Stability and Integration on $\mathfrak{so}(3)^*_-$

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Outline

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1. Introduction
2. Classification
3. Stability
4. Integration

Problem statement

Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$

- Classify quadratic Hamilton-Poisson systems
- Stability
- Integration via Jacobi elliptic functions.

Poisson structure

(minus) Lie-Poisson structure

$$\{F, G\}(p) = -p([dF(p), dG(p)]), \quad p \in \mathfrak{g}^*$$

- Hamiltonian vector field:

$$\vec{H}[F] = \{F, H\}$$

- Casimir function:

$$\{C, F\} = 0$$

Hamilton-Poisson Systems

Quadratic Hamilton-Poisson System $(\mathfrak{g}^*_-, H)$

- $H_{A,Q} : \; p \mapsto pA + pQp^\top, \; A \in \mathfrak{g}$
- Equations of motion:

$$\dot{p}_i = -p\,([E_i, dH(p)])$$

$$\vec{H} = \Pi_- \cdot \nabla H$$

- Q (PSD) quadratic form
- $A = 0$ - homogeneous
- $A \neq 0$ - inhomogeneous

$\mathfrak{so}(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^\top + A = \mathbf{0}\}$

- Basis:

$$E_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad E_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

- Commutator relations:

$$[E_2, E_3] = E_1 \qquad [E_3, E_1] = E_2 \qquad [E_1, E_2] = E_3$$

Poisson structure on $\mathfrak{so}(3)^*_-$

- $\Pi_- = \begin{bmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{bmatrix}$
- $C(p) = p_1^2 + p_2^2 + p_3^2$
- Automorphisms: $\mathrm{Aut}(\mathfrak{so}(3)^*_-) = \mathrm{Aut}(\mathfrak{so}(3)) = \mathrm{SO}(3)$

Outline

Affine equivalence

Definition

Systems G and H are affinely equivalent if
$\exists$ affine automorphism ψ
such that $\psi_* \vec{G} = \vec{H}$

Proposition

The following systems are equivalent to $H_{A,Q}$:

- $H_{A,Q} \circ \psi$: where ψ - linear Poisson automorphism
- $H_{A,rQ}$: where $r \neq 0$
- $H_{A,Q} + C$: where C- Casimir function

Classification on $\mathfrak{so}(3)^*_-$

$H(p) = pQp^\top$

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

$H(p) = pA + pQp^\top$

- αp_1
- $p_2 + \frac{1}{2}p_1^2$
- $p_1 + \frac{1}{2}p_1^2$
- $p_1 + \alpha p_2 + \frac{1}{2}p_1^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + \gamma p_3 + p_1^2 + \beta p_2^2$

Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

Homogeneous systems

Proof (sketch)

- Let $H_Q = pQp^\top$ (Q PSD 3×3 matrix).
- $\exists \Psi \in \mathrm{SO}(3)$ such that $\Psi Q \Psi^\top = \mathrm{diag}(a_1, a_2, a_3)$, where $a_1 \geq a_2 \geq a_3 \geq 0$
- Then $\frac{1}{a_1 - a_3}\left(\Psi Q \Psi^\top - a_3 C(p)\right) = \mathrm{diag}(1, \alpha, 0)$, $0 \leq \alpha \leq 1$.
- $\psi = \mathrm{diag}(-\sqrt{2}\sqrt{1-\alpha}, 2\sqrt{\alpha(1-\alpha)}, -\sqrt{2}\sqrt{\alpha})$ brings $H_\alpha(p) = p_1^2 + \alpha p_2^2$ into $H_1(p) = p_1^2 + \frac{1}{2}p_2^2$

$$\psi \cdot \vec{H}_\alpha = \vec{H}_1 \circ \psi : \begin{cases} \dot{p}_1 = 2\sqrt{2}\alpha\sqrt{1-\alpha}\, p_2 p_3 \\ \dot{p}_2 = 4\sqrt{\alpha(1-\alpha)}\, p_1 p_3 \\ \dot{p}_3 = 2\sqrt{2\alpha}(1-\alpha)\, p_1 p_2 . \end{cases}$$

Proof (sketch)

