CL-BASED METHOD TO OBTAIN NONLOCALLY RELATED SYSTEMS AND NONLOCAL SYMMETRIES

1. Construction of nonlocally related systems through conservation laws (CLs)
   --Use of CL to obtain nonlocally related system (potential system)
   --Use of *n* CLs to obtain up to $2^n - 1$ nonlocally related systems
   --How to find nonlocally related subsystems
   --Tree of nonlocally related systems

2. Examples
   --Nonlinear wave equations
   --Nonlinear telegraph equations
   --Planar gas dynamics equations

Limitation of local symmetries: **A given PDE as it stands does has no useful local symmetry or no useful conservation law (CL)**

Aim: **To extend existing methods to systems that are nonlocally related but equivalent to a given PDE system**

How to do this systematically?

Two natural ways:
*use of any CL
*use of any point symmetry

CONSTRUCTION OF NONLOCALLY RELATED SYSTEMS THROUGH CLs

Given any local CL

$$\mathrm{D}_x X(x,t,u,\partial u,\ldots,\partial^r u) + \mathrm{D}_t T(x,t,u,\partial u,\ldots,\partial^r u) = 0$$

of

$$R[u] = R(x,t,u,\partial u,\ldots,\partial^k u) = 0, \qquad (1)$$

one can form **equivalent** augmented *potential system P*

$$\frac{\partial v}{\partial t} = X(x,t,u,\partial u,\ldots,\partial^r u),$$
$$\frac{\partial v}{\partial x} = -T(x,t,u,\partial u,\ldots,\partial^r u),$$
$$R(x,t,u,\partial u,\ldots,\partial^k u) = 0$$

If (u,v) solves potential system P then u solves $R[u]=0$.

Conversely, if u solves $R[u]=0$, then there exists solution (u,v) of potential system P due to integrability condition $v_{xt}=v_{tx}$ being satisfied from CL.

But equivalence relationship is *nonlocal and non-invertible* since for any u solving $R[u]=0$, if (u,v) solves potential system P, then so does $(u,v+C)$ for any constant C.

Symmetry (CL) of $R[u]=0$ yields symmetry (CL) of potential system P.

Conversely, symmetry (CL) of potential system P yields symmetry (CL) of $R[u]=0$.

Suppose equivalent potential system P has point symmetry

$$\mathrm{X} = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \omega(x,t,u,v)\frac{\partial}{\partial u} + \phi(x,t,u,v)\frac{\partial}{\partial v}$$

Then X yields **nonlocal symmetry** of given PDE (1) iff

$$(\xi_v)^2 + (\tau_v)^2 + (\omega_v)^2 \not\equiv 0.$$

Hence through CL of (1), nonlocal symmetry of (1) can be obtained through point (local) symmetry of related potential system *P*. [Converse also true!]

Use of *n* CLs to obtain up to $2^n - 1$ nonlocally related systems

Now suppose, there are n multipliers $\{\Lambda_i(x,t,U,\partial U,\ldots,\partial^q U)\}$ yielding n independent CLs of $R[u]=0$.

Let v^i be potential variable $\leftrightarrow$ multiplier $\Lambda_i[U]$

Then one obtains n **singlet** potential systems
$P^i, \quad i=1,\ldots,n$

Moreover, one can consider potential systems

in **couplets** $\{P^i, P^j\}_{i,j=1}^n$ with two potential variables

in **triplets** $\{P^i, P^j, P^k\}_{i,j,k=1}^n$ with three potential variables

$,\ldots,$

in an ***n*-plet** $\left\{P^1,\ldots,P^n\right\}$ with n potential variables

Hence from n CLs, one obtains $2^n - 1$ *distinct potential systems!*

Starting from *any* potential system, one can continue process and if it has N "local" CLs, one can obtain up to $2^N - 1$ further distinct potential systems. One can tell in advance whether further potential systems are obtained.

In particular, one can show that if multipliers depend only on independent variables (x, t) **then no new potential system is obtained.**

Any potential system could yield new nonlocal symmetries or new nonlocal CLs for any other potential system or "given" PDE

Nonlocally related subsystems

Suppose one has given PDE system $S\{x,t,u^1,\ldots,u^M\}=0$ with indicated M dependent variables.

A **subsystem** excluding dependent variable, say u^M, $\underline{S}\{x,t,u^1,\ldots,u^{M-1}\}=0$ is *nonlocally related* to given system $S\{x,t,u^1,\ldots,u^M\}=0$ if u^M cannot be directly expressed from equations of $S\{x,t,u^1,\ldots,u^M\}=0$ in terms of x,t, remaining dependent variables $u^1,\ldots,u^{M-1}$, and their derivatives.

Subsystems for consideration can arise following interchange of dependent and independent variables of given system

$$S\{x,t,u^1,\ldots,u^M\}=0$$

Tree of Nonlocally Related Systems

Consequently, for given PDE system one obtains tree of nonlocally related (but equivalent) systems arising from CLs and subsystems.

Each system in such an extended tree is equivalent in sense that solution set for any system in tree can be found from solution set for any other system in tree through connection formula

Due to equivalence of solution sets and nonlocal relationship, it follows that any coordinate-independent method of analysis (quantitative, analytical, numerical, perturbation, etc.) when applied to any system in tree may yield simpler computations and/or results that cannot be obtained when method is directly applied to given system.

Note also that "given" system could be any system in tree!!

EXAMPLES

1. Nonlinear wave equation

Suppose given PDE is nonlinear wave equation

$$\mathbf{U}\{x,t,u\}=0: \quad u_{tt}=(c^2(u)u_x)_x$$

Directly, one obtains singlet potential system (multiplier is 1)

$$\mathbf{UV}\{x,t,u,v\}=0: \quad \begin{cases} v_x-u_t=0, \\ v_t-c^2(u)u_x=0 \end{cases}$$

By invertible point transformation (hodograph) $x=x(u,v), t=t(u,v)$,
UV potential system becomes

$$\mathbf{XT}\{x,t,u,v\}=0: \quad \begin{cases} x_v-t_u=0, \\ x_u-c^2(u)t_v=0 \end{cases}$$

One can show that there are only three more multipliers of form $\Lambda(x,t,u) = xt, x, t$ that yield CLs for **U** for an *arbitrary* wave speed $c(u)$.

This yields three more singlet potential systems given by

$$\mathbf{UA}\{x,t,u,a\} = 0: \quad \begin{cases} a_x - x[tu_t - u] = 0, \\ a_t - t[xc^2(u)u_x - \int c^2(u)du] = 0 \end{cases}$$

$$\mathbf{UB}\{x,t,u,b\} = 0: \quad \begin{cases} b_x - xu_t = 0, \\ b_t - [xc^2(u)u_x - \int c^2(u)du] = 0 \end{cases}$$

$$\mathbf{UW}\{x,t,u,w\} = 0: \quad \begin{cases} w_x - [tu_t - u] = 0, \\ w_t - t\int c^2(u)du = 0 \end{cases}$$

Nonlocally related subsystems arise from **UV** through **XT**:

$$\mathbf{T}\{u,v,t\} \equiv \mathbf{L}\{u,v,t\} = 0: \quad t_{vv} - c^{-2}(u)t_{uu} = 0$$

$$\mathbf{X}\{u,v,x\} = 0: \quad x_{vv} - (c^{-2}(u)x_u)_u = 0$$

One can show that symmetry classifications of these two PDEs are "equivalent". Hence concentrate on **T.**

One can show that *any* solution of **T** yields a multiplier for a CL. These include four multipliers of form

$$\Lambda(u,v,t) = c^2(u),\ uc^2(u),\ vc^2(u),\ uvc^2(u)$$

that yield CLs for **T** for arbitrary wave speed $c(u)$.

