Berezin transform of two arguments

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Geometry, Integrability and Qauntization

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Holomorphic in x and antiholomorpic in y

$$\partial_{\bar{x}} K_\alpha(x,y) = \partial_y K_\alpha(x,y) = 0,$$

where $\partial_{\bar{x}} := \left(\frac{1}{2}\frac{\partial}{\partial x_1^k} + \frac{i}{2}\frac{\partial}{\partial x_2^k}\right)_{k=1}^n$.

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Berezin transform: As $\alpha \to \infty$

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$$\begin{array}{ccc} \mathbb{C}^n & \to & \mathbb{R}^n \\ \text{Entire functions } \partial_{\bar{x}} f = 0 & \to & \text{Harmonic functions } \Delta f = 0. \end{array}$$

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Bergman Kernel

The reproducing property
For harmonic, integrable function, i.e. $\Delta f = 0$ on $\mathbb{R}^n$ it holds

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$$u = x\cdot y + \mathrm{i}\sqrt{|x|^2 |y|^2 - (x\cdot y)^2}.$$

Berezin transform of one argument

It was shown by M. Engliš in 2009 that as $\alpha \to \infty$

$$(B_\alpha f)(x) := \int_{\mathbb{R}^n} f(y) \frac{R_\alpha(x,y) R_\alpha(x,y)}{R_\alpha(x,x)} \mathrm{d}\mu_\alpha^n(y) \approx f(x) +$$

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for n even.

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$$(B_\alpha f)(0) \approx f(0) + \frac{1}{4\alpha} \Delta f(0) + \dots$$

Stokes phenomenon.

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Also not working in dimension one (n=1).

Harmonic Berezin transform of two argument

Question: Is there a limit as $\alpha \to \infty$

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for some v?

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$$t \cdot v = \lim_{\alpha\to\infty} \int_{\mathbb{R}^n} t \cdot y \frac{R_\alpha(x,y)R_\alpha(z,y)}{R_\alpha(x,z)} \mathrm{d}\mu^n_\alpha(y).$$

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Applying the Stokes theorem we have

$$\int_{\mathbb{R}^n} (t \cdot y)\ g(y) \mathrm{d}\mu_\alpha^n = \frac{1}{2\alpha} \int_{\mathbb{R}^n} t \cdot \nabla g(y) \mathrm{d}\mu_\alpha^n,$$

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$$= \frac{1}{2\alpha} \frac{1}{R_\alpha(x,z)} \int_{\mathbb{R}^n} R_\alpha(z,y) t \cdot \nabla_y R_\alpha(x,y) + R_\alpha(x,y) t \cdot \nabla_y R_\alpha(z,y) \mathrm{d}\mu_\alpha^n(y),$$

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since $\Delta R_\alpha = 0$ then by reproducing property

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$$= \frac{1}{2\alpha}\frac{1}{R_\alpha(x,z)}\int_{\mathbb{R}^n} R_\alpha(z,y)t\cdot\nabla_y R_\alpha(x,y) + R_\alpha(x,y)t\cdot\nabla_y R_\alpha(z,y)\mathrm{d}\mu_\alpha^n(y),$$

since $\Delta R_\alpha = 0$ then by reproducing property

$$= \frac{1}{2\alpha}\frac{t\cdot\nabla_z R_\alpha(x,z) + t\cdot\nabla_x R_\alpha(z,x)}{R_\alpha(x,z)}.$$

Remember that

$$R_\alpha(x,y) = \Phi_2\left(\begin{matrix} - \\ \frac{n}{2}-1\end{matrix};\begin{matrix}\frac{n}{2}-1 & \frac{n}{2}-1 \\ & - \end{matrix};\alpha u,\alpha\bar{u}\right),$$

