On Quadratic Hamilton-Poisson Systems

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Geometry, Integrability & Quantization
June 7–12, 2013, Varna, Bulgaria

Outline

Outline

Problem Statement

Context

- Quadratic Hamilton-Poisson systems: $H = p^\top Q\, p$
- 3D (minus) Lie-Poisson spaces: $\mathfrak{g}^*_-$

Problem

- Classification, under linear equivalence
- Stability
- Invariants

Motivation

Dynamical systems

- Euler's classic equations for the rigid body

Invariant optimal control / sub-Riemannian geometry

- Hamiltonian reduction $T^*\mathsf{G} \cong \mathsf{G} \times \mathfrak{g}^*_-$

Recent papers

- Tudoran 2009: The free rigid body dynamics
- Biggs & Remsing 2012: Cost-extended control systems
- J. Biggs & Holderbaum 2010, Sachkov 2008: optimal control
- Aron, Craioveanu, Dăniasă, Pop, Puta 2007–2010: QHP systems

Lie-Poisson formalism

(Minus) Lie-Poisson space $\mathfrak{g}^*_-$

$$\{F, G\}(p) = -p([dF(p), dG(p)]), \qquad p \in \mathfrak{g}^*$$

- Hamiltonian vector field: $\vec{H}[F] = \{F, H\}$
- Casimir function: $\{C, F\} = 0$

Quadratic Hamilton-Poisson system $(\mathfrak{g}^*_-, H_Q)$

- $H_Q(p) = p^\top Q\, p$
- Equations of motion:

$$\dot{p}_i = -p([E_i, dH(p)])$$

Outline

Linear equivalence

Definition

$(\mathfrak{g}^*_-, H_Q)$ and $(\mathfrak{h}^*_-, H_R)$ are linearly equivalent if

$\exists$ linear isomorphism $\psi : \mathfrak{g}^* \to \mathfrak{h}^*$

such that $\psi_* \vec{H}_Q = \vec{H}_R$.

Classification approach

Step 1. Classification by algebra

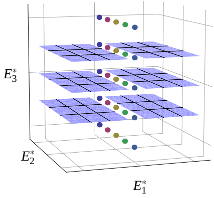
Step 2. General classification

Bianchi-Behr classification

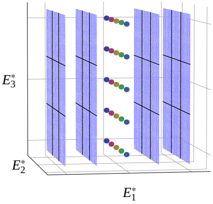
Type	Unimodular	Global Casimirs	Representatives
$\mathfrak{g}_{2.1} \oplus \mathfrak{g}_1$		p_3	$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$
$\mathfrak{g}_{3.1}$	•	p_1	$\mathfrak{h}_3$
$\mathfrak{g}_{3.2}$		—	
$\mathfrak{g}_{3.3}$		—	
$\mathfrak{g}_{3.4}^0$	•	$p_1^2 - p_2^2$	$\mathfrak{se}(1,1)$
$\mathfrak{g}_{3.4}^\alpha$		—	
$\mathfrak{g}_{3.5}^0$	•	$p_1^2 + p_2^2$	$\mathfrak{se}(2)$
$\mathfrak{g}_{3.5}^\alpha$		—	
$\mathfrak{g}_{3.6}$	•	$p_1^2 + p_2^2 - p_3^2$	$\mathfrak{sl}(2,\mathbb{R}),\ \mathfrak{so}(2,1)$
$\mathfrak{g}_{3.7}$	•	$p_1^2 + p_2^2 + p_3^2$	$\mathfrak{so}(3),\ \mathfrak{su}(2)$

Table: Three-dimensional real Lie algebras

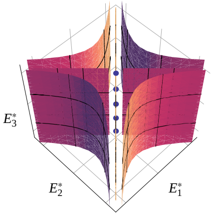
Coadjoint orbits (admit global Casimirs)



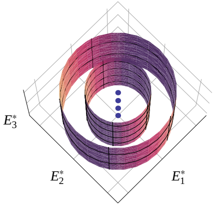
$\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R}$