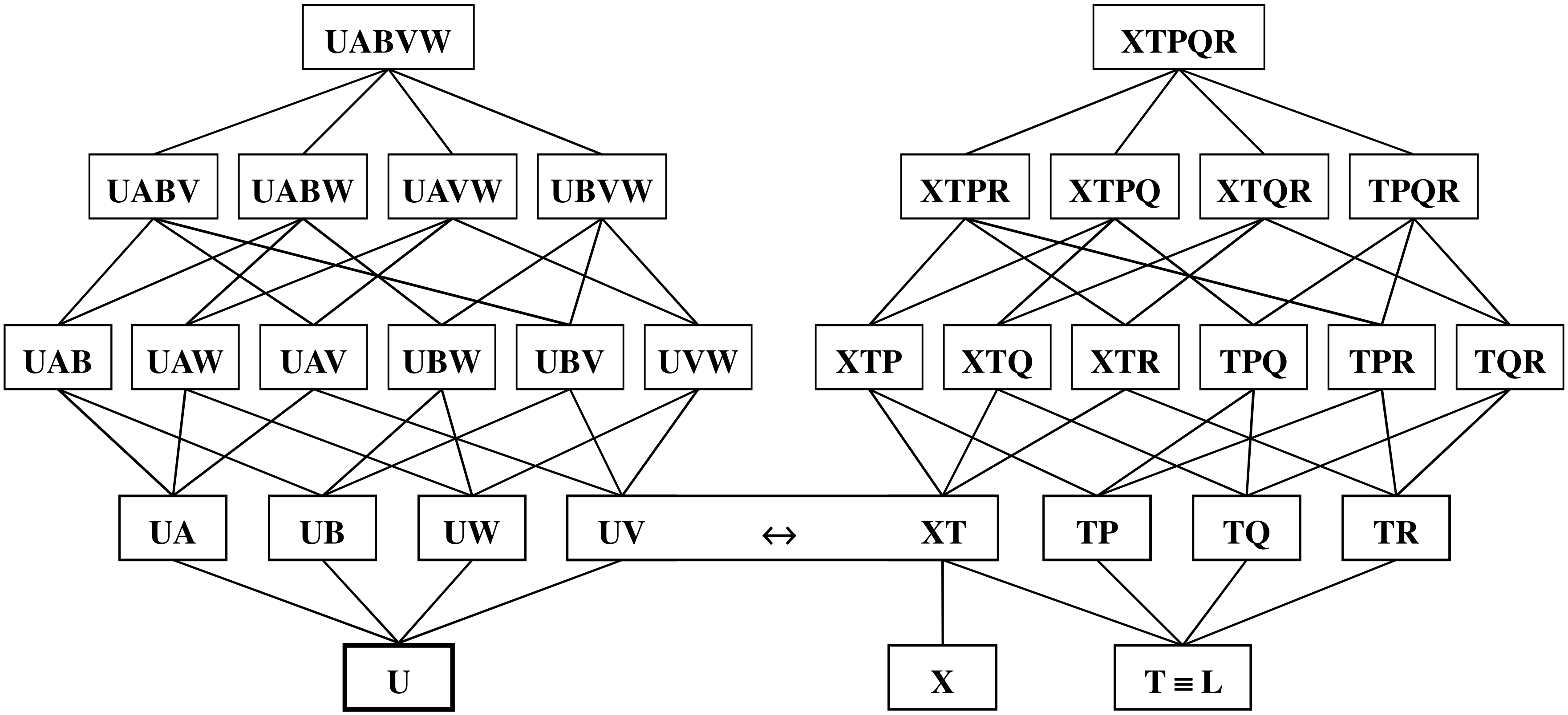
Resulting new singlet potential systems include

$$\mathbf{TP}\{u,v,t,p\}=0: \quad \begin{cases} p_v-(ut_u-t)=0, \\ p_u-uc^2(u)t_v=0 \end{cases}$$

$$\mathbf{TQ}\{u,v,t,q\}=0: \quad \begin{cases} q_v-vt_u=0, \\ q_u+c^2(u)(t-vt_v)=0 \end{cases}$$

$$\mathbf{TR}\{u,v,t,r\}=0: \quad \begin{cases} r_v-v(ut_u-t)=0, \\ r_u-uc^2(u)[vt_v-t]=0 \end{cases}$$

Consequently, one obtains following **(far from exhaustive)** tree of nonlocally related systems for nonlinear wave equation **U** for *arbitrary* wave speed $c(u)$


UABVW
XTPQR
UABV
UABW
UAVW
UBVW
XTPR
XTPQ
XTQR
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UAB
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UBW
UBV
UVW
XTP
XTQ
XTR
TPQ
TPR
TQR
UA
UB
UW
UV ↔ XT
TP
TQ
TR
U
X
T ≡ L

- Point symmetry classification of nonlinear wave equation **U** given in Ames, Lohner & Adams (1981)

- Point symmetry classifications of potential system **XT** and subsystem **T** given in B & Kumei (1987)

- Partial point symmetry classifications of potential systems **TP** and **TQ** can be adapted from results presented in Ma (1990).

- Complete point symmetry classifications of potential systems **UA, UB, UW, TP, TQ** given in B & Cheviakov (2007). Many nonlocal symmetries for nonlinear wave equation are found from each of these nonlocally related systems in terms of specific forms of nonlinear wave speed $c(u)$. In particular, following new nonlocal symmetries for nonlinear wave equation **U** were found:

For potential system **UB**, setting $F(u)=\int c^2(u)\,du$, one finds that if $F(u)$ satisfies ODE

$$\frac{F''(u)}{F'(u)^2}=\frac{4F(u)+2C_1}{(F(u)+C_2)^2+C_3},$$

with arbitrary constants C_1, C_2, C_3, then potential system **UB** has point symmetry

$$\mathrm{X}=(F(u)+C_1)x\frac{\partial}{\partial x}+b\frac{\partial}{\partial t}+\frac{(F(u)+C_2)^2+C_3}{F'(u)}\frac{\partial}{\partial u}$$
$$+(2C_2b-(C_2{}^2+C_3)t)\frac{\partial}{\partial b}$$

that is a nonlocal symmetry of nonlinear wave equation **U**.

For potential system **UW** if $c(u)$ satisfies ODE

$$\frac{c'(u)}{c(u)} = -\frac{2u + C_1}{u^2 + C_2},$$

with arbitrary constants C_1, C_2, then it has point symmetry

$$\mathrm{X} = w\frac{\partial}{\partial x} + (u + C_1)t\frac{\partial}{\partial t} + (u^2 + C_2)\frac{\partial}{\partial u} - C_2 x\frac{\partial}{\partial w},$$

that is a nonlocal symmetry of nonlinear wave equation **U**.

For potential system **TP**, if

$$c(u) = u^{-2} e^{1/u},$$

it has point symmetries

$$X_1 = (pu - 2tv(u+1))\frac{\partial}{\partial t} - 2u^2 v\frac{\partial}{\partial u} + (u^2 + e^{2/u})\frac{\partial}{\partial v} + tu^{-1}e^{2/u}\frac{\partial}{\partial p},$$

$$X_2 = t(u+1))\frac{\partial}{\partial t} + u^2\frac{\partial}{\partial u} - v\frac{\partial}{\partial v},$$

that are both nonlocal symmetries of nonlinear wave equation **U**

For potential system **TR,** new nonlocal symmetries are found for **U** from its point symmetries when

$$c(u) = u^{-4/3}.$$

Table 2
Cases in which nonlocal symmetries of the nonlinear wave equation **U** (1.1) arise

System	Nonlocal variable(s)	Condition on $c(u)$	Symmetries; remarks
UA (2.12)	a	No special cases	Nonlocal symmetries do not arise.
UB (2.13)	b	$c(u)=u^{-2/3}$	Linearizable by a point transformation.
		$\frac{F''(u)}{(F'(u))^2}=\frac{4F(u)+C_1}{(F(u)+C_2)^2+C_3}$ $(F(u)=\int c^2(u)\,du,\ C_1,C_2,C_3=\text{const})$	One nonlocal symmetry.
UW (2.14)	w	$c(u)=u^{-2}$	Linearizable by a point transformation.
		$\frac{c'(u)}{c(u)}=-\frac{2u+C_1}{u^2+C_2}$ $(C_1,C_2=\text{const})$	One nonlocal symmetry.
XT (2.5) or **UV** (2.3)	v	$[\frac{c'(u)}{c^3(u)}(\frac{c(u)}{c'(u)})'']'=0$	One or two nonlocal symmetries; adapted from [4].
TP (2.22)	v,p	$\frac{-(2uc^2+u^2cc')c'''+2u^2c(c'')^2}{c^3(uc'+2c)^2}+\frac{-(4c^2+u^2(c')^2-8ucc')c''+6(c-uc')(c')^2}{c^3(uc'+2c)^2}=\lambda^2,$ $\lambda=\text{const}$	One or two nonlocal symmetries; partially adapted from [6].
		$c(u)=u^{-2}$	Infinite number of nonlocal symmetries; there exists a point mapping into a system with constant coefficients.
TQ (2.23)	v,q	$c(u)=u^{-2/3};\ c(u)=u^{-2}$	Two nonlocal symmetries; partially adapted from [6].
TR (2.24)	v,r	$\frac{ucc''+(c-uc')c'}{(uc'+2c)^2}=\gamma^2=\text{const}$	Two nonlocal symmetries.
L (2.15)	v	$(\alpha'+H\alpha)'=\sigma^2\alpha c^2(u),\ \sigma=\text{const}$ $(H=c'(u)/c(u),\ \alpha^2=(H^2-2H')^{-1})$	One or two nonlocal symmetries; adapted from [4].
		$c(u)=u^{-2}$	Infinite number of nonlocal symmetries; there exists an invertible mapping into a system with constant coefficients [9].
X (2.16)	v	$\frac{(-2cc''+5(c')^2)c^2c''''+3c^3(c''')^2+16c^2(c'')^3}{c^3(2cc''-5(c')^2)^2}+\frac{-24c^2c'c''c'''+12c(c'c'')^2-10(c')^4c''}{c^3(2cc''-5(c')^2)^2}=\sigma^2,$ $\sigma=\text{const}$	One or two nonlocal symmetries; partially adapted from [6].

