Some Applications of Rodrigues' Vectors In Hyperbolic Geometry and Relativity

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Quaternion Construction

We represent $SU(2) \cong \mathbb{S}^3$ as the space of unit quaternions

$$\mathbb{S}^3 = \{\zeta = z + wj\,,\ z, w \in \mathbb{C},\ j^2 = -1,\ |\zeta|^2 = 1\}$$

that is the spin cover of the rotation group

$$0 \to \mathbb{Z}_2 \to SU(2) \to SO(3) \to 0$$

Vectors in $\mathbb{R}^3$ are mapped in $\mathfrak{su}(2)$ via expansion over basis

$$\mathbf{e}_1 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

and the metric structure is preserved

$$\mathbf{x} \to \mathbf{X} = x_1\,\mathbf{i} + x_2\,\mathbf{j} + x_3\,\mathbf{k}, \qquad (\mathbf{x}, \mathbf{x}) = \det \mathbf{X}$$

The Spin Cover

Elements of SU(2) expand in the same basis with the identity added.

$$\mathbb{S}^3 \ni \zeta = (\zeta_0, \boldsymbol{\zeta}) = \zeta_0 + \zeta_1\,\mathbf{i} + \zeta_2\,\mathbf{j} + \zeta_3\,\mathbf{k}, \qquad |\,\zeta\,|^2 = \det(\zeta) = 1$$

Then the adjoint action of $\mathbb{S}^3$ in its algebra $\mathbb{R}^3$

$$\mathrm{Ad}_{\zeta} : \ \mathbf{X} \longrightarrow \zeta\,\mathbf{X}\,\zeta^{-1}, \qquad \zeta^{-1} = \frac{\bar{\zeta}}{|\zeta|}$$

preserves the metric and thus projects to action of SO(3)

$$\mathcal{R}(\zeta) = (\zeta_0^2 - \boldsymbol{\zeta}^2)\mathcal{I} + 2\boldsymbol{\zeta}\otimes\boldsymbol{\zeta}^t + 2\zeta_0\boldsymbol{\zeta}^{\times}.$$

that is the famous *Rodrigues'* formula

$$\mathcal{R}(\mathbf{n},\varphi) = \cos\varphi\,\mathcal{I} + (1-\cos\varphi)\,\hat{\mathbf{n}}\otimes\hat{\mathbf{n}}^t + \sin\varphi\,\hat{\mathbf{n}}^{\times} \tag{1}$$

with the substitution

$$\zeta_0 = \cos\frac{\varphi}{2}, \qquad \boldsymbol{\zeta} = \sin\frac{\varphi}{2}\hat{\mathbf{n}}$$

The Vector Parameter

Projecting to the $\zeta_0 = 1$ hyperplane we obtain a $\mathbb{RP}^3$ valued parameter

$$\mathbf{c} = \frac{\zeta}{\zeta_0} = \tau\hat{\mathbf{n}}, \qquad \tau = \tan\frac{\varphi}{2} \tag{2}$$

known as *Rodrigues'* or *Gibbs*' vector.
With its help the transformation matrix can me written as

$$\mathcal{R}(\mathbf{c}) = \frac{(1-\mathbf{c}^2)\mathcal{I} + 2\,\mathbf{c}\otimes\mathbf{c}^t + 2\,\mathbf{c}^\times}{1+\mathbf{c}^2}. \tag{3}$$

Some advantages of this formalism:

1. topologically correct parametrization of $SO(3) \cong \mathbb{RP}^3$
2. carries information for both the axis and angular parameter
3. allows for rational expressions for the matrix entries
4. easier to work with compared to matrices

Composition Law

Quaternion multiplication

$$\zeta = \xi\,\eta = (\xi_0\eta_0 - (\boldsymbol{\xi}, \boldsymbol{\eta}),\ \eta_0\,\boldsymbol{\xi} + \xi_0\boldsymbol{\eta} + \boldsymbol{\xi} \times \boldsymbol{\eta})$$

naturally projects to vector parameter composition $\mathcal{R}(\mathbf{a})\,\mathcal{R}(\mathbf{b}) = \mathcal{R}(\langle \mathbf{a}, \mathbf{b}\rangle)$

$$\langle\, \mathbf{a}, \mathbf{b} \rangle = \frac{\mathbf{a} + \mathbf{b} + \mathbf{a} \times \mathbf{b}}{1 - (\,\mathbf{a}\,, \mathbf{b})}\,. \tag{4}$$

