{c}\right)^2$$

- The photon scatters electromagnetically from the electron at some indeterminate time during the interaction and the electron may be anywhere in the interaction region. So the electron-photon distance should be of order $r \simeq L$, which is the only distance scale in the problem. Since the photon energy is related to the momentum by $E = pc$ we may also express this as

$$\Delta x_G \simeq \frac{Gp}{c^3}$$

The electron momentum uncertainty must be of order of the photon momentum so that, by using the Planck length $l_p^2 = G\hbar/c^3$ as a parameter, we have

$$\Delta x_G \simeq \frac{G\Delta p}{c^3} = l_p^2 \frac{\Delta p}{\hbar}$$

- Adding this uncertainty to the Heisenberg relation we obtain the modified uncertainty relation

$$\Delta x \simeq \frac{\hbar}{\Delta p} + l_p^2 \frac{\Delta p}{\hbar}$$

This relation, referred descriptively as the generalized uncertainty principle (GUP), is invariant under

$$\frac{\Delta p l_p}{\hbar} \leftrightharpoons \frac{\hbar}{\Delta p l_p}$$

That is, it has a kind of momentum inversion symmetry (M. B. Green, J. H. Schwarz, and E. Witten, Superstring Theory, Cambridge University Press, 1987).

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- Previous heuristic arguments are based on *action-at-a-distance* Newtonian gravitational theory, with the additional *ad hoc* assumption that the energy of the photon produces a gravitational field. However, based on general relativity theory, a dimensional estimate free of such drawbacks can be done. The Einstein field equations of general relativity are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The left side has the units of inverse distance squared, since it is constructed from second derivatives and squares of first derivatives of the metric.

- Thus on dimensional grounds we may write the left hand side in terms of deviations $(h_{\mu\nu})$ of the metric from Lorentzian, in schematic order of magnitude dimensional form, as

$$LHS \simeq \frac{h_{\mu\nu}}{L^2}.$$

Similarly the energy-momentum tensor has the units of an energy density, so its components must be roughly equal to the photon energy over L^3. Thus we can write the right hand side of the field equations schematically as

$$RHS \simeq \left(\frac{8\pi G}{c^4}\right) \cdot \frac{E}{L^3} \simeq \frac{Gp}{c^3 L^3}$$

- Thus, we get an estimate for the deviation of the metric,

$$h_{\mu\nu} \simeq \frac{Gp}{Lc^3}$$

This deviation corresponds to a fractional uncertainty in all positions in the region L, which we identify with a fractional uncertainty in position, $\Delta x_G/L$. Thus the gravity uncertainty position is

$$\Delta x_G \simeq h_{\mu\nu} \cdot L \simeq \frac{Gp}{c^3}$$

where the characteristic size L doesn't appear anymore. Since the uncertainty in momentum of the electron must be comparable to the photon momentum, $\Delta p \simeq p$, and we obtain $\Delta x_G \simeq G\Delta p/c^3$

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- The photon-photon and photon-electron scatterings may occur through the creation and annihilation of virtual electron-positron pairs and may even lead to collective photon phenomena. Photons also interact gravitationally but the gravitational scattering of light by light has been much less studied.

  First studies go back to Tolman, Ehrenfest and Podolsky (1931) and to Wheeler (1955) who analysed the gravitational field of light beams and the corresponding geodesics in the linear approximation of Einstein equations. They also discovered that null rays behave differently according to whether they propagate parallel or antiparallel to a steady, long, straight beam of light, but they didn't provide a physical explanation of this fact.

Results of Tolman, Ehrenfest, Podolsky, Wheeler were clarified in part by Faraoni and Dumse 1999, in the setting of classical pure General Relativity, the general point of view being that gravitational interaction is mediated by a spin-2 particle.

More recently however, within the context of modern quantum field theories, it was proven (Fabbrichesi and Roland, 1992)that in supergravity and string theory, due to dimensional reduction, the effective 4-dimensional theory of gravity may show repulsive aspects because of the appearance of spin-1 graviphotons.

- In the usual treatment of gravitational waves only Fourier expandable solutions of d'Alembert equation are considered; then it is possible to choose a special gauge (TT-gauge) which kills the spin-0 and spin-1 components.