2. Nonlinear Telegraph Equation

Suppose given PDE is nonlinear telegraph (NLT) equation

$$\mathbf{U}\{x,t,u\}=0: \quad u_{tt}-(F(u)u_x)_x-(G(u))_x=0$$

Case (a) For *arbitrary* $F(u), G(u)$, one obtains two singlet potential systems

$$\mathbf{UV}_1\{x,t,u,v_1\}=0: \quad \begin{cases} v_{1x}-u_t=0, \\ v_{1t}-(F(u)u_x+G(u))=0 \end{cases}$$

$$\mathbf{UV}_2\{x,t,u,v_2\}=0: \quad \begin{cases} v_{2x}-(tu_t-u)=0, \\ v_{2t}-t(F(u)u_x+G(u))=0 \end{cases}$$

Case (b) For *arbitrary* $G(u)$, $F(u)=G'(u)$, one obtains two more singlet potential systems

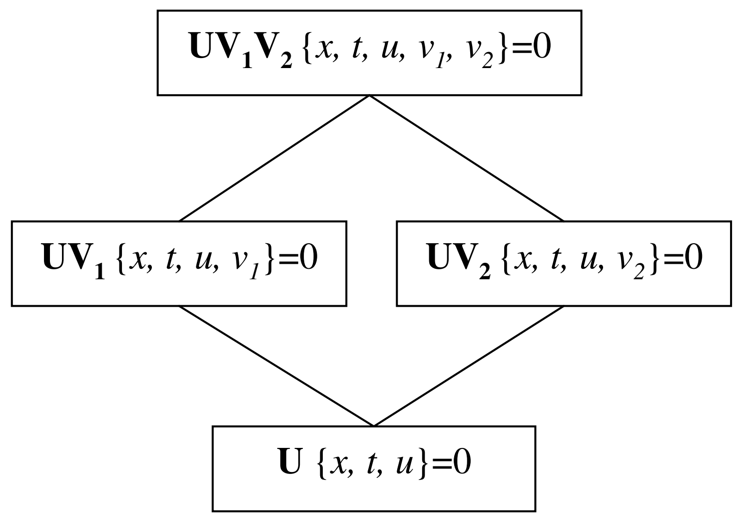
$$\mathbf{UB}_3\{x,t,u,b_3\}=0: \quad \begin{cases} b_{3x}-e^x u_t=0, \\ b_{3t}-e^x F(u)u_x=0 \end{cases}$$

$$\mathbf{UB}_4\{x,t,u,b_4\}=0: \quad \begin{cases} b_{4x}-e^x(tu_t-u)=0, \\ b_{4t}-te^x F(u)u_x=0 \end{cases}$$

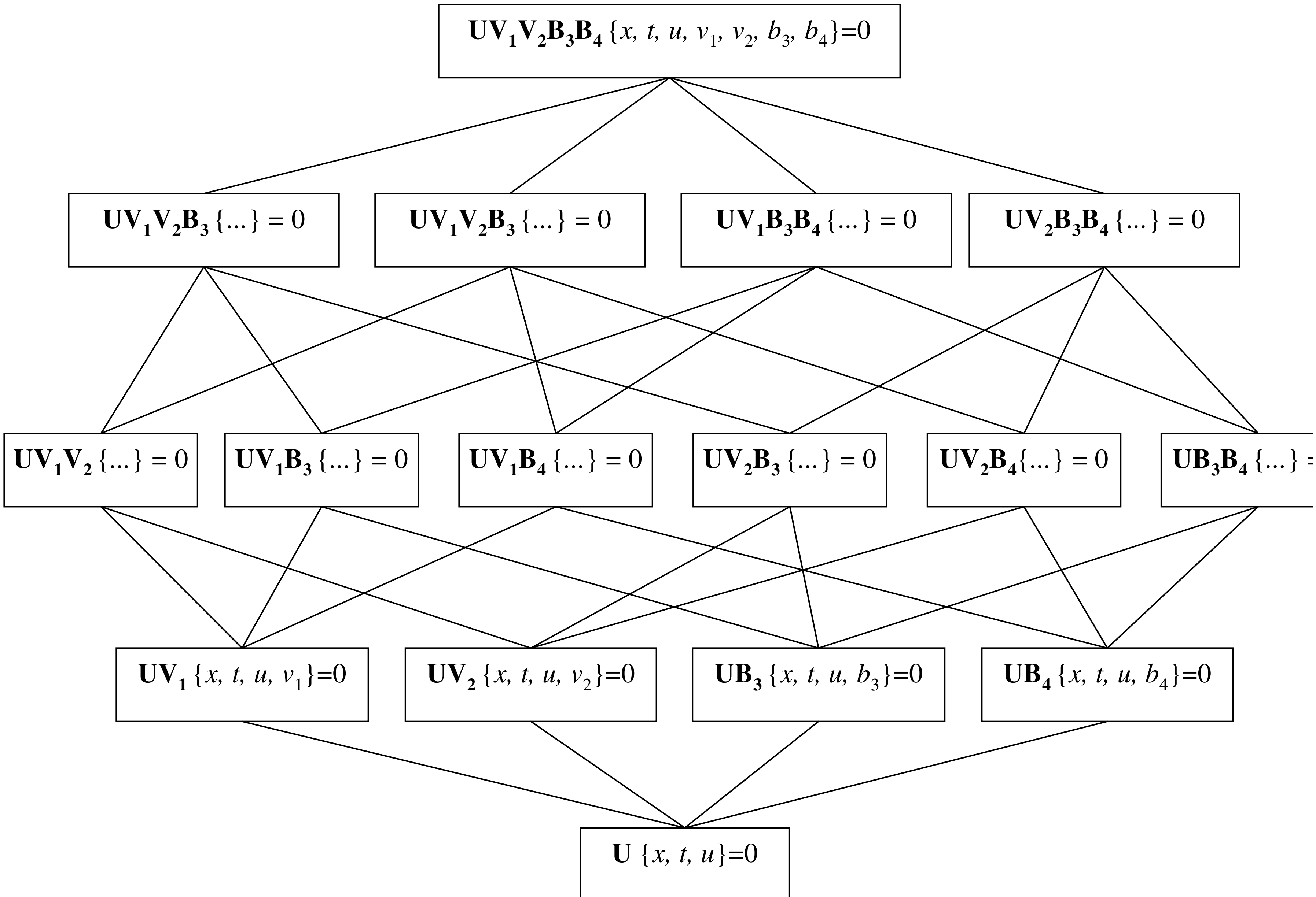
Case (c) $F(u)$ arbitrary, $G(u)=u$: In addition to first two singlet potential systems, there are two more:

$$\mathbf{UC}_3\{x,t,u,c_3\}=0: \quad \begin{cases} c_{3x}-((x-\tfrac{1}{2}t^2)u_t+tu)=0, \\ c_{3t}-(x-\tfrac{1}{2}t^2)(F(u)u_x+u)+\int F(u)du=0 \end{cases}$$

$$\mathbf{UC}_4\{x,t,u,c_4\}=0: \quad \begin{cases} c_{4x}+(\tfrac{1}{6}t^3-tx)u_t+(x-\tfrac{1}{2}t^2)u=0, \\ c_{4t}+(\tfrac{1}{6}t^3-tx)(F(u)u_x+u)+t\int F(u)du=0 \end{cases}$$



Tree of nonlocally related systems for NLT eqn for arbitrary $F(u)$, $G(u)$



Tree of nonlocally related systems for NLT eqn for arbitrary $G(u)$, $F = G'$

TABLE IV. Symmetries of the potential NLT systems for case for case (b): $F(u)=(\alpha+1)u^{\alpha}$, $G(u)=u^{\alpha+1}(\alpha\neq 0,-1)$.

System	$F(u)$	$G(u)$	Symmetries
$UV_1V_2B_3B_4$, **$UV_1V_2B_3$,** **$UV_1V_2B_4$,** **$UV_1B_3B_4$,** **$UV_2B_3B_4$,** **UV_1V_2, UV_1B_3,** **UV_1B_4, UV_2B_3,** **UV_2B_4, UB_3B_4,** **UV_1, UV_2,** **UB_3, UB_4,** **U**	$(\alpha+1)u^{\alpha}$	$u^{\alpha+1}$	$Y_1=-\frac{\alpha}{2}t\frac{\partial}{\partial t}+u\frac{\partial}{\partial u}+v_2\frac{\partial}{\partial v_2}+\frac{\alpha+2}{2}v_1\frac{\partial}{\partial v_1}+\frac{\alpha+2}{2}b_3\frac{\partial}{\partial b_3}+b_4\frac{\partial}{\partial b_4}$, $Y_2=\frac{\partial}{\partial x}+b_3\frac{\partial}{\partial b_3}+b_4\frac{\partial}{\partial b_4}$, $Y_3=\frac{\partial}{\partial t}+b_3\frac{\partial}{\partial b_4}+v_1\frac{\partial}{\partial v_2}$, $Y_4=\frac{\partial}{\partial v_1}$, $Y_5=\frac{\partial}{\partial v_2}$, $Y_6=\frac{\partial}{\partial b_3}$, $Y_7=\frac{\partial}{\partial b_4}$
	$-3u^{-4}$	u^{-3}	$Y_8=t^2\frac{\partial}{\partial t}+tu\frac{\partial}{\partial u}-v_2\frac{\partial}{\partial v_1}-b_4\frac{\partial}{\partial b_3}$
UV_1V_2	$3u^2$	u^3	$Y_9=3v_1\frac{\partial}{\partial x}+(tv_1-v_2+3u)\frac{\partial}{\partial t}-uv_1\frac{\partial}{\partial u}-v_1^2\frac{\partial}{\partial v_1}-v_1v_2\frac{\partial}{\partial v_2}$

TABLE VI. Nonlocal conservation laws of (4.1).