thus

$$\begin{aligned}(B_\alpha^2 t\cdot y)(x,z) \quad &= \quad \frac{\Phi_2\left(\begin{matrix} - \\ \frac{n}{2}\end{matrix};\begin{matrix}\frac{n}{2} & \frac{n}{2}-1 \\ & - \end{matrix};\alpha u,\alpha\bar{u}\right)}{\Phi_2\left(\begin{matrix} - \\ \frac{n}{2}-1\end{matrix};\begin{matrix}\frac{n}{2}-1 & \frac{n}{2}-1 \\ & - \end{matrix};\alpha u,\alpha\bar{u}\right)}\frac{t\cdot\nabla_x + t\cdot\nabla_z}{2}u \\ &+ \quad \frac{\Phi_2\left(\begin{matrix} - \\ \frac{n}{2}\end{matrix};\begin{matrix}\frac{n}{2}-1 & \frac{n}{2} \\ - & \end{matrix};\alpha u,\alpha\bar{u}\right)}{\Phi_2\left(\begin{matrix} - \\ \frac{n}{2}-1\end{matrix};\begin{matrix}\frac{n}{2}-1 & \frac{n}{2}-1 \\ & - \end{matrix};\alpha u,\alpha\bar{u}\right)}\frac{t\cdot\nabla_x + t\cdot\nabla_z}{2}\bar{u}.\end{aligned}$$

Hypergeometrization

$$R_\alpha(x,y) = {}_0P_1\left(\begin{matrix} - \\ \frac{n}{2} \end{matrix} ; \alpha x, y \right) = \sum_{m=0}^{\infty} \frac{\alpha^m}{(\frac{n}{2})_m} Z_m(x,y).$$

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Generally, for f function of a real argument $x \in \mathbb{R}$

$${}_pf_q\left(\begin{matrix} a_1 \dots a_p \\ c_1 \dots c_q \end{matrix} ; x \right) := \sum_{k=0}^{\infty} \frac{f^{(k)}(0)x^k}{k!} \frac{(a_1)_k \dots (a_p)_k}{(c_1)_k \dots (c_q)_k}.$$

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For a function of a vector argument $x \in \mathbb{R}^n$

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For a symmetric function $F(tx,y) = F(x,ty)$ $\forall t$ near 1

$${}_0F_1\left(\begin{array}{c} - \\ c \end{array}; x, y\right) := {}_0(F(\cdot,y))_1\left(\begin{array}{c} - \\ c \end{array}; x\right) = {}_0(F(x,\cdot))_1\left(\begin{array}{c} - \\ c \end{array}; y\right).$$

Examples

Monomials $f(x) = x^m$

$$ {}_0f_1\left(\begin{matrix} - \\ c \end{matrix}; x\right) = \frac{1}{(c)_m}x^m \qquad {}_1f_0\left(\begin{matrix} a \\ - \end{matrix}; x\right) = (a)_m x^m. $$

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asymptotic expansions of ${}_1F_1$

$$(1-x)^{-a} = (-x)^{-a}\left(1-\frac{1}{x}\right)^{-a}$$

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$$(1-x)^{-a} = (-x)^{-a}\left(1-\frac{1}{x}\right)^{-a} \approx (-x)^{-a} - a(-x)^{-a-1} + \frac{a(a+1)}{2}(-x)^{-a-2} + \dots.$$

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But

$${}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; x\right) = e^x {}_1F_1\left(\begin{matrix} c-a \\ c \end{matrix} ; -x\right)$$

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Correct answer

$${}_1F_1\left(\begin{matrix} a \\ c \end{matrix} ; x\right) \sim e^x\frac{\Gamma(c)}{\Gamma(a)}x^{a-c} + \frac{\Gamma(c)}{\Gamma(c-a)}(-x)^{-a}.$$

asymptotic expansions of Φ_2

$$f(t) := (1-tx)^{-b_1}(1-ty)^{-b_2} = (-tx)^{-b_1}(-ty)^{-b_2}\left(1-\frac{1}{tx}\right)^{-b_1}\left(1-\frac{1}{ty}\right)^{-b_2}.$$