  However there exist (see section 2 and 3) physically meaningful solutions (Peres 1959 Stephani 1996, Stephani, Kramer, MacCallum, Honselaers and Herlt 2003, Canfora, Vilasi and Vitale 2002) of Einstein equations which are not Fourier expandable and nevertheless whose associated energy is finite.

- For some of these solutions the standard analysis shows that spin-1 components cannot be killed (Canfora and Vilasi 2004, Canfora, Vilasi and Vitale 2004). In previous works it was shown that light is among possible sources of such spin-1 waves (Vilasi 2007) and this implies that repulsive aspects of gravity are possible within pure General Relativity, i.e. without involving spurious modifications (Vilasi et al, Class. Quant. Grav. 2011) .

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Geometric properties

In previous papers (Sparano, Vilasi, Vinogradov, Canfora 2000-2010) a family of exact solutions g of Einstein field equations, representing the gravitational wave generated by a beam of light, has been explicitly written

$$g = 2f(dx^2 + dy^2) + \mu\left[(w(x,y) - 2q)dp^2 + 2dpdq\right], \qquad (1)$$

where $\mu(x,y) = A\Phi(x,y) + B$ (with $\Phi(x,y)$ a harmonic function and A, B numerical constants), $f(x,y) = (\nabla\Phi)^2\sqrt{|\mu|}/\mu$, and $w(x,y)$ is solution of the *Euler-Darboux-Poisson equation:*

$$\Delta w + (\partial_x \ln|\mu|)\,\partial_x w + (\partial_y \ln|\mu|)\,\partial_y w = \rho,$$

$T_{\mu\nu} = \rho\delta_{\mu 3}\delta_{\nu 3}$ representing the energy-momentum tensor and Δ the Laplace operator in the (x,y) −plane.

Previous metric is invariant for the non Abelian Lie agebra $\mathcal{G}_2$ of Killing fields, generated by

$$X = \frac{\partial}{\partial p}, \qquad Y = \exp\left(p\right) \frac{\partial}{\partial q},$$

with $[X, Y] = Y, \quad g\left(Y, Y\right) = 0$ *and whose orthogonal distribution is integrable.*

Table:

	$\mathcal{D}^\perp$, $r=0$	$\mathcal{D}^\perp$, $r=1$	$\mathcal{D}^\perp$, $r=2$
$\mathcal{G}_2$	**integrable**	**integrable**	**integrable**
$\mathcal{G}_2$	**semi-integrable**	**semi-integrable**	**semi-integrable**
$\mathcal{G}_2$	**non-integrable**	**non-integrable**	non-integrable
$\mathcal{A}_2$	**integrable**	**integrable**	integrable
$\mathcal{A}_2$	semi-integrable	semi-integrable	semi-integrable
$\mathcal{A}_2$	non-integrable	non-integrable	non-integrable

In the particular case $s = 1$, $f = 1/2$ and $\mu = 1$, the above family is locally diffeomorphic to a subclass of Peres solutions and, by using the transformation

$$p = \ln |u| \qquad q = uv,$$

can be written in the form

$$g = dx^2 + dy^2 + 2dudv + \frac{w}{u^2}du^2, \tag{2}$$

with $\Delta w(x, y) = \rho$, and has the Lorentz invariant *Kerr-Schild* form:

$$g_{\mu\nu} = \eta_{\mu\nu} + V k_\mu k_\nu, \qquad k_\mu k^\mu = 0.$$

- **Wave Character**
  The wave character and the polarization of these gravitational fields has been analyzed in many ways. For example, the Zel'manov criterion (Zakharov 1973) was used to show that these are gravitational waves and the propagation direction was determined by using the Landau-Lifshitz pseudo-tensor. However, the algebraic Pirani criterion is easier to handle since it determines both the wave character of the solutions and the propagation direction at once. Moreover, it has been shown that, in the vacuum case, the two methods agree. To use this criterion, the Weyl scalars must be evaluated according to Petrov classification (Petrov 1969).