- Let $H_{A,Q} = pA + pQp^\top$.
- By Proposition, $H_{A,Q}$ is A-equivalent to (for some $p' \in \mathfrak{so}(3)^*_-$)

$$H_a(p) = p', \quad H_b(p) = p' + \frac{1}{2}p_1^2, \quad H_c(p) = p' + p_1^2 + \beta p_2^2$$

- Consider $\Psi \in \mathsf{SO}(3)$ s.t. $\Psi Q \Psi^\top = Q$. Let $H_{1,\alpha}(p) = \alpha p_1$, then $H_a(p) \cong H_{1,\alpha}$, $\alpha > 0$.
- Consider $\psi : p \mapsto \psi_0(p) + q$ s.t. $\psi_0 \cdot \vec{H}_{1,\alpha} = \vec{H}_{1,\beta} \circ \psi$:
- For $\psi_0 = [\psi_{ij}]$,

$$\begin{aligned} -\alpha\psi_{13}p_2 + \alpha\psi_{12}p_3 &= 0 \\ -\alpha\psi_{23}p_2 + \alpha\psi_{22}p_3 - \beta(\psi_{31}p_1 + \psi_{32}p_2 + \psi_{33}p_3 + q_3) &= 0 \\ -\alpha\psi_{33}p_2 + \alpha\psi_{32}p_3 - \beta(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3 + q_2) &= 0 \end{aligned}$$

Inhomogeneous systems

Proof (sketch)

- Consider $H_b(p) = p' + \frac{1}{2}p_1^2$.
- Then $\Psi = \begin{bmatrix} \pm 1 & 0 \\ 0 & S \end{bmatrix}$, $S \in O(2)$, for $\Psi Q \Psi^\top = Q$
- Thus $H_b(p)$ is equivalent to $(\alpha_1, \alpha_2 \geq 0)$

$$H_{2,\alpha}(p) = \alpha_1 p_1 + \alpha_2 p_2 + \frac{1}{2}p_1^2$$

- Let $\alpha_1 = 0$. Then $H_{2,\alpha} \cong H_{2,1}$, where $H_{2,1} = p_2 + \frac{1}{2}p_1^2$
- Indeed, $\vec{H}_{2,\alpha}$ and $\vec{H}_{2,1}$ are compatible with the affine isomorphism

$$p \mapsto p \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - \alpha_2^2 \\ 0 \end{bmatrix}^\top$$

Outline

Types of stability

- Lyapunov stable
  $\forall$ nbd N $\exists$ nbd N' s.t. $\mathcal{F}_t(N') \subset N$
- spectrally stable
  $\text{Re}(\lambda_i) \leq 0$ for eigenvalues of $DH(p_e)$
- Lyapunov stable $\implies$ spectrally stable

Energy methods

- Energy Casimir: $(\lambda_1, \lambda_2 \in \mathbb{R})$,

$$d(\lambda_1 H + \lambda_2 C)(p_e) = 0 \quad \text{and} \quad d^2(\lambda_1 H + \lambda_2 C)(p_e)\,|_{W \times W} < 0$$

$$W = \ker dH(p_e) \cap \ker dC(p_e)$$

- Continuous energy Casimir: $H^{-1}(H(p_e)) \cap C^{-1}(C(p_e)) = \{p_e\}$

Classification on $\mathfrak{so}(3)^*_-$

$H(p) = pQp^\top$

- $\frac{1}{2}p_1^2$
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- $\alpha_1 p_1 + \alpha_2 p_2 + \gamma p_3 + p_1^2 + \beta p_2^2$

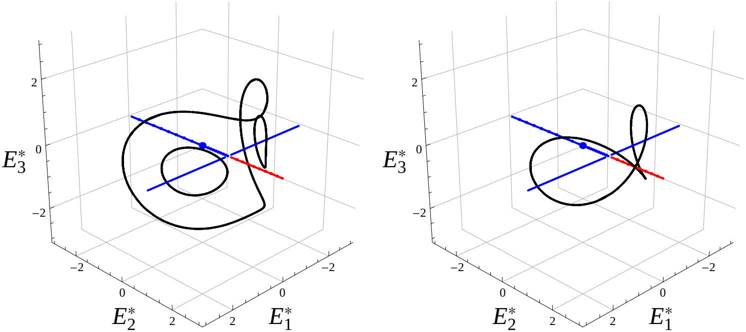
Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