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Correct answer

$$\begin{aligned}\Phi_2\left(\begin{matrix} - \\ c \end{matrix};\begin{matrix} b_1 \quad b_2 \\ - \end{matrix};x,y\right) &= \frac{\Gamma(c)}{\Gamma(b_1)}e^x x^{b_1+b_2-c}(x-y)^{-b_2} + \dots \\ &+ \frac{\Gamma(c)}{\Gamma(b_1)}e^y y^{b_1+b_2-c}(y-x)^{-b_1} + \dots \\ &+ \frac{\Gamma(c)}{\Gamma(c-b_1-b_2)}(-x)^{-b_1}(-y)^{-b_2} + \dots\end{aligned}$$

asymptotic of Φ_2

As $|\alpha| \to \infty$

$$
\begin{aligned}
\Phi_2\left(\begin{matrix} - \\ c \end{matrix} ; \begin{matrix} b_1 & b_2 \\ & - \end{matrix} ; \alpha x, \alpha y\right) &= \frac{\Gamma(c)}{\Gamma(b_1)} \alpha^{b_1 - c} x^{b_1 + b_2 - c} (x-y)^{-b_2} e^{\alpha x} O(1) \\
&+ \frac{\Gamma(c)}{\Gamma(b_2)} \alpha^{b_2 - c} y^{b_1 + b_2 - c} (y-x)^{-b_1} e^{\alpha y} O(1) \\
&+ \frac{\Gamma(c)}{\Gamma(c - b_1 - b_2)} \alpha^{-b_1 - b_2} (-x)^{-b_1} (-y)^{-b_2} O(1),
\end{aligned}
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&+ \frac{\Gamma(c)}{\Gamma(c - b_1 - b_2)} \alpha^{-b_1 - b_2} (-x)^{-b_1} (-y)^{-b_2} O(1),
\end{aligned}
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hence

$$
\begin{aligned}
\Phi_2\left(\begin{matrix} - & ; & \frac{n}{2} & \frac{n}{2} - 1 \\ \frac{n}{2} & ; & & - \end{matrix} ; \alpha u, \alpha \bar{u}\right) &= u^{\frac{n}{2} - 1} (u - \bar{u})^{1 - \frac{n}{2}} e^{\alpha u} O(1) \\
&+ \left(\frac{n}{2} - 1\right) \alpha^{-1} \bar{u}^{\frac{n}{2} - 1} (\bar{u} - u)^{-\frac{n}{2}} e^{\alpha \bar{u}} O(1) \\
&+ \frac{\Gamma(\frac{n}{2} - 1)}{\Gamma(1 - \frac{n}{2})} \alpha^{1-n} (-u)^{-\frac{n}{2}} (-\bar{u})^{1 - \frac{n}{2}} O(1),
\end{aligned}
$$

No go

$$(B_\alpha^2 t \cdot y)(x,z) \sim \frac{B_1 e^{\alpha u} + \bar{B}_1 e^{\alpha \bar{u}} + C_1 \alpha^{1-n}}{B_2 e^{\alpha u} + \bar{B}_2 e^{\alpha \bar{u}} + C_2 \alpha^{2-n}}.$$

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We must make $\alpha \in \mathbb{C}$ such that $\Re(\alpha u) > \Re(\alpha \bar{u})$, $\Re(\alpha u) > 0$ and $\Re(\alpha) > 0$.

No go

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$$\alpha = |\alpha| \, e^{i\theta}, \quad u = |x| \, |z| \, e^{i\varphi}, \sin\varphi \geq 0, \quad \cos\theta > 0,$$

No go

$$(B_{\alpha}^{2} t \cdot y)(x,z) \sim \frac{B_1 e^{\alpha u} + \bar{B}_1 e^{\alpha \bar{u}} + C_1 \alpha^{1-n}}{B_2 e^{\alpha u} + \bar{B}_2 e^{\alpha \bar{u}} + C_2 \alpha^{2-n}}.$$

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we get

$$\cos(\theta + \varphi) > \cos(\theta - \varphi) \qquad \cos(\theta + \varphi) > 0$$

No go

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$$\alpha = |\alpha| \, e^{i\theta}, \quad u = |x| \, |z| \, e^{i\varphi}, \sin\varphi \geq 0, \quad \cos\theta > 0,$$

we get

$$\begin{aligned} \cos(\theta+\varphi) &> \cos(\theta-\varphi) & \qquad \cos(\theta+\varphi) &> 0 \\ \cos\theta\cos\varphi - \sin\theta\sin\varphi &> \cos\theta\cos\varphi + \sin\theta\sin\varphi & \qquad \cos\theta\cos\varphi &> \sin\theta\sin\varphi \end{aligned}$$