Case	System	Subcase	Multipliers	Fluxes
(a)	$\mathbf{UV}_1$	$\beta=-1$	$\Lambda_1=x+\frac{v_1^2}{2}+\frac{u^{\alpha+2}}{\alpha+2},\Lambda_2=uv_1$	$X=-\left(\frac{u^{\alpha+2}}{\alpha+2}+\frac{v_1^2}{6}+x\right)v_1,$ $T=\left(\frac{u^{\alpha+2}}{(\alpha+2)(\alpha+3)}+\frac{v_1^2}{2}+x\right)u.$
$F(u)=u^\alpha$			$\Lambda_1=v_1,\Lambda_2=u.$	$X=-\frac{u^{\alpha+2}}{\alpha+2}-\frac{v_1^2}{2},$ $T=uv_1-t.$
$G(u)=u^\beta$		$\alpha=-1$ $\beta=-1$	$\Lambda_1=\frac{v_1^3}{3}+2(x+u)v_1+t,$ $\Lambda_2=(v_1^2+u+2x)u.$	$X=-\frac{v_1^4}{12}-(x+u)v_1^2-tv_1-\frac{u^2}{2}-2xu,$ $T=\left(u+\frac{v_1^2}{3}\right)uv_1+2xuv_1+t(u-2x).$
			$\Lambda_1=v_1^4/12+(u+x)v_1^2+tv_1+2xu+x^2+\frac{u^2}{2},$ $\Lambda_2=\left(\frac{v_1^3}{3}+t+uv_1+2xv_1\right)u.$	$X=-\frac{v_1^5}{60}-\frac{(x+u)v_1^3}{3}-\frac{(tv_1+u^2)v_1}{2}-(2u+x)xv_1-tu,$ $T=-\frac{t^2}{2}+\left(\frac{u}{3}+v_1^2+2x\right)\frac{u^2}{2}+\frac{uv_1^4}{12}+(xv_1+t)uv_1+x^2u.$
	$\mathbf{UV}_2$	$\beta=-1$	$\Lambda_1=-\frac{v_2}{t^2},\Lambda_2=\frac{u}{t}.$	$X=-\frac{v_2^2}{2t^2}-\frac{u^{\alpha+2}}{\alpha+2},T=\frac{uv_2-t^2}{t}.$

(b) $F(u)=(\alpha+1)u^{\alpha}$ $G(u)=u^{\alpha+1}$	$\mathbf{UV}_1$	$\alpha\neq-1$ $\alpha\neq-2$	$\Lambda_1=e^{x}u^{\alpha+1},\Lambda_2=e^{x}v_1,$	$X=-e^{x}u^{\alpha+1}v_1,$ $T=e^{x}\left(\frac{u^{\alpha+2}}{\alpha+2}+\frac{v_1^2}{2}\right).$
	$\mathbf{UV}_2$	$\alpha=-4$	$\Lambda_1=-e^{x}\frac{t}{u^3},\Lambda_2=e^{x}v_2.$	$X=e^{x}\frac{tv_2}{u^3},T=e^{x}\left(\frac{t^2}{u^2}-v_2^2\right).$
	$\mathbf{UB}_3$	$\alpha\neq-1$	$\Lambda_1=-u^{\alpha+1},\Lambda_2=e^{-x}b_3.$	$X=-u^{\alpha+1}b_3,T=e^{x}\frac{u^{\alpha+2}}{\alpha+2}+e^{-x}\frac{b_3^2}{2}.$
	$\mathbf{UB}_4$	$\alpha=-4$	$\Lambda_1=-\frac{t}{u^3},\Lambda_2=e^{-x}b_4.$	$X=-\frac{tb_4}{u^3},T=\frac{1}{2}e^{-x}b_4^2-e^{x}\frac{t^2}{2u^2}.$
(c) $F(u)=u^{\alpha}$ $G(u)=u$	$\mathbf{UV}_1$	$\alpha=1$	$\Lambda_1=\frac{t^4}{12}-xt^2+tv_1-\frac{u^2}{2}+x^2,$ $\Lambda_2=-\frac{t^3}{3}+t(u+2x)-v_1.$	$X=\left(\frac{v_1}{2}-xt+\frac{t^3-2tu}{6}\right)u^2$ $-\left(tv_1+\frac{t^4}{12}-xt^2+x^2\right)v_1,$ $T=-\frac{u^3}{6}+\left(\frac{t^4}{12}+x^2-xt^2+tv_1\right)u$ $+\left(2xt-\frac{v}{2}-\frac{t^3}{3}\right)v_1.$
			$\Lambda_1=\frac{t^3}{6}-xt+v_1$ $\Lambda_2=-\frac{t^2}{2}+u+x.$	$X=\left(\frac{t^2}{2}-\frac{u}{3}-x\right)u^2+\left(2xt-\frac{t^3}{3}-\frac{v_1}{2}\right)v_1,$ $T=\left(\frac{t^3}{3}-2xt\right)u+(u+2x-t^2)v_1.$
	$\mathbf{UV}_2$	$\alpha=1$	$\Lambda_1=\frac{t^2}{4}-x+\frac{v_2-x^2}{t^2},$ $\Lambda_2=t-\frac{u+2x}{t}.$	$X=\frac{u^3}{3}+\frac{2x-t^2}{2}u^2+\frac{v_2^2}{2t^2}+\frac{(t^4-4x(t^2+x))v_2}{4t^2},$ $T=-\frac{uv_2}{t}-\frac{(t^4-4x(t^2+x))u}{4t}-\frac{(2x-t^2)v_2}{t}.$
	$\mathbf{UC}_3$	$\alpha=1$	$\Lambda_1=-\frac{t^2-2x}{80}+\frac{2xt^2+5u^2}{40(t^2-2x)}+\frac{4x^3+5tc_3}{10(t^2-2x)^2},$ $\Lambda_2=\frac{3t^5-20c_3}{40(t^2-2x)^2}-\frac{t(2x+u)}{4(t^2-2x)}.$	$X=-\frac{(t^2-2x)(tu^2+2c_3)}{64}+\frac{t(u^3+3tc_3)}{48}+\frac{t^4(tu^2-10c_3)+20u^2c_3}{160(t^2-2x)}+\frac{t(t^5+5c_3)c_3}{40(t^2-2x)^2},$ $T=\frac{(t^4-4x^2)u}{64}+\frac{u^3-3t^4u-6tc_3}{96}+\frac{t(t^5+10c_3)u}{80(t^2-2x)}+\frac{(t^5+5c_3)c_3}{40(t^2-2x)^2}.$

Consider now classification problem for nonlinear telegraph (NLT) equation

$$u_{tt}-(F(u)u_x)_x-(G(u))_x=0. \qquad (1)$$

For **any** $(F(u),G(u))$ pair, one naturally has potential systems

$$\begin{aligned} R_1[u,v]&=v_t-F(u)u_x-G(u)=0,\\ R_2[u,v]&=v_x-u_t=0; \end{aligned} \qquad (2)$$

$$\begin{aligned} H_1[u,v,w]&=R_1[u,v]=0,\\ H_2[u,v,w]&=w_t-v=0,\\ H_3[u,v,w]&=w_x-u=0. \end{aligned} \qquad (3)$$

For **specific** $(F(u),G(u))$ pairs, CL classification problem for (2), etc. can yield additional CLs and hence further potential systems for consideration [B & Temuerchaolu, J. Math. Anal. Appl. **310**, 459 (2005)]

NONLOCAL SYMMETRIES

Potential system (2) has point symmetry

$$X = \xi(x,t,U,V)\frac{\partial}{\partial x} + \tau(x,t,U,V)\frac{\partial}{\partial t} + \eta(x,t,U,V)\frac{\partial}{\partial U} + \phi(x,t,U,V)\frac{\partial}{\partial V}$$