Equations of motion

$$
\begin{aligned}
\dot{p}_1 &= -p_3 \\
\dot{p}_2 &= p_1 p_3 \\
\dot{p}_3 &= p_1(1 - p_2)
\end{aligned}
$$



Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

Each state $e^\mu = (\mu, 1, 0)$ is stable, $\mu \in \mathbb{R}$

- Let $H_\lambda = \lambda_1 H + \lambda_2 C$.
- For $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{2}$

$$dH_\lambda(e^\mu) = 0 \quad \text{and} \quad d^2H_\lambda(e^\mu) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$W = \ker dH(e^\mu) \cap \ker dC(e^\mu) = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ -\mu \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Thus ($\mu \neq 0$)

$$d^2H_\lambda(e^\mu)|_{W\times W} = \begin{bmatrix} -\mu^2 & 0 \\ 0 & -1 \end{bmatrix}$$

Stability: $H(p) = p_2 + \frac{1}{2}p_1^2$

Each state $e^\mu = (0, \mu, 0)$ is stable for $\mu \leq 1$

- For $\mu < 1$, the Hessian is given by

$$d^2 H_\lambda(\mu, 0, 0) = \begin{bmatrix} -(\frac{1-\mu}{\mu}) & 0 & 0 \\ 0 & -\frac{1}{\mu} & 0 \\ 0 & 0 & -\frac{1}{\mu} \end{bmatrix}$$

- For $\mu = 0$

$$C^{-1}(C(e^0)) = (0, 0, 0)$$

- For $\mu = 1$

$$H^{-1}(H(e^1)) \cap C^{-1}(C(e^1)) = (0, 1, 0)$$

- e^μ is (spectrally) unstable for $\mu > 1$.

Classification on $\mathfrak{so}(3)^*_-$

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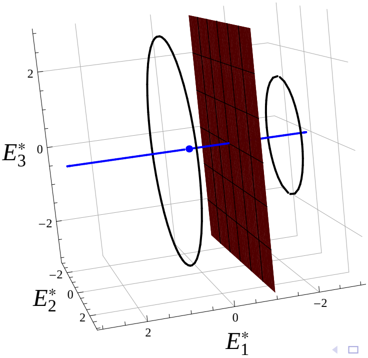
Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

Stability: $H(p) = p_1 + \frac{1}{2}p_1^2$

Equations of motion

$$\begin{aligned} \dot{p}_1 &= 0 \\ \dot{p}_2 &= (1 + p_1)p_3 \\ \dot{p}_3 &= -(1 + p_1)p_2. \end{aligned}$$



Stability: $H(p) = p_1 + \frac{1}{2}p_1^2$

Each state $e^{\mu,\nu} = (-1, \mu, \nu)$ is unstable

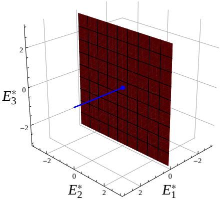
- Let $\epsilon = \sqrt{\mu^2 + \nu^2}$
- Consider open ball $\mathcal{B}_\epsilon$ centred at $e^{\mu,\nu}$
- An integral curve of $\vec{H}$, for any $\delta > 0$,

$$p(t) = (-1 + \delta, \mu \cos(\delta t) + \nu \sin(\delta t), \nu \cos(\delta t) - \mu \sin(\delta t))$$

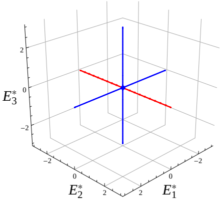
- $\forall\ V \subset \mathcal{B}_\epsilon$, $e_2^{\mu,\nu} \in V$, there $\exists\, \delta > 0$ s.t. $p(0) \in V$
- Furthermore

$$\|p(\frac{\pi}{3\delta}) - e_2^{\mu,\nu}\| = \sqrt{\delta^2 + \mu^2 + \nu^2} > \epsilon$$

