SYMMETRY-BASED METHOD FOR OBTAINING NONLOCALLY RELATED SYSTEMS AND NONLOCAL SYMMETRIES

Consider the nonlinear reaction diffusion equation

$$u_t - u_{xx} = Q(u) \qquad (1)$$

One can show that for any nonlinear reaction term $Q(u)$, eqn (1) has no local CLs.

Note that (1) is invariant under translations in x and t.

Consider invariance under translations in x. After an interchange of the variables x and u, eqn (1) becomes the invertibly equivalent PDE

$$x_t = \frac{x_{uu} - Q(u)x_u^3}{x_u^2} \quad (2)$$

Now consider the ***intermediate PDE system***

$$
\begin{aligned}
v &= x_u, \\
w &= x_t, \qquad\qquad (3) \\
w &= \frac{v_u - Q(u)v^3}{v^2}.
\end{aligned}
$$

The intermediate PDE system (3) is locally related to PDE (1). Now consider the subsystem (***inverse potential system***) of (3) obtained by excluding x, i.e.,

$$
\begin{aligned}
v_t &= w_u, \\
w &= \frac{v_u - Q(u)v^3}{v^2}
\end{aligned} \qquad (4)
$$

The intermediate system (3) (and hence the given PDE (1)) is nonlocally related to the inverse potential system (4) since the intermediate system (3) is the potential system of the PDE system (4) with the potential variable x arising from the first eqn of (4) which is a CL.

Hence from the above, we see that any point symmetry of a given PDE system naturally yields a nonlocally related system.

Moreover, excluding w from the inverse potential system (4), one obtains the scalar PDE

$$v_t = \left(\frac{v_u - Q(u)v^3}{v^2} \right)_u \qquad (5)$$

which is clearly nonlocally related to the given PDE (1) since (1) has no local CLs!

Construction of a nonlocally related system from a point symmetry

Consider PDE system

$$R^{\sigma}(x,t,u,\partial u,\ldots,\partial^{l}u)=0,\ \sigma=1,\ldots,s, \quad (6)$$

where $u=(u^{1},\ldots,u^{m})$.

Suppose PDE system (6) has point symmetry

$$\mathrm{X}=\xi(x,t,u)\frac{\partial}{\partial x}+\tau(x,t,u)\frac{\partial}{\partial t}+\eta^{i}(x,t,u)\frac{\partial}{\partial u^{i}}$$

Let $X(x,t,u), T(x,t,u), U^1(x,t,u), \ldots, U^m(x,t,u)$

be corresponding canonical coordinates so that point symmetry X becomes $\mathrm{Y} = \dfrac{\partial}{\partial U^1}$.

Then PDE system (6) becomes PDE system invariant under translations in U^1:

$$\hat{R}^{\sigma}(X, T, \hat{U}, \partial U, \ldots, \partial^l U) = 0, \sigma = 1, \ldots, s, \quad (7)$$

where $\hat{U} = (U^2, \ldots, U^m); U = (U^1, \ldots, U^m)$

Now consider the ***intermediate PDE system***

$$\begin{aligned}
&\alpha = U_T^1, \\
&\beta = U_X^1, \qquad\qquad (8) \\
&\tilde{R}^\sigma(X, T, \hat{U}, \alpha, \beta, \partial\hat{U}, \ldots, \partial^{l-1}\alpha, \partial^{l-1}\beta, \partial^l\hat{U}) = 0,
\end{aligned}$$

where $\tilde{R}^\sigma(X, T, \hat{U}, \alpha, \beta, \partial\hat{U}, \ldots, \partial^{l-1}\alpha, \partial^{l-1}\beta, \partial^l\hat{U})$ is obtained from $\hat{R}^\sigma(X, T, \hat{U}, \partial U, \ldots, \partial^l U)$ after making the appropriate substitutions, $\sigma = 1, \ldots, s$.

By construction the intermediate system (8) is locally equivalent to the given PDE system (6).

Excluding dependent variable U^1 from intermediate system (8), one obtains the equivalent **inverse potential system**

$$\begin{aligned} &\alpha_X = \beta_T, \\ &\tilde{R}^\sigma(X, T, \hat{U}, \alpha, \beta, \partial\hat{U}, \ldots, \partial^{l-1}\alpha, \partial^{l-1}\beta, \partial^l\hat{U}) = 0 \end{aligned} \qquad (9)$$

The inverse potential system (9) is nonlocally related to the given PDE system (6) since the intermediate system (8) is the potential system for the inverse potential system (9) with potential variable U^1 arising from the displayed CL of (9).

The following theorem has been proved.

Theorem: Any point symmetry of a given PDE system (6) yields an equivalent nonlocally related PDE system (inverse potential system) given by PDE system (9).

This theorem can be extended to the situation of three or more independent variables. Here the resulting inverse potential system has curl-type CLs.

The special situation when the given PDE is an evolutionary scalar PDE

Theorem: **Suppose a given PDE is an evolutionary scalar PDE invariant under a point symmetry. W.l.o.g., the given PDE is of the form**

$$u_t = F(x, t, u_1, \ldots, u_l), \quad (10)$$

where $u_i = \frac{\partial^i u}{\partial x^i}$. **Let** $\beta = u_x$. **Then the scalar PDE**

$$\beta_t = D_x F(x, t, \beta, \ldots, \beta_{l-1}) \quad (11)$$

is a locally related subsystem of the corresponding inverse potential system resulting from the invariance of PDE (10) under translations in u.

Example: nonlinear wave equation

Consider PDE

$$u_{tt} = (c^2(u)u_x)_x \qquad (12)$$

and its corresponding potential system

$$\begin{aligned} v_x &= u_t, \\ v_t &= c^2(u)u_x. \end{aligned} \qquad (13)$$

The invariance of (13) under translations in t and v shows that (13) is invariant under a point symmetry with infinitesimal generator

$$\mathrm{X} = \frac{\partial}{\partial v} - \frac{\partial}{\partial t}.$$

Corresponding canonical coordinates are

$$X = x, T = u, U = t + v, V = v \tag{14}$$

with the potential system (13) invariant under translations in U. The point transformation (14) maps the potential system into the invertibly related PDE system

$$\begin{aligned} &V_X U_T - V_T U_X - 1 = 0, \\ &V_T + c^2(T)(U_X - V_X) = 0. \end{aligned} \tag{15}$$

Excluding dependent variable V from (15) yields the subsystem (a scalar PDE)

$$\begin{aligned} &U_{TT} + c^4(T)U_{XX} \\ &+ c^2(T)[2U_{TX}U_T U_X - U_{XX}U_T^2 - U_{TT}U_X^2 - 2U_{TX}] \\ &+ 2c(T)c'(T)[U_X^2 U_T - U_X] = 0. \end{aligned} \tag{16}$$

When $c(u) = u^{-2}$, PDE (16) has the point symmetry

$$U^2 \frac{\partial}{\partial U} + TU \frac{\partial}{\partial T} - \frac{U}{T^3} \frac{\partial}{\partial X}$$

which yields the previously unknown nonlocal symmetry of both (12) and (13) given partially by

$$u(t+v) \frac{\partial}{\partial u} - \frac{t+v}{u^3} \frac{\partial}{\partial x}$$

in terms of the (x, u) components of the admitted symmetry (it is nonlocal in at least one of the t- and v- components)

REFERENCES

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