In the Newmann-Penrose formulation (Penrose 60) of Petrov classification, we need a *tetrad* basis with two real null vector fields and two real spacelike (or two complex null) vector fields. Then, if the metric belongs to type **N** of the Petrov classification, it is a gravitational wave propagating along one of the two real null vector fields (Pirani criterion). Let us observe that ∂_x and ∂_y are spacelike real vector fields and ∂_v is a null real vector but ∂_u is not. With the transformation $x \mapsto x, \quad y \mapsto y, \quad u \mapsto u, \quad v \mapsto v + w(x,y)/2u$, whose Jacobian is equal to one, the metric (2) becomes:

$$g = dx^2 + dy^2 + 2dudv + dw(x,y)dln|u|. \tag{3}$$

Since ∂_x and ∂_y are spacelike real vector fields and ∂_u and ∂_v are null real vector fields, the above set of coordinates is the right one to apply for the Pirani's criterion.

Since the only nonvanishing components of the Riemann tensor, corresponding to the metric (3), are

$$R_{iuju} = \frac{2}{u^3}\partial^2_{ij}w(x,y), \qquad i,j = x,y$$

these gravitational fields belong to Petrov type **N** (Zakharov 73). Then, according to the Pirani's criterion, previous metric does indeed represent a gravitational wave propagating along the null vector field ∂_u.

It is well known that linearized gravitational waves can be characterized entirely in terms of the linearized and gauge invariant Weyl scalars. The non vanishing Weyl scalar of a typical spin-2 gravitational wave is Ψ_4. Metrics (3) also have as non vanishing Weyl scalar Ψ_4.

- **Spin**

  Besides being an exact solution of Einstein equations, the metric (3) is, for $w/u^2 << 1$, also a solution of linearized Einstein equations, thus representing a perturbation of Minkowski metric $\eta = dx^2 + dy^2 + 2dudv = dx^2 + dy^2 + dz^2 - dt^2$ (with $u = (z-t)/\sqrt{2} \quad v = (z+t)/\sqrt{2}$) with the perturbation, generated by a light beam or by a photon wave packet moving along the z-axis, given by

$$h = dw(x,y)dln|z-t|,$$

  whose non vanishing components are

$$h_{0,1} = -h_{13} = -\frac{w_x}{(z-t)} \quad h_{0,2} = -h_{23} = -\frac{w_y}{(z-t)}$$

- A transparent method to determine the spin of a gravitational wave is to look at its physical degrees of freedom, *i.e.* the components which contribute to the energy (Dirac 75). One should use the Landau-Lifshitz (pseudo)-tensor t^{μ}_{ν} which, in the asymptotically flat case, agrees with the Bondi flux at infinity canfora, Vilasi and Vitale 2004). It is worth to remark that the canonical and the Landau-Lifchitz energy-momentum pseudo-tensors are true tensors for Lorentz transformations. Thus, any Lorentz transformation will preserve the form of these tensors; this allows to perform the analysis according to the Dirac procedure. A globally square integrable solution $h_{\mu\nu}$ of the wave equation is a function of $r = k_{\mu}x^{\mu}$ with $k_{\mu}k^{\mu} = 0$.

- With the choice $k_\mu = (1, 0, 0, -1)$, we get for the energy density t_0^0 and the energy momentum t_0^3 the following result:

$$16\pi t_0^0 = \frac{1}{4}\left(u_{11} - u_{22}\right)^2 + u_{12}^2, \qquad t_0^0 = t_0^3$$

where $u_{\mu\nu} \equiv dh_{\mu\nu}/dr$. Thus, the physical components which contribute to the energy density are $h_{11} - h_{22}$ and h_{12}. Following the analysis of Dirac 1975, we see that they are eigenvectors of the infinitesimal rotation generator $\mathcal{R}$, in the plane $x - y$, belonging to the eigenvalues $\pm 2i$. The components of $h_{\mu\nu}$ which contribute to the energy thus correspond to spin-2.

- In the case of the prototype of spin−1 gravitational waves (3), both Landau-Lifchitz energy-momentum pseudo-tensor and Bel-Robinson tensor (1958) single out the same wave components and we have:

$$\tau_0^0 \sim c_1(h_{0x,x})^2 + c_2(h_{0y,x})^2, \quad t_0^0 = t_0^3$$

where c_1 e c_2 constants, so that the physical components of the metric are h_{0x} and h_{0y}. Following the previous analysis one can see that these two components are eigenvectors of $i\mathcal{R}$ belonging to the eigenvalues ± 1. In other words, metrics (3), which are not pure gauge since the Riemann tensor is not vanishing, represent spin−1 gravitational waves propagating along the $z-$axis at light velocity.