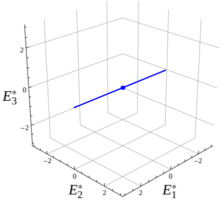
Stability summary



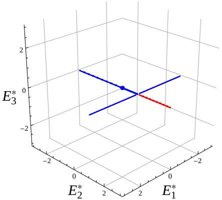
(d) $\frac{1}{2}p_1^2$



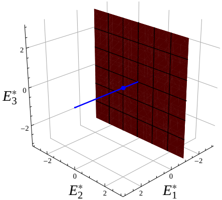
$p_1^2 + \frac{1}{2}p_2^2$



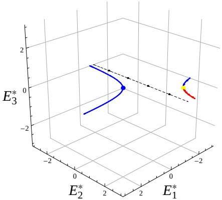
αp_1



(e) $p_2 + \frac{1}{2}p_1^2$



$p_1 + \frac{1}{2}p_1^2$



$p_1 + \alpha p_2 + \frac{1}{2}p_1^2$

Outline

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- $\alpha_1 p_1 + \alpha_2 p_2 + \gamma p_3 + p_1^2 + \beta p_2^2$

Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

Integration: $H(p) = p_2 + \frac{1}{2}p_1^2$

Equations of motion

$$\dot{p}_1 = -p_3, \qquad \dot{p}_2 = p_1 p_3, \qquad \dot{p}_3 = p_1(1 - p_2)$$

Separation: Let $h_0 = H(p(0))$ and $c_0 = C(p(0))$

- Level sets $H^{-1}(h_0)$ and $C^{-1}(c_0)$ are tangent if their gradients are parallel
- For $\lambda \in \mathbb{R} \backslash \{0\}$

$$\nabla H(p) = \lambda \nabla C(p) \iff \begin{bmatrix} p_1 \\ 1 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 2p_1 \\ 2p_2 \\ 2p_3 \end{bmatrix}$$

$$p_1 = 2\lambda p_1, \quad p_2 = \frac{1}{2\lambda}, \quad p_3 = 0$$

Integration: $H(p) = p_2 + \frac{1}{2}p_1^2$

Seperation

- For $\lambda \neq \frac{1}{2}$: $p_3 = p_1 = 0$, $p_2 \in \mathbb{R}\backslash\{1\}$:

$$h_0 = p_2 \quad \text{and} \quad c_0 = p_2^2 \implies c_0 = h_0^2$$

- Therefore we consider:

$$c_0 < h_0^2, \qquad c_0 = h_0^2, \qquad c_0 > h_0^2$$

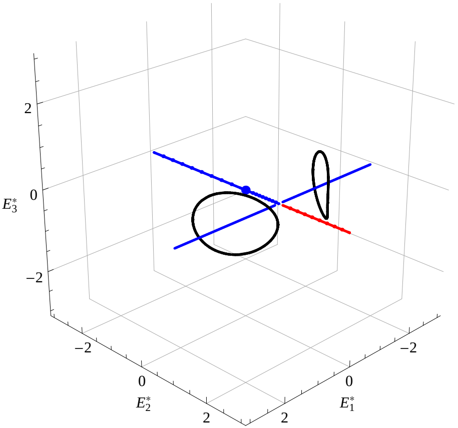
- For $\lambda = \frac{1}{2}$: $p_3 = 0$, $p_2 = 1$, $p_1 \in \mathbb{R}$:

$$h_0 = 1 + \frac{1}{2}p_1^2 \quad \text{and} \quad c_0 = p_1^2 + 1 \implies c_0 = 2h_0 - 1$$

- We further consider:

$$c_0 = 2h_0 - 1, \qquad c_0 > 2h_0 - 1$$

Integration: $H(p) = p_2 + \frac{1}{2}p_1^2$: $0 < 2h_0 - 1 < c_0 < h_0^2$



Theorem: $H(p) = p_2 + \frac{1}{2}p_1^2$

For $0 < 2h_0 - 1 < c_0 < h_0^2$, $\sigma \in \{-1, 1\}$,

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{2\delta}\,\dfrac{1 + k\,\mathrm{sn}\,(\Omega\,t,\,k)}{\mathrm{dn}\,(\Omega\,t,\,k)} \\ \bar{p}_2(t) = h_0 + \delta - \dfrac{2\delta}{1 - k\,\mathrm{sn}\,(\Omega\,t,\,k)} \\ \bar{p}_3(t) = \sigma k\Omega\sqrt{2\delta}\,\dfrac{\mathrm{cn}\,(\Omega\,t,\,k)}{k\,\mathrm{sn}\,(\Omega\,t,\,k) - 1} \end{cases}$$