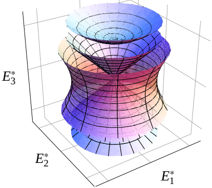
$\mathfrak{h}_3$



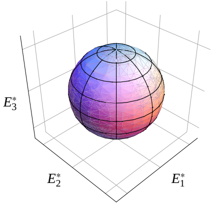
$\mathfrak{se}(1,1)$



$\mathfrak{se}(2)$



$\mathfrak{so}(2,1)$



$\mathfrak{so}(3)$

Classification by algebra (positive semidefinite quadratic systems)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$
p_1^2
p_2^2
$p_1^2 + p_2^2$
$(p_1 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$
p_1^2
p_3^2
$p_1^2 + p_3^2$
$(p_1 + p_2)^2$
$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$
p_1^2
p_3^2
$p_1^2 + p_3^2$
$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$
p_3^2
$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$
p_2^2
p_3^2
$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$
p_1^2
$p_1^2 + \frac{1}{2}p_2^2$

Proof sketch 1/4

Proposition

The following systems on $\mathfrak{g}_-^*$ are equivalent to H_Q:

($\mathfrak{E}1$) $H_Q \circ \psi$, *where* $\psi : \mathfrak{g}_-^* \to \mathfrak{g}_-^*$ *is a linear Poisson automorphism;*

($\mathfrak{E}2$) H_{rQ}, *where* $r \neq 0$;

($\mathfrak{E}3$) $H_Q + C$, *where* C *is a Casimir function.*

Case: $(\mathfrak{h}_3)_-^*$

- Casimir: p_1^2
- Linear Poisson automorphisms: $\begin{bmatrix} yw - zv & 0 & 0 \\ x & y & z \\ u & v & w \end{bmatrix}$

Proof sketch 2/4

- $H_Q(p) = p^\top Q\, p, \qquad Q = \begin{bmatrix} a_1 & b_1 & b_2 \\ b_1 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$
- Suppose $a_3 > 0$. Then $\psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{b_2}{a_3} & -\frac{b_3}{a_3} & 1 \end{bmatrix} \in \mathrm{Aut}\,((\mathfrak{h}_3)^*_-)$,

$$\psi^\top Q\, \psi = \begin{bmatrix} a_1 - \frac{b_2^2}{a_3} & b_1 - \frac{b_2 b_3}{a_3} & 0 \\ b_1 - \frac{b_2 b_3}{a_3} & a_2 - \frac{b_3^2}{a_3} & 0 \\ 0 & 0 & a_3 \end{bmatrix} = \begin{bmatrix} a_1' & b_1' & 0 \\ b_1' & a_2' & 0 \\ 0 & 0 & a_3 \end{bmatrix}.$$

- If $a_2' = 0$, then $H_Q \sim H(p) = p_3^2$.
- Suppose $a_2' > 0$. Then $\exists \psi' \in \mathrm{Aut}\,((\mathfrak{h}_3)^*_-)$, such that $\psi'^\top \psi^\top Q\, \psi\, \psi' = \mathrm{diag}\,(a_1'', 1, 1)$. Thus $H_Q \sim H(p) = p_2^2 + p_3^2$.

Proof sketch 3/4

- Suppose $a_3 = 0$. Likewise, $H_Q \sim H(p) = p_3^2$.
- Remains to be shown: $H_1(p) = p_3^2$ and $H_2 = p_2^2 + p_3^2$ distinct.
- Suppose $\exists\, \psi$ such that $\psi \cdot \vec{H}_1 = \vec{H}_2 \circ \psi$. Then

$$\begin{bmatrix} -2\psi_{12}p_1p_3 \\ -2\psi_{22}p_1p_3 \\ -2\psi_{32}p_1p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\left(\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3\right)\left(\psi_{31}p_1 + \psi_{32}p_2 + \psi_{33}p_3\right) \\ 2\left(\psi_{11}p_1 + \psi_{12}p_2 + \psi_{13}p_3\right)\left(\psi_{21}p_1 + \psi_{22}p_2 + \psi_{23}p_3\right) \end{bmatrix}.$$

  Yields contradiction.