.if and only if

$$\begin{aligned}
&\xi_V - \tau_U = 0,\\
&\eta_U - \phi_V + \xi_x - \tau_t = 0,\\
&G(U)[\eta_V + \tau_x] + \eta_t - \phi_x = 0,\\
&\xi_U - F(U)\tau_V = 0,\\
&\phi_U - G(U)\tau_U - F(U)\eta_V = 0,\\
&G(U)\xi_V + \xi_t - F(U)\tau_x = 0,\\
&F(U)[\phi_V - \tau_t + \xi_x - \eta_U - 2G(U)\tau_V] - F'(U)\eta = 0,\\
&G(U)[\phi_V - \tau_t - G(U)\tau_V] - F(U)\eta_x - G'(U)\eta + \phi_t = 0,
\end{aligned}$$

holds for **arbitrary** values of x,t,U,V

Theorem 1 [B, Temuerchaolu & Sahadevan, JMP **46**, 023505 (2005)] Potential system (2) yields nonlocal symmetry of NLT eqn (1) if and only if

$$(c_3 u + c_4)F'(u) - 2(c_1 - c_2 - G(u))F(u) = 0,$$

$$(c_3 u + c_4)G'(u) + G^2(u) - (c_1 - 2c_2 + c_3)G(u) - c_5 = 0.$$

In linearizable case: $c_1 = 0,\ \ c_5 = c_2(c_3 - c_2).$

For any such pair $(F(u), G(u))$, (u,v) potential system has point symmetry

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \phi\frac{\partial}{\partial v}$$

with

$$
\begin{aligned}
\xi &= c_1 x + \int F(u)\,du, \\
\tau &= c_2 t + v, \\
\eta &= c_3 u + c_4, \\
\phi &= c_5 t + (c_1 - c_2 + c_3)v
\end{aligned}
$$

Modulo translations and scalings in u and G and scalings in F (involving 5/7 parameters), one obtains six distinct classes of ODEs for $(F(u), G(u))$ where scalar (u) eqn (1) has potential symmetry.

Classification Table for Potential Symmetries

relationship	$G(u)$	$F(u)$
$F(u)=\frac{u^{\beta}}{\alpha}G'(u)$	$\frac{u^{2\alpha}-1}{u^{2\alpha}+1}$	$\frac{4u^{2\alpha+\beta-1}}{(u^{2\alpha}+1)^2}$
	$\frac{u^{2\alpha}+1}{u^{2\alpha}-1}$	$-\frac{4u^{2\alpha+\beta-1}}{(u^{2\alpha}-1)^2}$
$F(u)=\frac{u^{\beta}}{\alpha}G'(u)$	$\tan(\alpha\ln u)$	$u^{\beta-1}\sec^2(\alpha\ln u)$
$F(u)=u^{\beta}G'(u)$	$(\ln u)^{-1}$	$-u^{\beta-1}(\ln u)^{-2}$
$F(u)=e^{2\beta u}G'(u)$	$\tan u$	$e^{2\beta u}\sec^2 u$
$F(u)=e^{2\beta u}G'(u)$	$\tanh u$	$e^{2\beta u}\operatorname{sech}^2 u$
	$\coth u$	$-e^{2\beta u}\operatorname{csch}^2 u$
$F(u)=e^{2\beta u}G'(u)$	u^{-1}	$-u^{-2}e^{2\beta u}$

Modulo scalings and translations, two distinct linearization cases occur:

Case 1.

$$v_t - F(u)u_x = 0,$$
$$v_x - u_t = 0$$

admits

$$X = A(u,v)\frac{\partial}{\partial x} + B(u,v)\frac{\partial}{\partial t}$$

with

$$A_u - F(u)B_v = 0,$$
$$A_v - B_u = 0.$$

(hodograph transf)

Case 2.

$$v_t - u^{-2}u_x - u^{-1} = 0, v_x - u_t = 0$$

admits

$$X = -u^{-1}A(\hat{u}, v)\frac{\partial}{\partial x} + B(\hat{u}, v)\frac{\partial}{\partial t} + A(\hat{u}, v)\frac{\partial}{\partial u},$$

$$\hat{u} = x + \log u,$$

with

$$A_v + B_{\hat{u}} = 0,$$
$$A_{\hat{u}} + B_v - A = 0.$$

Point Symmetry Classification of (u) Scalar NLT

Scalar NLT eqn

$$u_{tt} = [F(u)u_x]_x + [G(u)]_x$$

has point symmetry

$$x^* = x + \varepsilon\xi(x,t,u) + O(\varepsilon^2),$$

$$t^* = t + \varepsilon\tau(x,t,u) + O(\varepsilon^2),$$

$$u^* = u + \varepsilon\eta(x,t,u) + O(\varepsilon^2),$$

iff

$$X^{(2)}(u_{tt} - (F(u)u_x)_x - G(u)_x) = 0$$

for *any* soln of scalar eqn where $X^{(2)}$ is second extension of

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}$$

This leads to **determining equations**

$$
\begin{aligned}
&\xi_u = \tau_x = \tau_u = \eta_{uu} = \xi_t = 0, \\
&2F(u)[-\tau_t + \xi_x] - F'(u)\eta = 0, \\
&\eta_{tt} - F(u)\eta_{xx} - G'(u)\eta_x = 0, \\
&2\eta_{tu} - \tau_{tt} = 0, \\
&F(u)[2\eta_{xu} - \xi_{xx}] + \xi_{tt} + 2F'(u)\eta_x - G'(u)[\xi_x - 2\tau_t] + G''(u)\eta = 0,
\end{aligned}
$$

which must hold for *arbitrary* values of x, t, and u.

For arbitrary $(F(u), G(u))$, scalar NLT eqn is invariant under translations in x and t.

Classes of $(F(u), G(u))$ yielding point symmetries of scalar NLT eqn

$G(u)$	$F(u)$	Infinitesimals
e^{u}	$e^{(\alpha+1)u}$	$(\xi,\tau,\eta)=(2\alpha x,[\alpha-1]t,2)$
$u^{\alpha+\beta+1}$	u^{α}	$(\xi,\tau,\eta)=(2\beta x,[\alpha+2\beta]t,-2u)$
u^{-1}	u^{-2}	above $+(\xi,\tau,\eta)=(e^{x},0,-ue^{x})$
$\ln u$	u^{α}	$(\xi,\tau,\eta)=(2[\alpha+1]x,[\alpha+2]t,2u)$
u	$e^{\alpha u}$	$(\xi,\tau,\eta)=(2\alpha x,\alpha t,2)$
u^{-3}	u^{-4}	above $+(\xi,\tau,\eta)=(0,t^{2},tu)$

Theorem 2 Each point symmetry of (u,v) NLT potential system that is a nonlocal symmetry of NLT scalar (u) eqn yields a contact symmetry of NLT (w) potential eqn given by

$$w_{tt} = F(w_x)w_{xx} + G(w_x)$$

Theorem 3 A point symmetry of NLT scalar (u) eqn yields a point symmetry of (u,v) NLT potential system for all cases except when $(F(u), G(u)) = (u^{-4}, u^{-3})$.

CONSERVATION LAWS

$(\xi(x,t,U,V),\phi(x,t,U,V))$ are multipliers for CL of NLT potential system iff

$$E_U\left(\xi R_1[U,V]+\phi R_2[U,V]\right)\equiv 0,$$
$$E_V\left(\xi R_1[U,V]+\phi R_2[U,V]\right)\equiv 0,$$

for **arbitrary** diff. functions $(U(x,t),(V(x,t))$. This yields determining eqns:

$$\begin{aligned}
&\phi_V-\xi_U=0,\\
&\phi_U-F(U)\xi_V=0,\\
&\phi_x-\xi_t-G(U)\xi_V=0,\\
&F(U)\xi_x-\phi_t-[G(U)\xi]_U=0.
\end{aligned}\qquad (4)$$

Then for any solution of (4), conserved densities are

$$X = -\int_a^U \xi(x,t,s,b)ds - \int_b^V \phi(x,t,U,s)ds - G(a)\int^x \xi(s,t,a,b)ds,$$

$$T = \int_a^U \phi(x,t,s,b)ds + \int_b^V \xi(x,t,U,s)ds.$$

Classification results for CLs

Solution of determining system reduces to study of system of two functions

$$d(U) = G'^2 F''' - 3G'G''F'' + [3G''^2 - G'G''']F',$$

$$h(U) = G'^2 G^{(4)} - 4G'G''G''' + 3G''^3$$

Three cases arise:

$$d(U) = h(U) \equiv 0,$$

$$d(U) \not\equiv 0, h(U) \equiv 0,$$

$$d(U) \not\equiv 0, h(U) \not\equiv 0.$$

Case I: $F(u)$ is arbitrary

$F(u)$	$G(u)$	Multipliers
arb	u	$(\xi,\phi)=\left(t, x-\frac{1}{2}t^2\right)$ $(\xi,\phi)=\left(1,-t\right)$
arb	$1/u$	$(\xi,\phi)=\left(U,V\right)$ $(\xi,\phi)=(UV,\frac{1}{2}V^2+x+\int^U sF(s)ds$

Case II: $h(u) \neq 0, d(u) \neq 0$

$\gamma F - G' = \frac{\alpha}{\gamma}(G+\beta)^2$	$(\xi_1, \phi_1) = Ae^{\sqrt{\alpha}(\beta t + V)}(1, \frac{\sqrt{\alpha}}{\gamma}(G+\beta))$ $(\xi_2, \phi_2) = (\xi_1, -\phi_1)(x, -t, U, -V)$ $[A = \exp(\gamma x + \frac{\alpha}{\gamma}\int^{U}(G(s)+\beta)ds]$
$\gamma F - G' = \frac{\alpha}{\gamma}$	$(\xi_1, \phi_1) = e^{\gamma x + \sqrt{\alpha} t}(1, \frac{\sqrt{\alpha}}{\gamma})$ $(\xi_2, \phi_2) = (\xi_1(x, -t), -\phi_1(x, -t))$
$\gamma F = G'$	$(\xi, \phi) = e^{\gamma x}(t, \frac{1}{\gamma})$ $(\xi, \phi) = e^{\gamma x}(V, \frac{1}{\gamma} G(U))$ $(\xi, \phi) = e^{\gamma x}(1, 0)$

Case III: $d(u)=0, h(u)=0$

Using symmetry analysis (substitution + invariance under solvable three-parameter group), ODE $h(u)=0$
can be solved in terms of elementary functions (for $G(u)$).