Here $\Omega = \sqrt{h_0 - 1 + \delta}$, $k = \frac{\sqrt{h_0 - 1 - \delta}}{\sqrt{h_0 - 1 + \delta}}$, and $\delta = \sqrt{h_0^2 - c_0}$

Proof (Sketch)

- $$H(p) = p_2 + \tfrac{1}{2}p_1^2 \qquad \text{and} \qquad C(p) = p_1^2 + p_2^2 + p_3^2$$
- $\frac{d}{dt}\bar{p}_2 = p_1 p_3 = \sqrt{2(h_0 - \bar{p}_2)(c_0 - 2(h_0 - \bar{p}_2) - \bar{p}_2^2)}$
- Transform it into standard form, letting $s = \frac{\bar{p}_2 - r_1}{\bar{p}_2 - r_2}$,

$$\sqrt{2}\,t = \frac{1}{(r_1 - r_2)\sqrt{A_1 A_2}} \int_0^{\frac{\bar{p}_2(t)-r_1}{\bar{p}_2(t)-r_2}} \frac{ds}{\sqrt{\left(s^2 + \frac{B_2}{A_2}\right)\left(s^2 + \frac{B_1}{A_1}\right)}}$$

$$A_1 = \frac{1}{4\delta} > 0 \qquad A_2 = \frac{h_0 - 1 - \delta}{2\delta} > 0$$

$$B_1 = -\frac{1}{4\delta} < 0 \qquad B_2 = -\frac{h_0 - 1 + \delta}{2\delta} < 0$$

$$r_1 = h_0 + \delta \qquad r_2 = h_0 - \delta.$$

Proof (Sketch)

- Applying the integral formula

$$\int_a^x \frac{dt}{\sqrt{(t^2 - a^2)(t^2 - b^2)}} = \tfrac{1}{a}\,\mathrm{dc}^{-1}\left(\tfrac{1}{a}\,x,\ \tfrac{b}{a}\right), \quad b < a \leq x$$

we obtain

$$\bar{p}_2(t) = \frac{r_2\sqrt{-\frac{B_2}{A_2}}\,\mathrm{dc}\left((r_1 - r_2)\sqrt{2A_1A_2}\sqrt{-\frac{B_2}{A_2}}\,t,\ \frac{1}{\sqrt{-\frac{B_2}{A_2}}}\right) - r_1}{\sqrt{-\frac{B_2}{A_2}}\,\mathrm{dc}\left((r_1 - r_2)\sqrt{2A_1A_2}\sqrt{-\frac{B_2}{A_2}}\,t,\ \frac{1}{\sqrt{-\frac{B_2}{A_2}}}\right) - 1}.$$

- Hence

$$\bar{p}_2(t) = h_0 + \frac{\delta\,(k + \mathrm{dc}\,(\Omega\,t,\ k))}{k - \mathrm{dc}\,(\Omega\,t,\ k)}.$$

Proof (Sketch)

- Using $\frac{1}{\mathrm{sn}\,(x+K,k)} = \mathrm{dc}\,(x,k)$ and a translation in t we get

$$\bar{p}_2(t) = h_0 + \delta + \frac{2\delta}{k\,\mathrm{sn}\,(\Omega\,t,\;k) - 1}\,.$$

- Solve for $\bar{p}_1(t)$ and $\bar{p}_3(t)$ using constants of motion.
- Verify $\frac{d}{dt}\bar{p}(t) = H(\bar{p}(t))$, for example

$$\begin{aligned}&\frac{d}{dt}\bar{p}_2(t) - \bar{p}_1(t)\bar{p}_3(t)\\ &= \frac{2k\delta(-1+\sigma^2)\Omega\,\mathrm{cn}\,(\Omega\,t,\;k)\,\mathrm{dn}\,(\Omega\,t,\;k)}{(-1+k\,\mathrm{sn}\,(\Omega\,t,\;k))^2}\,.\end{aligned}$$

which is zero for $\sigma \in \{-1, 1\}$.