Case: $(\mathfrak{so}\,(3)^*_-)$

Casimir: $p_1^2 + p_2^2 + p_3^2$ Automorphisms: $\mathrm{SO}\,(3)$

- Orthogonal matrices diagonalize PSD quadratic forms
- Consequently $H \sim p_1^2$ or $H \sim p_1^2 + \alpha p_2^2$, $0 < \alpha < 1$
- $\psi = \mathrm{diag}\left(-\sqrt{2}\sqrt{1-\alpha}, 2\sqrt{\alpha(1-\alpha)}, -\sqrt{2}\sqrt{\alpha}\right)$ brings $p_1^2 + \alpha\, p_2^2$ into $p_1^2 + \frac{1}{2}p_2^2$

Proof sketch 4/4

Case: $\mathfrak{so}(2,1)^*_-$

Casimir: $K = p_1^2 + p_2^2 - p_3^2$ Automorphisms: $\mathrm{SO}(2,1)$

- Direct application of automorphisms ($\mathfrak{E}1$) not fruitful
- Using rotation: $Q' = \rho_3(\theta)^\top Q \, \rho_3(\theta) = \begin{bmatrix} a_1 & 0 & b_2 \\ 0 & a_2 & b_3 \\ b_2 & b_3 & a_3 \end{bmatrix}$.
- Assume $a_1, a_2 \neq 0$. Then $Q' + xK$ has a Cholesky decomposition

$$Q' + xK = R^\top R, \qquad R = \begin{bmatrix} r_1 & 0 & r_3 \\ 0 & r_2 & r_4 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{for some } x \geq 0.$$

- Use automorphisms to normalise R.
- After normalization, we can apply similar approach to $R^\top R$.

General classification

- Consider equivalence of systems on different spaces
  — direct computation with MATHEMATICA

Types of systems

- Linear: integral curves contained in lines
  (sufficient: has two linear constants of motion)
- Planar: integral curves contained in planes, not linear
  (sufficient: has one linear constant of motion)
- otherwise: non-Planar

Classification by algebra (positive semidefinite quadratic systems)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

Linear systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Linear systems (3 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

Linear systems

$\mathsf{L}(1),\ \mathsf{L}(2),\ \mathsf{L}(3)$

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

$1 : p_2^2$

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$2 : (p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

$3 : p_2^2$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$
p_1^2
p_2^2
$p_1^2 + p_2^2$
$(p_1 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$
p_1^2
p_3^2
$p_1^2 + p_3^2$
$(p_1 + p_2)^2$
$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$
p_1^2
p_3^2
$p_1^2 + p_3^2$
$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$
p_3^2
$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$
p_2^2
p_3^2
$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$
p_1^2
$p_1^2 + \frac{1}{2}p_2^2$

Planar systems (5 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$1 : p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$2 : p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

$3 : p_3^2$

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$5 : (p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

$4 : p_3^2$

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Non-planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

$(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Non-planar systems (2 classes)

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$	$\mathfrak{se}(1,1)^*_-$	$\mathfrak{so}(2,1)^*_-$
p_1^2	p_1^2	p_1^2
p_2^2	p_3^2	p_3^2
$p_1^2 + p_2^2$	$p_1^2 + p_3^2$	$p_1^2 + p_3^2$
$(p_1 + p_3)^2$	$(p_1 + p_2)^2$	$(p_2 + p_3)^2$
$p_2^2 + (p_1 + p_3)^2$	$(p_1 + p_2)^2 + p_3^2$	$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$	$\mathfrak{se}(2)^*_-$	$\mathfrak{so}(3)^*_-$
p_3^2	p_2^2	p_1^2
$p_2^2 + p_3^2$	p_3^2	$p_1^2 + \frac{1}{2}p_2^2$
	$p_2^2 + p_3^2$	

Non-planar systems

$(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$

p_1^2

p_2^2

$p_1^2 + p_2^2$

$(p_1 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$\mathfrak{se}(1,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_1 + p_2)^2$

1 : $(p_1 + p_2)^2 + p_3^2$

$\mathfrak{so}(2,1)^*_-$

p_1^2

p_3^2

$p_1^2 + p_3^2$

$(p_2 + p_3)^2$

$p_2^2 + (p_1 + p_3)^2$

$(\mathfrak{h}_3)^*_-$

p_3^2

$p_2^2 + p_3^2$

$\mathfrak{se}(2)^*_-$

p_2^2

p_3^2

2 : $p_2^2 + p_3^2$

$\mathfrak{so}(3)^*_-$

p_1^2

$p_1^2 + \frac{1}{2}p_2^2$

Remarks

Interesting features

- Systems on $(\mathfrak{h}_3)^*_-$ or $\mathfrak{so}\,(3)^*_-$
  — equivalent to ones on $\mathfrak{se}\,(2)^*_-$
- Systems on $(\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-$ or $(\mathfrak{h}_3)^*_-$
  — planar or linear
- Systems on $(\mathfrak{h}_3)^*_-$, $\mathfrak{se}\,(1,1)^*_-$, $\mathfrak{se}\,(2)^*_-$ and $\mathfrak{so}\,(3)^*_-$
  — may be realized on multiple spaces.