Then note that $F(u) = G(u)$ + const is a particular soln of resulting linear ODE $d(u) = 0 \Rightarrow$ general soln.

Consequently, for $F(u)=\beta_1 G^2(u)+\beta_2 G(u)+\beta_3,\ {\beta_2}^2 \neq 4\beta_1\beta_3$

there are four highly nontrivial CLs when

$$G(u)=u, \quad 1/u, \quad e^u, \quad \tanh u, \quad \tan u.$$

[In case of "perfect square" ${\beta_2}^2 = 4\beta_1\beta_3$, there are two CLs.]

3. Planar Gas Dynamics (PGD) Equations

Suppose given PDE system is the planar gas dynamics (PGD) equations. In *Eulerian description,* one has **Euler system**

$$\mathbf{E}\{x,t,v,p,\rho\}=0: \quad \begin{cases} \rho_t+(\rho v)_x=0, \\ \rho(v_t+vv_x)+p_x=0, \\ \rho(p_t+vp_x)+B(p,\rho^{-1})v_x=0; \end{cases}$$

in terms of entropy density $S(p,\rho)$, constitutive function $B(p,\rho^{-1})$ is given by

$$B(p,\rho^{-1})=-\rho^2 S_\rho / S_p$$

In *Lagrangian description*, in terms of Lagrange mass coordinates $s = t,\ y = \int_{x_0}^{x} \rho(\xi)d\xi$, one has

Lagrange system

$$\mathbf{L}\{y, s, v, p, q\} = 0: \quad \begin{cases} q_s - v_y = 0, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0 \end{cases}$$

We now show that potential system framework yields a direct connection between Euler and Lagrange systems. As well, we derive other equivalent descriptions!

Use Euler system as given system. First equation is a CL and through it, introduce potential r and obtain potential system

$$\mathbf{G}\{x,t,v,p,\rho,r\}=0: \quad \begin{cases} r_x-\rho=0, \\ r_t+\rho v=0, \\ \rho(v_t+vv_x)+p_x=0, \\ \rho(p_t+vp_x)+B(p,\rho^{-1})v_x=0 \end{cases}$$

In order to obtain nonlocally-related subsystem, first consider interchange of dependent and independent variables in $\mathbf{G}$ with $r = y$ and $t = s$ as independent variables; x, v, p, ρ as dependent variables and let $q = 1/\rho$ to obtain 1:1 equivalent system

$$\mathbf{G}_0\{y,s,x,v,p,\rho\}=0: \quad \begin{cases} x_y-q=0, \\ x_s-v=0, \\ v_s+p_y=0, \\ p_s+B(p,q)v_y=0 \end{cases}$$

Nonlocally related subsystem of $\mathbf{G_0}$ is obtained by excluding x through
$x_{ys} = x_{sy}$ to obtain **Lagrange system** $\mathbf{L}\{y, s, v, p, q\} = 0$

Second conservation law of Euler system obtained with multipliers

$$(\Lambda_1, \Lambda_2, \Lambda_3) = (v,1,0)$$

yields second potential variable w. Couplet potential system containing both variables r and w is given by

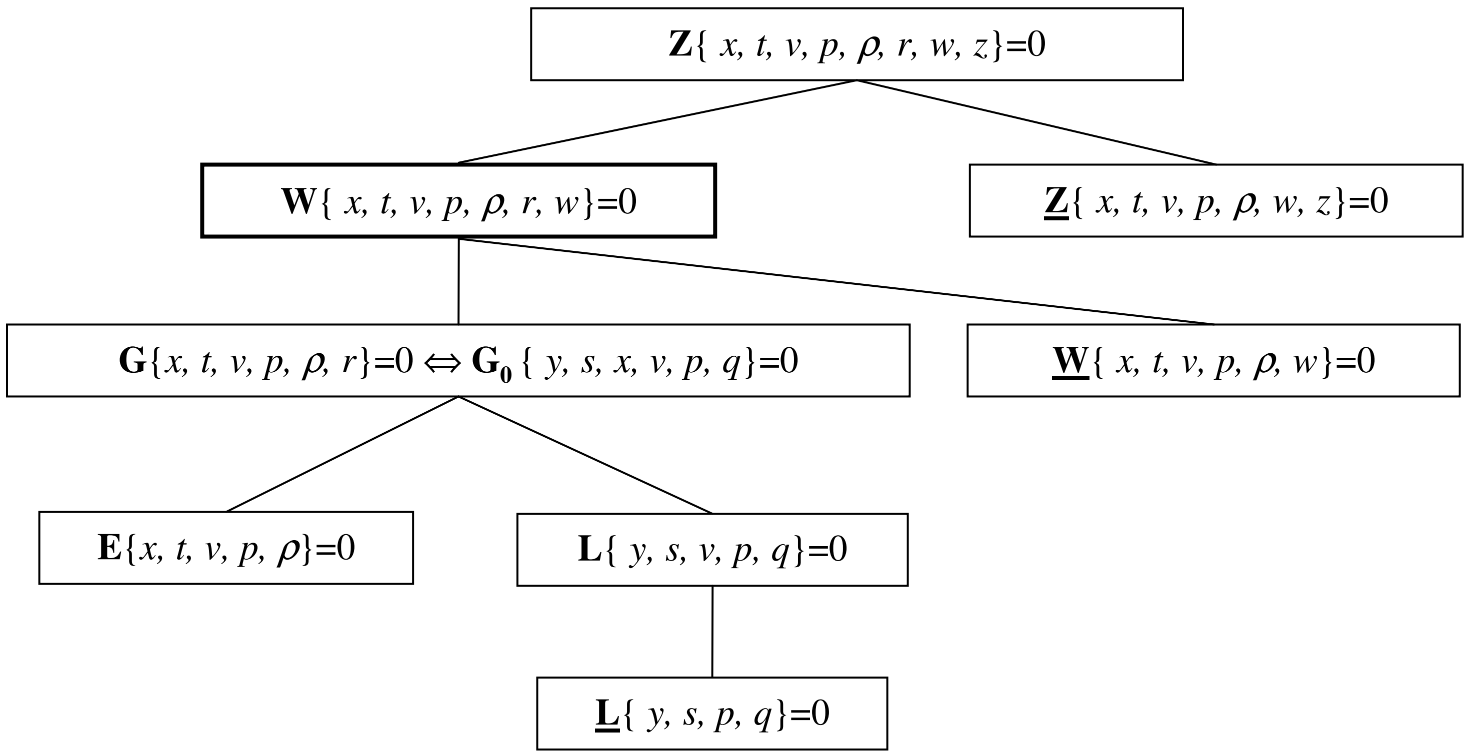
$$\mathbf{W}\{x, t, v, p, \rho, r, w\} = 0: \quad \begin{cases} r_x - \rho = 0, \\ r_t + \rho v = 0, \\ w_x + r_t = 0, \\ w_t + p + v w_x = 0, \\ \rho(p_t + v p_x) + B(p, \rho^{-1}) v_x = 0 \end{cases}$$

From third equation of **W**, introduce third potential variable z to obtain potential system

$$\mathbf{Z}\{x,t,v,p,\rho,r,w,z\}=0: \quad \begin{cases} r_x-\rho=0, \\ r_t+\rho v=0, \\ z_t-w=0, \\ z_x+r=0, \\ w_t+p+vw_x=0, \\ \rho(p_t+vp_x)+B(p,\rho^{-1})v_x=0 \end{cases}$$

Lagrangian system **L**, has nonlocally related subsystem

$$\underline{\mathbf{L}}\{y,s,p,q\}=0: \quad \begin{cases} q_{ss}+p_{yy}=0, \\ p_s+B(p,q)q_s=0 \end{cases}$$