Proof (Sketch)

Verify $p(t) = \bar{p}(t + t_0)$

- We have $p_2(0) + \frac{1}{2}p_1(0)^2 = h_0$ and $p_1(0)^2 + p_2(0)^2 + p_3(0)^2 = c_0$.
- Thus

$$1 - \sqrt{1 + c_0 - 2h_0} \leq p_2(0) \leq 1 + \sqrt{1 + c_0 - 2h_0}.$$

- Now $\bar{p}_2(\frac{K}{\Omega}) = 1 - \sqrt{1 + c_0 - 2h_0}$ and $\bar{p}_2(\frac{3K}{\Omega}) = 1 + \sqrt{1 + c_0 - 2h_0}$.
- Thus $\exists t_1 \in [\frac{K}{\Omega}, \frac{3K}{\Omega}]$ such that $\bar{p}_2(t_1) = p_2(0)$.
- Then

$$\min_{p_2} (h_0 - p_2(0)) = h_0 - 1 - \sqrt{1 + c_0 - 2h_0} > 0.$$

Thus $p_1(0) \neq 0$. Let $\sigma = \operatorname{sgn}(p_1(0))$.

Proof (Sketch)

- Then, we have

$$p_1(0)^2 = 2h_0 - 2p_2(0) = 2h_0 - 2\bar{p}_2(t_1) = \bar{p}_1(t_1)^2.$$

- As $\operatorname{sgn}(p_1(0)) = \sigma = \operatorname{sgn}(\bar{p}_1(t_1))$, $p_1(0) = \bar{p}_1(t_1)$.
- Also,

$$\begin{aligned} p_3(0)^2 &= c_0 - p_1(0)^2 - p_2(0)^2 \\ &= c_0 - \bar{p}_1(t_1)^2 - \bar{p}_2(t_1)^2 = \bar{p}_3(t_0)^2. \end{aligned}$$

  Thus $p_3(0) = \pm\bar{p}_3(t_1)$.
- Now

$$\bar{p}_1(-t+\tfrac{2K}{\Omega}) = \bar{p}_1(t), \quad \bar{p}_2(-t+\tfrac{2K}{\Omega}) = \bar{p}_2(t), \quad \bar{p}_3(-t+\tfrac{2K}{\Omega}) = -\bar{p}_3(t).$$

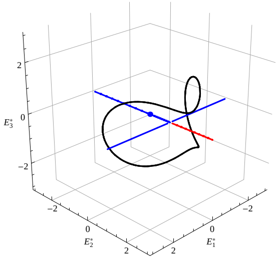
- Thus there exists $t_0 \in \mathbb{R}$ such that $p(0) = \bar{p}(t_0)$.

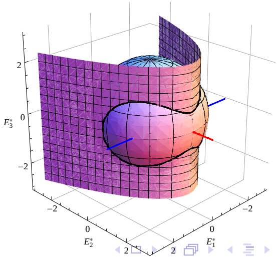
Theorem: $H(p) = p_2 + \frac{1}{2}p_1^2$

For $h_0^2 < c_0$, $\sigma \in \{-1, 1\}$

$$\begin{cases} \bar{p}_1(t) = \sigma\sqrt{2}\sqrt{h_0 + \delta - 1}\,\mathrm{cn}\,(\Omega\, t,\ k) \\ \bar{p}_2(t) = h_0 - (h_0 + \delta - 1)\,\mathrm{cn}\,(\Omega\, t,\ k)^2 \\ \bar{p}_3(t) = \sigma\sqrt{2}\sqrt{h_0 + \delta - 1}\,\Omega\,\mathrm{dn}\,(\Omega\, t,\ k)\,\mathrm{sn}\,(\Omega\, t,\ k)\,. \end{cases}$$

Here $\Omega = \sqrt{\delta}$, $k = \sqrt{\frac{h_0+\delta-1}{2\delta}}$, and $\delta = \sqrt{1 + c_0 - 2h_0}$