(for $\mathfrak{so}\,(2,1)^*_-$ exception is $\mathsf{P}\,(5)$)

Outline

Types of stability

Stability of equilibrium point z_e

- (Lyapunov) stable
  $\forall$ nbd N $\exists$ nbd N' s.t. $\mathscr{F}_t(N') \subset N$
- spectrally stable
  $\mathrm{Re}\,(\lambda_i) \leq 0$ for eigenvalues of $D\vec{H}(z_e)$
- weakly assymptotically stable [Ortega et al. 2005]
  stable & $\exists$ nhd N s.t. $\mathscr{F}_t(N) \subset \mathscr{F}_s(N)$ whenever $t > s$.

weak asymptotic stab $\implies$ (Lyapunov) stab $\implies$ spectral stab

Methods (Positive results)

- Energy Casimir:
  $$d(H + C)(z_e) = 0 \qquad \text{and} \qquad d^2(H + C)(z_e) > 0$$
- Extended Energy Casimir:
  $$d(\lambda_0 H + \lambda_1 C + \lambda_2 F)(z_e) = 0 \text{ and } d^2(\lambda_0 H + \lambda_1 C + \lambda_2 F)(z_e) > 0$$

Linear systems

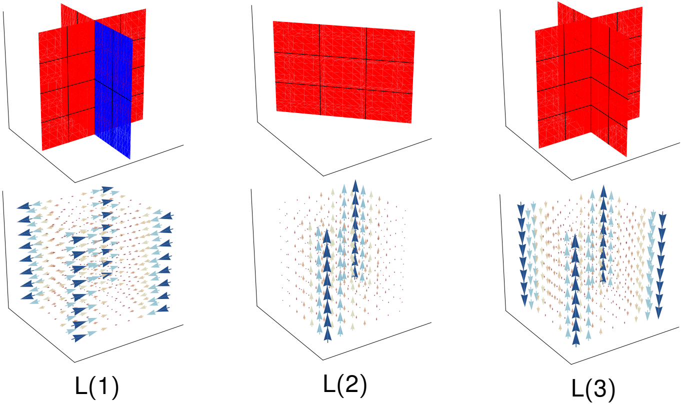


Figure: Equilibria (and vector fields) for linear systems

Planar systems

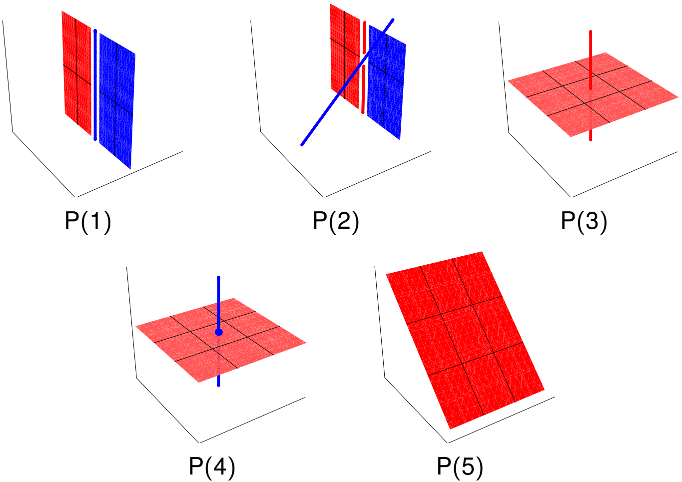
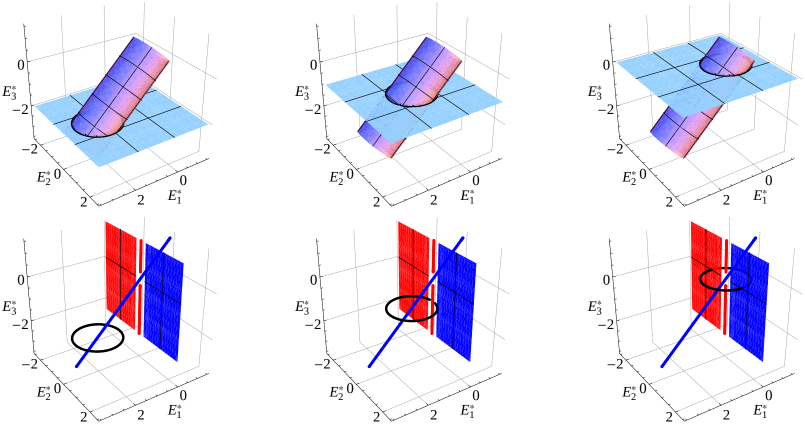