- **Summarizing**

  Globally square integrable spin-1 *gravitational waves propagating on a flat background are always pure gauge.*

- *Spin*-1 *gravitational waves which are not globally square integrable are not pure gauge.* It is always possible to write metric (3) in an apparently transverse gauge (Stefani 96); however since these coordinates are no more harmonic this transformation is not compatible with the linearization procedure.

- What truly distinguishes spin-1 from spin-2 gravitational waves is the fact that in the spin-1 case the Weyl scalar has a non trivial dependence on the transverse coordinates (x, y) due to the presence of the harmonic function. This could led to observable effects on length scales larger than the *characteristic length scale* where the harmonic function changes significantly.

- Indeed, the Weyl scalar enters in the geodesic deviation equation implying a non standard deformation of a ring of test particles breaking the invariance under of π rotation around the propagation direction. Eventually, one can say that there should be distinguishable effects of spin-1 waves at suitably large length scales.
- It is also worth to stress that the results of Aichelburg and Sexl 1971, Felber 2008 and 2010, van Holten 2008 suggest that the sources of asymptotically flat PP_waves (which have been interpreted as spin-1 gravitational waves Canfora, Vilasi and Vitale 2002 and 2004) repel each other. Thus, in a field theoretical perspective (see Appendix), *pp*-gravitons" must have spin-1 .

- **Gravitoelectromagnetism**

Hereafter the spatial part of four-vectors will be denoted in bold and the standard symbols of three-dimensional vector calculus will be adopted.

Metric (3) can be written in the GEM form

$$g = (2\Phi^{(g)} - 1)dt^2 - 4(\mathbf{A}^{(g)} \cdot d\mathbf{r})dt + (2\Phi^{(g)} + 1)d\mathbf{r} \cdot d\mathbf{r}, \qquad (4)$$

where

$$\mathbf{r} = (x, y, z)\,, \quad 2\Phi^{(g)} = h_{00}, \quad 2A_i^{(g)} = -h_{0i}.$$

- **Gravito-Lorentz gauge**

In terms of $\Phi^{(g)}$ and $\mathbf{A}^{(g)}$ the harmonic gauge condition reads

$$\frac{\partial \Phi^{(g)}}{\partial t} + \frac{1}{2}\nabla \cdot \mathbf{A}^{(g)} = 0, \tag{5}$$

and, once the gravitoelectric and gravitomagnetic fields are defined in terms of g-potentials, as

$$\mathbf{E}^{(\mathbf{g})} = -\nabla \Phi^{(g)} - \frac{1}{2}\frac{\partial \mathbf{A}^{(g)}}{\partial t}, \qquad \mathbf{B}^{(\mathbf{g})} = \nabla \wedge \mathbf{A}^{(g)},$$

one finds that the linearized Einstein equations resemble Maxwell equations. Consequently, being the dynamics fully encoded in Maxwell-like equations, this formalism describes the physical effects of the vector part of the gravitational field.

- **Gravito-Faraday tensor**

Gravitational waves can be also described in analogy with electromagnetic waves, the gravitoelectric and the gravitomagnetic components of the metric being

$$E^{(g)}_{\mu} = F^{(g)}_{\mu 0}; \qquad B^{(g)\mu} = -\varepsilon^{\mu 0\alpha\beta} F^{(g)}_{\alpha\beta}/2 \quad ,$$

where

$$\begin{aligned} F^{(g)}_{\mu\nu} &= \partial_\mu A^{(g)}_\nu - \partial_\nu A^{(g)}_\mu \\ A^{(g)}_\mu &= -h_{0\mu}/2 = (-\Phi^{(g)}, \mathbf{A}^{(g)}). \end{aligned}$$

- **Geodesic motion**

The first order geodesic motion for a *massive particle* moving with velocity $v^\mu = (1, \underline{\mathrm{v}}), |\mathbf{v}| << \mathbf{1}$, in a light beam gravitational field characterized by gravitoelectric $\mathbf{E}^{(g)}$ and gravitomagnetic $\mathbf{B}^{(g)}$ fields, is described (at first order in $|\mathbf{v}|$) by the *acceleration*:

$$\mathbf{a}^{(g)} = -\mathbf{E}^{(g)} - 2\mathbf{v} \wedge \mathbf{B}^{(g)}.$$

with $\mathbf{E}^{(g)} = (w_x, w_y, 0)/4u^2, \quad \mathbf{B}^{(g)} = (w_y, -w_x, 0)/4u^2,$ so that the gravitational acceleration of a massive particle is given by

$$\mathbf{a}^{(g)} = -[w_x(1+2v_z)\mathbf{i} + w_y(1+2v_z)\mathbf{j} - (w_x v_x + w_y v_y)\mathbf{k}]/4u^2. \quad (6)$$