No go

$$(B_\alpha^2 t \cdot y)(x,z) \sim \frac{B_1 e^{\alpha u} + \bar{B}_1 e^{\alpha \bar{u}} + C_1 \alpha^{1-n}}{B_2 e^{\alpha u} + \bar{B}_2 e^{\alpha \bar{u}} + C_2 \alpha^{2-n}}.$$

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$$\begin{aligned} \cos(\theta+\varphi) &> \cos(\theta-\varphi) & \cos(\theta+\varphi) &> 0 \\ \cos\theta\cos\varphi - \sin\theta\sin\varphi &> \cos\theta\cos\varphi + \sin\theta\sin\varphi & \cos\theta\cos\varphi &> \sin\theta\sin\varphi \\ 2\sin\theta\sin\varphi &< 0 \end{aligned}$$

No go

$$(B_\alpha^2 t\cdot y)(x,z)\sim \frac{B_1e^{\alpha u}+\bar{B}_1e^{\alpha\bar{u}}+C_1\alpha^{1-n}}{B_2e^{\alpha u}+\bar{B}_2e^{\alpha\bar{u}}+C_2\alpha^{2-n}}.$$

For $x\cdot z>0$ or n even, the limit does not exists!... when $\alpha\in\mathbb{R}$.
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$$\alpha=|\alpha|\,e^{i\theta},\quad u=|x|\,|z|\,e^{i\varphi}, \sin\varphi\ge 0,\quad \cos\theta>0,$$

we get

$$\begin{aligned} \cos(\theta+\varphi) &> \cos(\theta-\varphi) & \cos(\theta+\varphi)&>0\\ \cos\theta\cos\varphi-\sin\theta\sin\varphi &> \cos\theta\cos\varphi+\sin\theta\sin\varphi & \cos\theta\cos\varphi &> \sin\theta\sin\varphi\\ 2\sin\theta\sin\varphi &< 0 & &\\ \text{for }\ \sin\varphi>0 \qquad \sin\theta &< 0 & \tan\theta &< \mathrm{cotg}\varphi. \end{aligned}$$

No go

$$(B_\alpha^2 t \cdot y)(x,z) \sim \frac{B_1 e^{\alpha u} + \bar{B}_1 e^{\alpha \bar{u}} + C_1 \alpha^{1-n}}{B_2 e^{\alpha u} + \bar{B}_2 e^{\alpha \bar{u}} + C_2 \alpha^{2-n}}.$$

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we get

$$\begin{aligned} \cos(\theta+\varphi) &> \cos(\theta-\varphi) & \cos(\theta+\varphi) &> 0 \\ \cos\theta\cos\varphi - \sin\theta\sin\varphi &> \cos\theta\cos\varphi + \sin\theta\sin\varphi & \cos\theta\cos\varphi &> \sin\theta\sin\varphi \\ 2\sin\theta\sin\varphi &< 0 & & \\ \text{for } \sin\varphi > 0 \qquad \sin\theta &< 0 & \tan\theta &< \mathrm{cotg}\varphi. \end{aligned}$$

Always possible for noncolinear x, z!

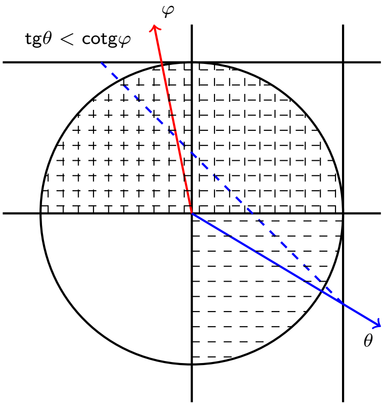

Figure: Angles.

The point

For $\alpha \in \mathbb{C}$ such that $\Re(\alpha u) > \Re(\alpha \bar{u})$, $\Re(\alpha u) > 0$ and $\Re(\alpha) > 0$ as $|\alpha| \to \infty$

$$(B_{\alpha}^{2} t \cdot y)(x, z) \to \frac{t \cdot \nabla_x + t \cdot \nabla_z}{2} u$$

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$$= \frac{t \cdot \nabla_x + t \cdot \nabla_z}{2} \left(x \cdot z + i\sqrt{|x|^2 |z|^2 - (x \cdot z)^2} \right)$$