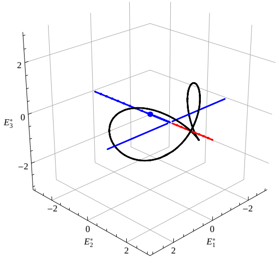


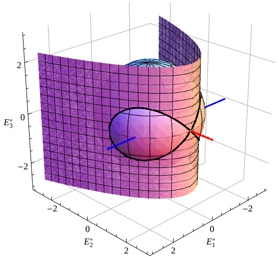


Theorem: $H(p) = p_2 + \frac{1}{2}p_1^2$

Limitng $c_0 \to h_0^2$, $h_0 > 1$, $\sigma \in \{-1, 1\}$

$$\begin{cases} \bar{p}_1(t) = 2\sigma\sqrt{h_0 - 1}\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right) \\ \bar{p}_2(t) = h_0 - 2(h_0 - 1)\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)^2 \\ \bar{p}_3(t) = 2\sigma(h_0 - 1)\ \operatorname{sech}\left(\sqrt{h_0 - 1}\, t\right)\ \tanh\left(\sqrt{h_0 - 1}\, t\right). \end{cases}$$





Classification on $\mathfrak{so}(3)^*_-$

$H(p) = pQp^\top$

- $\frac{1}{2}p_1^2$
- $p_1^2 + \frac{1}{2}p_2^2$

$H(p) = pA + pQp^\top$

- αp_1
- $p_2 + \frac{1}{2}p_1^2$
- $p_1 + \frac{1}{2}p_1^2$
- $p_1 + \alpha p_2 + \frac{1}{2}p_1^2$
- $\alpha_1 p_1 + \alpha_2 p_2 + \gamma p_3 + p_1^2 + \beta p_2^2$

Conditions

- $\alpha > 0$
- $\alpha_1, \alpha_2 \geq 0$
- $\gamma \in \mathbb{R}$
- $0 < \beta < 1.$

Integration: $H(p) = p_1 + \alpha p_2 + \frac{1}{2}p_1^2$

Equations of motion

$$\begin{aligned} \dot{p}_1 &= -\alpha p_3 \\ \dot{p}_2 &= (1 + p_1)p_3 \\ \dot{p}_3 &= \alpha p_1 - (1 + p_1)p_2. \end{aligned}$$

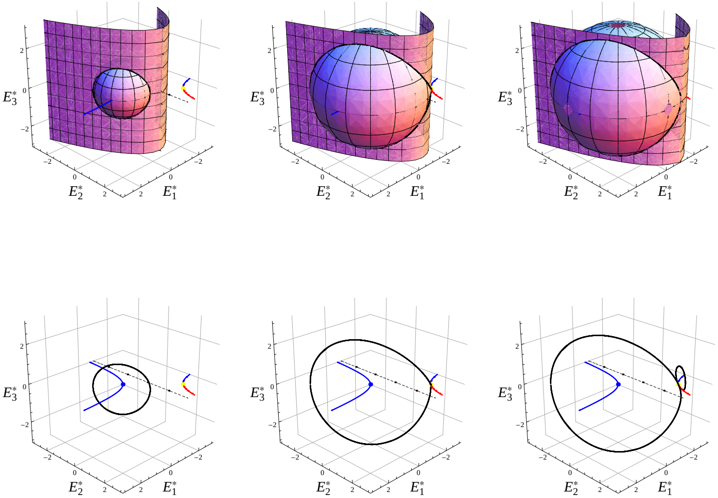
- Consider $H(p) = p_1 + p_2 + \frac{1}{2}p_1^2$
- Transform equation

$$\frac{d\bar{p}_1}{dt} = -\frac{1}{2}\sqrt{4c_0 - 4h_0^2 + 8h_0\bar{p}_1 - 8\bar{p}_1^2 + 4h_0\bar{p}_1^2 - 4\bar{p}_1^3 - \bar{p}_1^4}$$

- Decompose into

$$\left(\frac{d\bar{p}_1}{dt}\right)^2 = (\gamma_1 p_1^2 + \gamma_2 p_1 + \gamma_3)(\eta_1 p_1^2 + \eta_2 p_1 + \eta_3)$$

Integration: $H(p) = p_1 + \alpha p_2 + \frac{1}{2}p_1^2$



Conclusion

Summary

- Classification of Hamilton-Poisson systems on $\mathfrak{so}(3)^*_-$
- Stability nature of equilibria
- Integration of several systems

Outlook

- Integration of remaining systems
- Associated optimal control problems on $\mathsf{SO}(3)$
- Systems on $\mathsf{so}(4)^*_-$