Figure: Equilibria of planar systems

P(2) : $((\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-, p_2^2 + (p_1 + p_3)^2)$



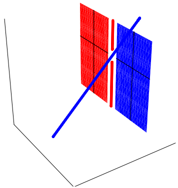
(a) $c_0 < -\sqrt{h_0} < 0$ (b) $c_0 = -\sqrt{h_0} < 0$ (c) $-\sqrt{h_0} < c_0 < \sqrt{h_0}$

Figure: Planar system P(2)

P(2) : $((\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-, p_2^2 + (p_1 + p_3)^2)$

$e_1^{\eta,\mu} = (0, \eta, \mu) \neq 0, \ \eta < 0$

Linearization $D\vec{H}$ has eigenvalues $\{0, 0, -2\eta\}$. Spectrally unstable.



$e_1^{\eta,\mu} = (0, \eta, \mu), \ \eta > 0, \mu \neq 0$ (Case $\mu = 0$ similar.)

- $H_\lambda = F, \ F = p_1^2$
- $\vec{H}[F](p) = -4p_1^2 p_2 \leq 0$ for p in some nhd of $(0, \eta, \mu)$
- $d\,H_\lambda(e_1^{\eta,\mu}) = 0$ and $d^2 H_\lambda(e_1^{\eta,\mu}) = \mathrm{diag}\,(2, 0, 0)$
- $d\,H(e_1^{\eta,\mu}) = \begin{bmatrix} 2\mu & 2\eta & 2\mu \end{bmatrix}$ and $d\,C^2(e_1^{\eta,\mu}) = \begin{bmatrix} 0 & 0 & \mu \end{bmatrix}$
- $d^2\,H_\lambda(e_1^{\eta,\mu})$ is PD on $W = \ker d\,H(e_1^{\eta,\mu}) \cap \ker d\,C^2(e_1^{\eta,\mu})$
- Weakly asymptotically stable

P(2) : $((\mathfrak{aff}(\mathbb{R}) \oplus \mathbb{R})^*_-, p_2^2 + (p_1 + p_3)^2)$

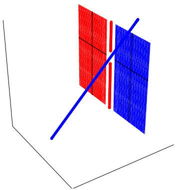
$e_2^\mu = (\mu, 0, -\mu), \ \mu \neq 0$ (Case $\mu = 0$ similar.)

$$H_\lambda = H, \qquad dH_\lambda(e_2^\mu) = 0, \qquad d^2 H_\lambda(e_2^\mu) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

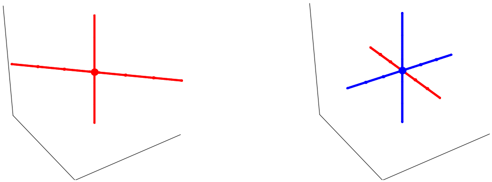
- $d\, C^2(e_2^\mu) = \begin{bmatrix} 0 & 0 & -2\mu \end{bmatrix}$
- $d^2 H_\lambda(e_2^\mu)$ is PD on $W = \ker d\, C^2(e_2^\mu)$ — stable

$e_1^{0,\mu} = (0, 0, \mu), \ \mu \neq 0$

- $p(t) = (\frac{-2\mu}{1+4\mu^2 t^2}, \frac{4\mu^2 t}{1+4\mu^2 t^2}, \mu)$ is integral curve
- $\forall$ nhd N of $e_1^{0,\mu}$ $\exists$ $t_0 < 0$ s.t. $p(t_0) \in N$
- $\|p(0) - e_1^{0,\mu}\| = 2|\mu|$
- Unstable



Non-planar systems



H(1) H(2)