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$$= \frac{t \cdot (x + z)}{2} + i \frac{x \cdot t(|z|^2 - x \cdot z) + z \cdot t(|x|^2 - x \cdot z)}{\sqrt{|x|^2 |z|^2 - (x \cdot z)^2}}.$$

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$$v := x \frac{u - |z|^2}{u - \bar{u}} + z \frac{u - |x|^2}{u - \bar{u}}.$$

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$$v := x \frac{u - |z|^2}{u - \bar{u}} + z \frac{u - |x|^2}{u - \bar{u}}.$$

$$v \cdot \bar{v} = 2(|x|^2 + |z|^2).$$

Polynomial case

Let p_M be a polynomial of degree M then

$$(B_\alpha^2 p_M)(x,z) = \sum_{\beta,\beta_2,\gamma,\gamma_2} C(\beta,\beta_2,\gamma,\gamma_2,\alpha,x,z)$$

$$\frac{\Phi_2\left(\begin{matrix} \frac{n}{2}-1+\beta \\ \frac{n}{2}-1+\gamma \quad \frac{n}{2}-1+\gamma_2 \end{matrix} ; \begin{matrix} \frac{n}{2}-1+\beta_2 \quad \frac{n}{2}-1+\beta_2 \\ - \end{matrix} ; \alpha u, \alpha\bar{u} \right)}{\Phi_2\left(\begin{matrix} - \\ \frac{n}{2}-1 \end{matrix} ; \begin{matrix} \frac{n}{2}-1 \quad \frac{n}{2}-1 \\ - \end{matrix} ; \alpha u, \alpha\bar{u} \right)}.$$

Actually

$$\int_{\mathbb{R}^n} p_M(y) R_\alpha(x,y) R_\alpha(z,y) \mathrm{d}\mu_\alpha^n(y) =$$

$$\sum_{j_1 \cdots j_{10}} \frac{\left(\frac{n}{2}-1\right)_{j_2+j_3+j_5+j_7+2j_8+j_9+2j_{10}}}{\left(\frac{n}{2}-1\right)_{j_2+2j_3+2j_5+2j_7+4j_8+j_9+3j_{10}}} \frac{\left(\frac{n}{2}-1\right)_{j_4+\cdots+j_{10}}}{\left(\frac{n}{2}-1\right)_{j_4+j_5+2(j_6+\cdots+j_{10})}}$$

$$\frac{|x|^{2(j_3+j_5+j_7+2j_8+j_{10})} \, |z|^{2(j_6+\cdots+j_{10})} \, 2^{-2j_1-2j_3+j_7} (-1)^{j_3+j_5+j_6+j_8+j_9} \alpha^{-2j_1+j_5+j_7+2j_8+j_9+2j_{10}}}{j_1! \ldots j_{10}!}$$

$$\Delta_t^{j_2+j_3+j_6+j_7+j_8} (x \cdot \nabla_t)^{j_2+j_9+j_{10}} (z \cdot \nabla_t)^{j_4+j_5} p_M(t)|_{t=0}$$

$$\Phi_2\left(\begin{matrix} & \frac{n}{2}-1+j_2+j_3+j_5+j_7+2j_8+j_9+2j_{10} & \\ \frac{n}{2}-1+j_2+2j_3+2j_5+2j_7+4j_8+j_9+3j_{10} & & \frac{n}{2}-1+j_4+j_5+2(j_6+\cdots+j_{10}) \end{matrix} ; \right.$$

$$\left. \begin{matrix} \frac{n}{2}-1+j_4+\cdots+j_{10} & \frac{n}{2}-1+j_4+\cdots+j_{10} \\ - & \end{matrix} ; \alpha u, \alpha\bar{u} \right),$$

where the summation indecies are non-negative integers bound by the following inequality

$$2j_1 + j_2 + 2j_3 + j_4 + j_5 + 2j_6 + 2j_7 + 2j_8 + j_9 + j_{10} \leq M.$$

As $|\alpha| \to \infty$

$$\Phi_2\left(\begin{matrix} & a & \\ c_1 & & c_2 \end{matrix} ; \begin{matrix} b & & b \\ & - & \end{matrix} ; \alpha u, \alpha \bar{u}\right) \sim \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b)\Gamma(a)} \alpha^{b+a-c_1-c_2} u^{2b+a-c_1-c_2} (u-\bar{u})^{-b} e^{\alpha u},$$

where $\alpha \in \mathbb{C}$ such that $\Re(\alpha u) > \Re(\alpha \bar{u})$ and $\Re(\alpha u) > 0$.