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Rather than geodesic orbits, the motion of spinning particles, should be described by Papapetrou equations

$$\frac{D}{D\tau}(mv^{\alpha} + v_{\sigma}\frac{DS^{\alpha\sigma}}{D\tau}) + \frac{1}{2}R^{\alpha}_{\sigma\mu\nu}v^{\sigma}S^{\mu\nu} = 0,$$

where $S^{\mu\nu}$ is the *angular momentum tensor* of the spinning particle and

$$S^{\alpha} = \frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}v_{\beta}S_{\rho\sigma}$$

defines the *spin four-vector* of the particle.

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- The relativistic interaction of a photon with an electron can be described by the geodesic motion of the electron in the light gravitational field. For a flow of radiation of a null *em* field along the z-axis, the electromagnetic (em) energy-momentum tensor macroscopic components $T_{\mu\nu} = F_{\mu\alpha}F_{\nu}^{\alpha} + \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$ reduce to

$$T_{00} = \frac{\rho}{z-ct}, \quad T_{03} = T_{30} = -\frac{\rho}{z-ct}, \quad T_{33} = \frac{\rho}{z-ct}$$

where $\rho = \left(E^2 + B^2\right)/2$ represents the amplitude of the field, $i.e.$ the density of radiant energy at point of interest. They are just the components in the coordinates (t, x, y, z) of the energy-momentum tensor $T = \rho du^2$ of section 53.

We assume then that the energy density is a constant ρ_0 within a certain radius $0 \leq r = \sqrt{x^2 + y^2} \leq r_0$ and vanishes outside. Thus, the source represents a cylindrical beam with width r_0 and constitutes a simple generalization of a single null particle.

Introducing back the standard coupling constant of Einstein tensor with matter energy-momentum tensor, we have:

$$\Delta w(x, y) = \frac{8\pi G}{c^4}\rho. \tag{7}$$

- The cylindrical symmetry implies that $w(x, y)$ will depend only on the distance r from the beam. A solution $w(r)$ of Poisson equation (7) satisfying the continuity condition at $r = r_0$ can be easily written as

$$w(r) = \frac{4\pi G}{c^4}\rho_0 r^2 \quad r \leq r_0 \tag{8}$$

$$w(r) = \frac{8\pi G}{c^4}\rho_0 r_0^2 \left[\ln\left(\frac{r}{r_0}\right) + \frac{1}{2}\right] \qquad r > r_0 \tag{9}$$

Or also

$$w(r) = \frac{4\pi G\rho_0}{c^4} r_0^2 W(r) = \frac{4\pi G E r_0^2}{c^4 L} W(r) = \frac{4\pi G p r_0^2}{c^3 L} W(r) \qquad (10)$$

where we have assumed that the cylinder has length L

$$W(r) = \begin{cases} r^2/r_0^2 & r < r_o \\ 1 + \ln\left(\frac{r}{r_0}\right)^2 & r > r_0 \end{cases} \qquad (11)$$

- so that the gravitational acceleration at the space-time point (t, x, y, z) of a massive particle is given by

$$\mathbf{a}^{(g)} = \frac{4\pi G p r_0^2}{c^3 L(z-ct)^2}\left[\frac{\mathbf{v}\cdot\nabla W(r)}{c}\mathbf{k} - (1 + 2\frac{\mathbf{v}\cdot\mathbf{k}}{c})\nabla W(r)\right] \qquad (12)$$

i.e.

$$\frac{d^2\mathbf{r}}{dc^2t^2} = -\frac{4\pi G p r_0^2}{c^3 L(z-ct)^2}(1 + 2\frac{\mathbf{v}\cdot\mathbf{k}}{c})\frac{\mathbf{r}}{r^2}, \qquad \frac{d^2 z}{dc^2t^2} = \frac{2 G p r_0^2}{c^3 L(z-ct)^2}\frac{\mathbf{r}\cdot\mathbf{v}}{r^2} \qquad (13)$$

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