Figure: Equilibria states of non-planar systems

Outline

Equilibria and invariants

$\mathcal{E}$-equivalence

Systems $(\mathfrak{g}_-^*, H)$ and $((\mathfrak{g}')_-^*, H')$ are $\mathcal{E}$-equivalent if there exists a linear isomorphism $\psi : \mathfrak{g}_-^* \to (\mathfrak{g}')_-^*$ such that $\psi \cdot \mathcal{E}_s = \mathcal{E}'_s$ and $\psi \cdot \mathcal{E}_u = \mathcal{E}'_u$

Proposition

- *Non-planar systems equivalent* $\iff$ *$\mathcal{E}$-equivalent*
- *Planar systems equivalent* $\iff$ *$\mathcal{E}$-equivalent*
- *Linear systems equivalent* $\iff$ *$\mathcal{E}$-equivalent*

Equilibrium index

Set of equilibria is union of i lines and j planes.

Pair (i, j): equilibrium index

Taxonomy

Algebra	Class	Equilibrium Index (lines, planes)	Normal Form(s)
$\mathfrak{se}(1,1)$	linear	$(0,1)$	L(2)
		$(0,2)$	L(3)
	planar	$(1,1)$	P(3)
	non-planar	$(2,0)$	H(1)
		$(3,0)$	H(2)
$\mathfrak{se}(2)$	linear	$(0,2)$	L(3)
	planar	$(1,1)$	P(4)
	non-planar	$(3,0)$	H(2)
$\mathfrak{so}(2,1)$	planar	$(1,1)$	P(3); P(4)
		$(0,1)$	P(5)
	non-planar	$(2,0)$	H(1)
		$(3,0)$	H(2)
$\mathfrak{so}(3)$	planar	$(1,1)$	P(4)
	non-planar	$(3,0)$	H(2)

Outline

Conclusion

Summary

- Classification of PSD quadratic systems in 3D
- Behaviour of equilibria
- Invariants & taxonomy

Outlook

- Integration
- Relax restrictions: PSD, global Casimir
- Affine case: $H_{A,Q} = p(A) + p^\top Q\, p$
- 4D case

R.M. Adams, R. Biggs and C.C. Remsing
On some quadratic Hamilton-Poisson systems
Appl. Sci. **15**(2013), (to appear).

A. Aron, C. Dăniasă and M. Puta
Quadratic and homogeneous Hamilton-Poisson system on $\mathfrak{so}(3)^*$
Int. J. Geom. Methods Mod. Phys. **4**(2007), 1173–1186.

A. Aron, M. Craioveanu, C. Pop and M. Puta
Quadratic and homogeneous Hamilton-Poisson systems on $A^*_{3,6,-1}$
Balkan J. Geom. Appl. **15**(2010), 1–7.

A. Aron, C. Pop and M. Puta
Some remarks on $(\mathfrak{sl}(2,\mathbb{R}))^*$ and Kahan's Integrator
An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. **53**(suppl.)(2007), 49–60.

J. Biggs and W. Holderbaum
Integrable quadratic Hamiltonian on the Euclidean group of motions
J. Dyn. Control Syst. **16**(3)(2010), 301–317.

R. Biggs and C.C. Remsing
On the equivalence of cost-extended control systems on Lie groups
In Recent Researches in Automatic Control, Systems Science and Communications (H.R. Karimi, ed.). WSEAS Press (2012) 60–65.

A. Krasiński et al
The Bianchi classification in the Schücking-Behr approach
Gen. Relativ. Gravit. **35**(2003), 475–489.

G.M. Mubarakzyanov
On solvable Lie algebras (in Russian)
Izv. Vyssh. Uchebn. Zaved. **32**(1)(1963), 114–123.

J-P. Ortega, V. Planas-Bielsa and T.S. Ratiu
Asymptotic and Lyapunov stability of constrained and Poisson equilibria
J. Diff. Equations **214** (2005), 92–127.

J. Patera, R.T. Sharp, P. Wintemitz, H. Zassenhaus
Invariants of low dimensional Lie algebras
J. Math. Phys. **17**(1976), 986–994.

R.M. Tudoran
The free rigid body dynamics: generalized versus classic.
arXiv:1102.2725v5 [math-ph].

R.M. Tudoran and R.A. Tudoran
On a large class of three-dimensional Hamiltonian systems
J. Math. Phys. **50**(2009) 012703.