As $|\alpha| \to \infty$

$$\Phi_2\left(\begin{array}{cc} & a & \\ c_1 & & c_2 \end{array};\begin{array}{ccc} b & & b \\ & - & \end{array};\alpha u, \alpha\bar{u}\right) \sim \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b)\Gamma(a)}\alpha^{b+a-c_1-c_2}u^{2b+a-c_1-c_2}(u-\bar{u})^{-b}e^{\alpha u},$$

where $\alpha \in \mathbb{C}$ such that $\Re(\alpha u) > \Re(\alpha\bar{u})$ and $\Re(\alpha u) > 0$.

$$\Phi_2\left(\begin{array}{cc} & a & \\ c_1 & & c_2 \end{array};\begin{array}{ccc} b_1 & & b_2 \\ & - & \end{array};x, y\right) = \sum_{j,k=0}^{\infty}\frac{(a)_{j+k}}{(c_1)_{j+k}(c_2)_{j+k}}\frac{(b_1)_j(b_2)_k}{j!k!}x^j y^k.$$

As $|\alpha| \to \infty$

$$\Phi_2\left(\begin{matrix} & a & \\ c_1 & & c_2 \end{matrix};\begin{matrix} b & & b \\ & - & \end{matrix};\alpha u, \alpha\bar{u}\right) \sim \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b)\Gamma(a)}\alpha^{b+a-c_1-c_2}u^{2b+a-c_1-c_2}(u-\bar{u})^{-b}e^{\alpha u},$$

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We get

$$(B_\alpha^2 p)(x,z) \to p(v).$$

Colinear case

For $t > 0$ as $\alpha \to +\infty$

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$$(B^2_\alpha f)(x, tx) \quad := \quad \int\limits_{\mathbb{R}^n} f(y) \frac{R_\alpha(x, y) R_\alpha(tx, y)}{R_\alpha(tx, x)} \mathrm{d}\mu^n_\alpha(y)$$

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$$\begin{aligned} (B^2_\alpha f)(x, tx) &:= \int_{\mathbb{R}^n} f(y) \frac{R_\alpha(x, y) R_\alpha(tx, y)}{R_\alpha(tx, x)} \mathrm{d}\mu^n_\alpha(y) \\ (B^2_\alpha f(|y|^2))(x, -tx) &\to f(0) \qquad (n \text{ odd}). \end{aligned}$$