Tree of nonlocally related systems for PGD equations

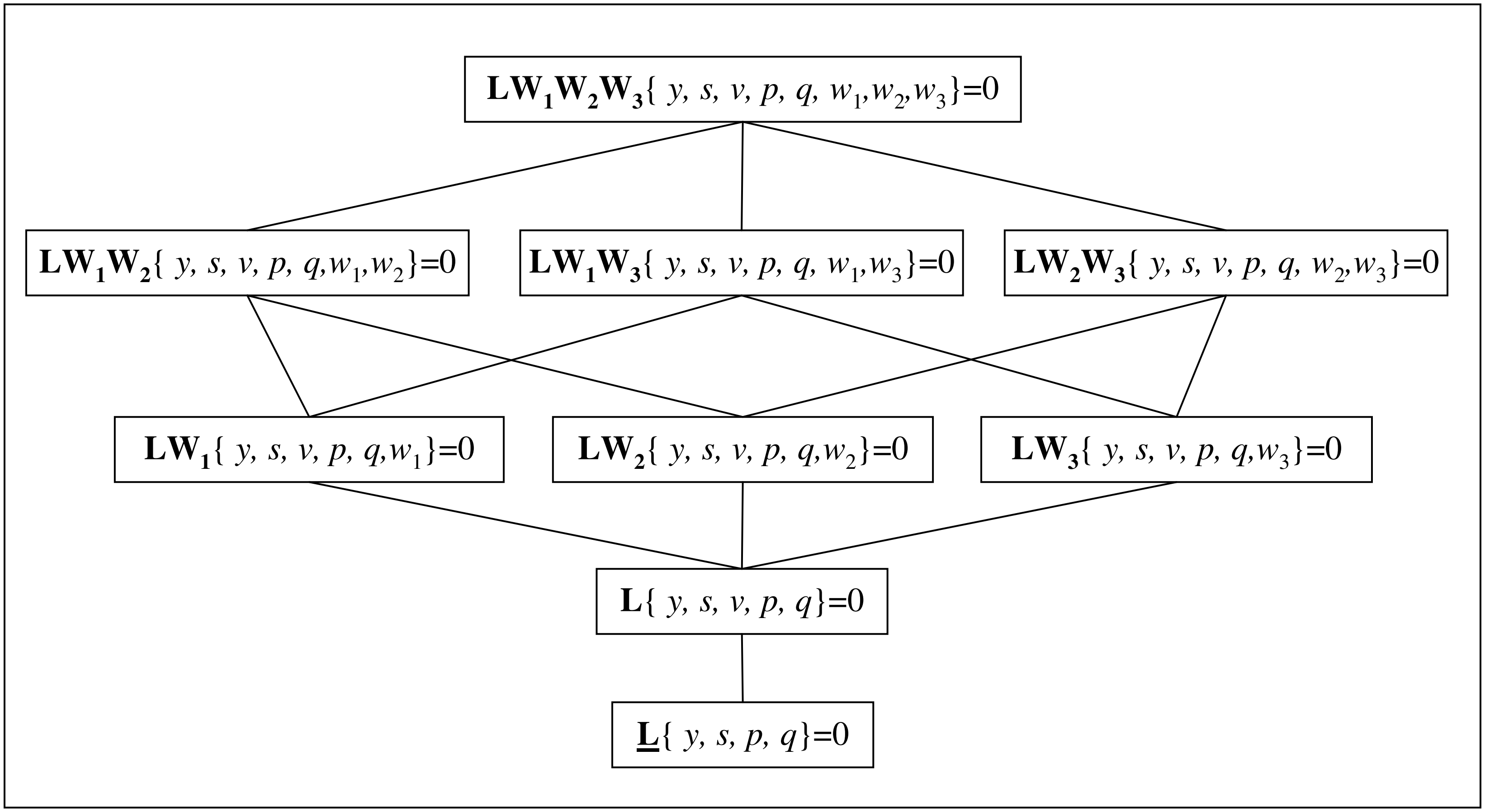
Starting from Lagrange system $\mathbf{L}$, one can obtain three singlet potential systems from three sets of multipliers

$$(\mu_1(y,s),\mu_2(y,s),\mu_3(y,s)) = (1,0,0),(0,1,0),(y,s,0)$$

$$\mathbf{G}=0 \Leftrightarrow \mathbf{G}_0 = \mathbf{LW}_1\{y,s,v,p,q,w_1\}=0: \begin{cases} w_{1_y} - q = 0, \\ w_{1_s} - v = 0, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0 \end{cases}$$

$$\mathbf{LW}_2\{y,s,v,p,q,w_2\}=0: \begin{cases} q_s - v_y = 0, \\ w_{2_y} - v = 0, \\ w_{2_s} + p = 0, \\ p_s + B(p,q)v_y = 0 \end{cases}$$

$$\mathbf{LW}_3\{y,s,v,p,q,w_3\}=0: \begin{cases} w_{3_y} - sv - yq = 0, \\ w_{3_s} + sp - yv = 0, \\ v_s + p_y = 0, \\ p_s + B(p,q)v_y = 0 \end{cases}$$



Extension of tree of nonlocally related systems for Lagrange system for PGD eqns

Two more CLs arise for Lagrange system **L**, when one considers multipliers of form

$$\mu_i(y,s,V,P,Q),\ i=1,2,3.$$

In general, one can show that

$$\begin{aligned}\mu_1 &= \alpha y - \beta P + B(P,Q)\mu_3 + \delta,\\ \mu_2 &= \alpha s + \beta V + \nu,\\ \mu_3 &= \mu_3(y,P,Q),\end{aligned}$$

where $\alpha, \beta, \nu, \delta$ are abitrary constants and $\mu_3(y,P,Q)$ is any solution of PDE

$$\frac{\partial \mu_3}{\partial Q} - \frac{\partial}{\partial P}(B(P,Q)\mu_3) + \beta = 0$$

The two extra CLs arise (for *arbitrary* constitutive function $B(p,q)$)

(1) from conservation of energy

$$\frac{\partial}{\partial s}(\tfrac{1}{2}v^2 + K(p,q)) + \frac{\partial}{\partial y}(pv) = 0$$

where $K(p,q)$ is a solution of eqn

$$K_q - B(p,q)K_p + p = 0$$

(2) from conservation of entropy

$$\frac{\partial}{\partial s}S(p,q) = 0$$

where $S(p,q)$ is a solution of eqn

$$S_q - B(p,q)S_p = 0$$

For multipliers restricted to dependence on independent variables (y,s), no further potential systems (just the first three) arise in case of Lagrange PGD system $\mathbf{L}$ with generalized polytropic equation of state

$$B(p,q)=\frac{M(p)}{q}, \quad M''(p)\neq 0$$

TABLE VII. Symmetries of the generalized polytropic PGD system (2.10), (5.1).

System	$M(p)$	Symmetries
L, **LW**$_1$,**LW**$_2$,**LW**$_3$, **LW**$_1$**W**$_2$,**LW**$_1$**W**$_3$, **LW**$_2$**W**$_3$, **LW**$_1$**W**$_2$**W**$_3$	(i) Arbitrary	$Z_1 = \frac{\partial}{\partial s} + w_2 \frac{\partial}{\partial w_3}$, $Z_2 = \frac{\partial}{\partial y} + w_1 \frac{\partial}{\partial w_3}$, $Z_3 = \frac{\partial}{\partial v} + s \frac{\partial}{\partial w_1} + y \frac{\partial}{\partial w_2} + sy \frac{\partial}{\partial w_3}$, $Z_4 = -y \frac{\partial}{\partial y} + 2q \frac{\partial}{\partial q} + v \frac{\partial}{\partial v} + w_1 \frac{\partial}{\partial w_1}$, $Z_5 = s \frac{\partial}{\partial s} + y \frac{\partial}{\partial y} + w_1 \frac{\partial}{\partial w_1} + w_2 \frac{\partial}{\partial w_2} + 2w_3 \frac{\partial}{\partial w_3}$, $Z_6 = \frac{\partial}{\partial w_1}$, $Z_7 = \frac{\partial}{\partial w_2}$, $Z_8 = \frac{\partial}{\partial w_3}$,
L, **LW**$_2$	(ii) $-p \ln p$	$Z_9 = y \frac{\partial}{\partial y} + 2p \frac{\partial}{\partial p} + \frac{2q}{\ln p} \frac{\partial}{\partial q} + v \frac{\partial}{\partial v} + 2w_2 \frac{\partial}{\partial w_2}$.
	(iii) $\gamma p + \alpha p^{(\gamma+1)/\gamma}$ $\gamma \neq 0, -1$	$Z_{10} = \frac{(\gamma+1)y}{2\gamma} \frac{\partial}{\partial y} + p \frac{\partial}{\partial p} - \frac{q}{\delta p^{1/\gamma} + \gamma} \frac{\partial}{\partial q} + \frac{(\gamma-1)v}{2\gamma} \frac{\partial}{\partial v} + w_2 \frac{\partial}{\partial w_2}$.
	(iv) $1 + \alpha e^p$, $\alpha = \pm 1$	$Z_{11} = \frac{\partial}{\partial p} + \frac{\alpha e^p}{1+\alpha e^p} q \frac{\partial}{\partial q} - s \frac{\partial}{\partial w_2}$, $Z_{12} = y \frac{\partial}{\partial p} + \frac{\alpha e^p}{1+\alpha e^p} yq \frac{\partial}{\partial q} - s \frac{\partial}{\partial v} - sy \frac{\partial}{\partial w_2}$.
LW$_2$	(ii) $-p \ln p$	$Z_{13} = y^2 \frac{\partial}{\partial y} + yp \frac{\partial}{\partial p} - \left(3 - \frac{1}{\ln p}\right) yq \frac{\partial}{\partial q} - (yv - w_2) \frac{\partial}{\partial v} + yw_2 \frac{\partial}{\partial w_2}$.
	(iii) $\gamma p + \delta p^{(\gamma+1)/\gamma}$ $\gamma \neq 0, -1$	$Z_{14} = y^2 \frac{\partial}{\partial y} + yp \frac{\partial}{\partial p} - \left(3 - \frac{\delta}{\gamma} \frac{p^{1/\gamma}}{\delta p^{1/\gamma} + \gamma}\right) yq \frac{\partial}{\partial q} - (yu - w_2) \frac{\partial}{\partial v} + yw_2 \frac{\partial}{\partial w_2}$.