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\begin{aligned}
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(B^2_\alpha f(|y|^2))(x, -tx) \quad &\to \quad f(0) \qquad (n \text{ odd}). \\
(B^2_\alpha f(|y|^2))(x, tx) \quad &\to \quad f(t\,|x|^2) \qquad (\forall t \quad n \text{ even}).
\end{aligned}
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(B_\alpha^2 f(|y|^2))(x, -tx) &\to f(0) \qquad (n \text{ odd}). \\
(B_\alpha^2 f(|y|^2))(x, tx) &\to f(t\,|x|^2) \qquad (\forall t \quad n \text{ even}). \\
(B_\alpha^2 e^{z\cdot y})(x, t, x) &\to e^{\frac{t}{2}\bar{u}_{z,x} + \frac{1}{2} u_{z,x}} {}_1F_1\left(\begin{matrix} \frac{n}{2} - 1 \\ n - 2 \end{matrix} ; \mathrm{i} v_{z,x}(t-1) \right).
\end{aligned}
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(B^2_\alpha f(|y|^2))(x,-tx) &\to f(0) \qquad (n \text{ odd}).\\
(B^2_\alpha f(|y|^2))(x,tx) &\to f(t\,|x|^2) \qquad (\forall t \quad n \text{ even}).\\
(B^2_\alpha e^{z\cdot y})(x,t,x) &\to e^{\frac{t}{2}\bar{u}_{z,x}+\frac{1}{2}u_{z,x}}\,{}_1F_1\left(\begin{array}{c}\frac{n}{2}-1\\ n-2\end{array};\mathrm{i}v_{z,x}(t-1)\right).\\
(B^2_\alpha e^{z\cdot y})(x,tx) &\to e^{z\cdot x}\Phi_2\left(\begin{array}{c}-\\ n-2\end{array};\begin{array}{cc}\frac{n}{2}-1 & \frac{n}{2}-1\\ & -\end{array};\frac{t-1}{2}u,\frac{t-1}{2}\bar{u}\right).
\end{aligned}
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(B^2_\alpha f)(x,tx) &:= \int_{\mathbb{R}^n} f(y)\frac{R_\alpha(x,y)R_\alpha(tx,y)}{R_\alpha(tx,x)}\mathrm{d}\mu^n_\alpha(y)\\
(B^2_\alpha f(|y|^2))(x,-tx) &\to f(0) \qquad (n \text{ odd}).\\
(B^2_\alpha f(|y|^2))(x,tx) &\to f(t\,|x|^2) \qquad (\forall t \quad n \text{ even}).\\
(B^2_\alpha e^{z\cdot y})(x,t,x) &\to e^{\frac{t}{2}\bar{u}_{z,x}+\frac{1}{2}u_{z,x}}\,{}_1F_1\left(\begin{matrix}\frac{n}{2}-1\\ n-2\end{matrix};\mathrm{i}v_{z,x}(t-1)\right).\\
(B^2_\alpha e^{z\cdot y})(x,tx) &\to e^{z\cdot x}\Phi_2\left(\begin{matrix}-\\ n-2\end{matrix};\begin{matrix}\frac{n}{2}-1 & \frac{n}{2}-1\\ - & \end{matrix};\frac{t-1}{2}u,\frac{t-1}{2}\bar{u}\right).\\
(B^2_\alpha e^{z\cdot y})(x,0) &\to \Phi_2\left(\begin{matrix}-\\ \frac{n}{2}-1\end{matrix};\begin{matrix}\frac{n}{2}-1 & \frac{n}{2}-1\\ - & \end{matrix};\frac{u}{2},\frac{\bar{u}}{2}\right) = R_{\frac{1}{2}}(x,z).
\end{aligned}
$$

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For $t > 0$ as $\alpha \to +\infty$

$$
\begin{aligned}
(B^2_\alpha f)(x, tx) \quad &:= \quad \int_{\mathbb{R}^n} f(y) \frac{R_\alpha(x,y) R_\alpha(tx,y)}{R_\alpha(tx,x)} \mathrm{d}\mu^n_\alpha(y) \\
(B^2_\alpha f(|y|^2))(x, -tx) \quad &\to \quad f(0) \qquad (n \text{ odd}). \\
(B^2_\alpha f(|y|^2))(x, tx) \quad &\to \quad f(t\,|x|^2) \qquad (\forall t \quad n \text{ even}). \\
(B^2_\alpha e^{z\cdot y})(x, t, x) \quad &\to \quad e^{\frac{t}{2}\bar{u}_{z,x} + \frac{1}{2} u_{z,x}} {}_1F_1\left(\begin{array}{c} \frac{n}{2} - 1 \\ n-2 \end{array} ; \mathrm{i} v_{z,x}(t-1) \right). \\
(B^2_\alpha e^{z\cdot y})(x, tx) \quad &\to \quad e^{z\cdot x} \Phi_2 \left(\begin{array}{c} - \\ n-2 \end{array} ; \begin{array}{cc} \frac{n}{2} - 1 & \frac{n}{2} - 1 \\ - & \end{array} ; \frac{t-1}{2} u, \frac{t-1}{2} \bar{u} \right). \\
(B^2_\alpha e^{z\cdot y})(x, 0) \quad &\to \quad \Phi_2 \left(\begin{array}{c} - \\ \frac{n}{2} - 1 \end{array} ; \begin{array}{cc} \frac{n}{2} - 1 & \frac{n}{2} - 1 \\ - & \end{array} ; \frac{u}{2}, \frac{\bar{u}}{2} \right) = R_{\frac{1}{2}}(x, z).
\end{aligned}
$$

When $t < 0$ the limit mostly does not exists.

Applications

What are the applications of the Berezin transform of two argument?

References

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