TABLE VIII. Point symmetries of the subsystem $\underline{\mathrm{L}}$ (2.19) of the generalized polytropic PGD system (2.10), (5.1).

$M(p)$	Symmetries
(i) Arbitrary	$\underline{Z_1}=\frac{\partial}{\partial s},\ \underline{Z_2}=\frac{\partial}{\partial y},$ $\underline{Z_4}=-y\frac{\partial}{\partial y}+2q\frac{\partial}{\partial q},\ \underline{Z_5}=s\frac{\partial}{\partial s}+y\frac{\partial}{\partial y}.$
(ii) $-p\ln p$	$\underline{Z_9}=y\frac{\partial}{\partial y}+2p\frac{\partial}{\partial p}+\frac{2q}{\ln p}\frac{\partial}{\partial q},$ $\underline{Z_{13}}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\big(3-\frac{1}{\ln p}\big)yq\frac{\partial}{\partial q}.$
(iii) $\gamma p+\delta p^{(\gamma+1)/\gamma}$	$\underline{Z_{10}}=\frac{(\gamma+1)y}{2\gamma}\frac{\partial}{\partial y}+p\frac{\partial}{\partial p}-\frac{q}{\delta p^{1/\gamma}+\gamma}\frac{\partial}{\partial q},$ $\underline{Z_{14}}=y^2\frac{\partial}{\partial y}+yp\frac{\partial}{\partial p}-\big(3-\frac{\alpha}{\gamma}\frac{p^{1/\gamma}}{\delta p^{1/\gamma}+\gamma}\big)yq\frac{\partial}{\partial q}.$
$\gamma=3$	$\underline{Z_{15}}=\frac{1}{3}s^2\frac{\partial}{\partial s}-sp\frac{\partial}{\partial p}+\frac{1}{\delta p^{4/3}+3}spq\frac{\partial}{\partial q}.$
(iv) $1+\alpha e^p$	$\underline{Z_{11}}=\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}q\frac{\partial}{\partial q},$ $\underline{Z_{12}}=y\frac{\partial}{\partial p}+\frac{\alpha e^p}{1+\alpha e^p}yq\frac{\partial}{\partial q}.$

Remarks

- Extended trees hold for *arbitrary* constitutive function
- Either Euler system or Lagrange system can be given system—tree will not change
- In Akhatov, Gazizov & Ibragimov (1991) a complete group classification with respect to constitutive function was given separately for Euler and Lagrange systems but connections between systems were heuristic
- To systematically construct nonlocal symmetries of Euler and Lagrange systems one needs to do group classification problem for *all* systems in extended tree with respect to constitutive function (as well as consider other possible extended trees for specific constitutive functions followed by point symmetry analysis)

- E.g., for Chaplygin gas $[B(p,q] = -p/q]$, subsystem

$$\underline{\mathbf{L}}\{y, s, p, q\} = 0$$

has point symmetry (not in AGI)

$$\mathrm{X} = -y^2 \frac{\partial}{\partial y} - py \frac{\partial}{\partial p} + 3yq \frac{\partial}{\partial q},$$

which yields nonlocal symmetry for **E** and **L.**

Further extended trees for PGD eqns for specific constitutive functions

Example A: For $B(p,1/\rho)=\rho(1+e^{p})$, system $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ has family of CLs:

$$D_t\left(\frac{f(r)e^{p}}{1+e^{p}}\right)+D_x\left(\frac{f(r)ve^{p}}{1+e^{p}}\right)=0,$$

for arbitrary *f*(*r*). Such a CL can be used to replace 4th eqn of $\mathbf{G}\{x,t,v,p,\rho,r\}=0$ to introduce potential c and potential system

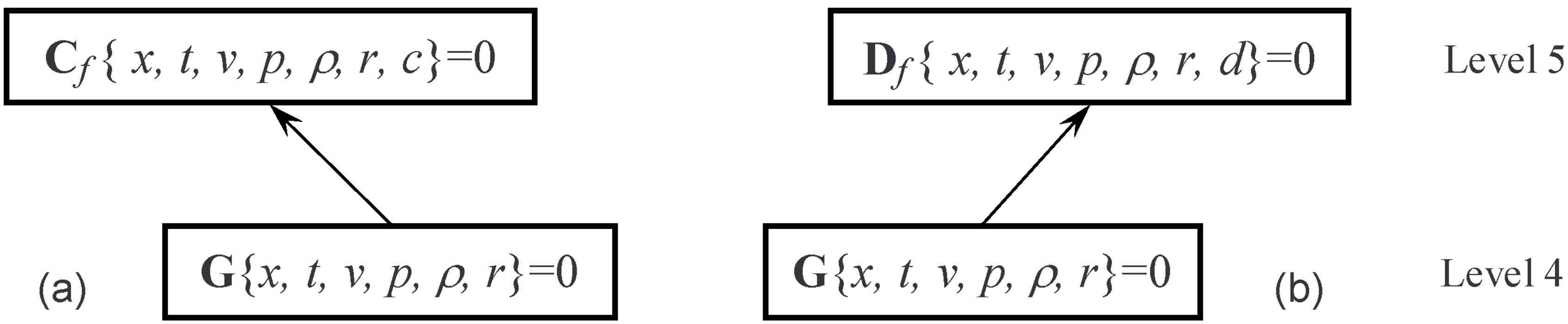
$$\mathbf{C}_f\{x,t,v,p,\rho,r,c\}=0:\begin{cases} r_x-\rho=0,\\ r_t+\rho v=0,\\ r_x(v_t+vv_x)+p_x=0,\\ c_x+e^{p}f(r)/(1+e^{p})=0,\\ c_t-ve^{p}f(r)/(1+e^{p})=0.\end{cases}$$

Example B: For Chaplygin gas, $B(p,1/\rho) = -p\rho$, system $\mathbf{G}\{x,t,v,p,\rho,r\} = 0$ has family of CLs**:**

$$D_t\left(\frac{f(r)}{p}\right) + D_x\left(\frac{f(r)v}{p}\right) = 0$$

for arbitrary $f(r)$ to yield family of potential systems

$$\mathbf{D}_f\{x,t,v,p,\rho,r,d\} = 0 : \begin{cases} r_x - \rho = 0, \\ r_t + \rho v = 0, \\ r_x(v_t + vv_x) + p_x = 0, \\ d_x + f(r)/p = 0, \\ d_t - vf(r)/p = 0. \end{cases}$$


$\mathbf{C}_f\{x, t, v, p, \rho, r, c\}=0$
$\mathbf{D}_f\{x, t, v, p, \rho, r, d\}=0$
Level 5
(a)
$\mathbf{G}\{x, t, v, p, \rho, r\}=0$
$\mathbf{G}\{x, t, v, p, \rho, r\}=0$
(b)
Level 4

One can show that new nonlocal symmetries arise for Chaplygin gas Euler system only when $f(r) = r, f(r) = \text{const.}$
For $f(r) = r,$ this Chaplygin gas system has symmetries

$$X_{\mathbf{D}_1} = \left(-\frac{t^3}{6} + dt\right)\frac{\partial}{\partial x} + \left(d - \frac{t^2}{2}\right)\frac{\partial}{\partial v} + rt\frac{\partial}{\partial p} - \frac{rt\rho}{p}\frac{\partial}{\partial p},$$

$$X_{\mathbf{D}_2} = \left(-\frac{t^2}{2} + d\right)\frac{\partial}{\partial x} + -t\frac{\partial}{\partial v} + r\frac{\partial}{\partial p} - \frac{r\rho}{p}\frac{\partial}{\partial p}.$$

- Symmetry $X_{\mathbf{D}_1}$ is nonlocal for both Euler and Lagrange systems
- Symmetry $X_{\mathbf{D}_2}$ is nonlocal for Euler system but local for Lagrange system
- Hence in AGI, symmetry $X_{\mathbf{D}_1}$ was not obtained

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Construction of Potential Systems

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