Similarly, the vector parameter of $\mathcal{R}(\mathbf{a})\,\mathcal{R}(\mathbf{b})\,\mathcal{R}(\mathbf{c})$ is given by

$$\langle \mathbf{a}, \mathbf{b}, \mathbf{c}\rangle = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} + (\,\mathbf{a} \times \mathbf{b}) \times \mathbf{c} - (\,\mathbf{a}\,, \mathbf{b})\,\mathbf{c}}{1 - (\,\mathbf{a}\,, \mathbf{b}) - (\,\mathbf{a}\,, \mathbf{c}) - (\,\mathbf{b}\,, \mathbf{c}) - (\,\mathbf{a}\,, \mathbf{b} \times \mathbf{c})}\,.$$

Moreover the latter constitutes a representation since

$$\langle\, \mathbf{c},\, 0\,\rangle = \langle\, 0,\, \mathbf{c}\,\rangle = \mathbf{c}, \qquad \langle\, \mathbf{c},\, -\mathbf{c}\,\rangle = 0. \tag{5}$$

Recommended Readings

- Fedorov F., *The Lorentz Group* (in Russian), Science, Moskow 1979.
- Gibbs, J., *Elements of Vector Analisys*, New Haven, Tuttle, Morehouse, & Taylor, CT (1881).
- Kuvshinov V., Tho N., *Local Vector Parameters of Groups, The Cartan Form and Applications to Gauge and Chiral Field Theory*, Physics of Elementary Particles and the Nucleus, **25** (1994) 603-648.
- Pina E., *Rotations with Rodrigues' Vector*, Eur. J. Phys. **32** (2011) 1171-1178.
- Rodrigues O., *Des lois géométriques qui regissent les déplacéments d'un systéme solide dans l'espace, et de la variation des coordonnées provenant de ces déplacéments considérés indépendamment des causes qui peuvent les produire*, J. Math. Pures Appl. **5** (1840) 380-440.

Euler and *Bryan* Angles

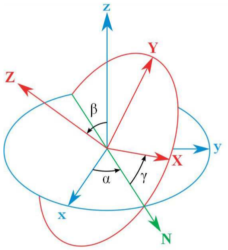

Figure: The classical *Euler* angles as thought in school.

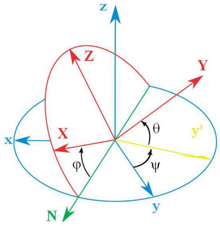

Figure: *Bryan* angles.

Various Approaches to the Problem

- Brezov D., Mladenova C. and Mladenov I., *Vector Decompositions of Rotations*, J. Geom. Symmetry Phys. **28** (2012) 67-103
- P. Davenport. Rotations About Nonorthogonal Axes, (1973)
- G. Piovan, F. Bullo. On Coordinate-Free Rotation Decomposition Euler Angles About Arbitrary Axes, (2012)
- K. Wohlhart: Decomposition of a Finite Rotation into Three Consecutive Rotations About Given Axes, (1992)

Our Approach

We use $\mathcal{R}(\mathbf{c}) = \mathcal{R}(\tau_3\hat{\mathbf{c}}_3)\mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}(\tau_1\hat{\mathbf{c}}_1), \quad |\hat{\mathbf{c}}_k| = 1$ in a scalar product

$$(\hat{\mathbf{c}}_3, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_3, \mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\,\hat{\mathbf{c}}_1)$$

which leads to

$$(\sigma_{31} + \kappa_{31} - 2\kappa_{12}\kappa_{23})\,\tau_2^2 + 2\omega\,\tau_2 + \sigma_{31} - \kappa_{31} = 0 \tag{6}$$

where we use the notations

$$\kappa_{ij} = (\hat{\mathbf{c}}_i, \hat{\mathbf{c}}_j)\,, \quad \sigma_{ij} = (\hat{\mathbf{c}}_i, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_j)\,, \quad \omega = (\hat{\mathbf{c}}_1, \hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3).$$

The equation has real solutions

$$\tau_2^{\pm} = \frac{-\omega \pm \sqrt{\Delta}}{\sigma_{31} + \kappa_{31} - 2\kappa_{12}\kappa_{23}} \tag{7}$$

as long as

$$\Delta = \begin{vmatrix} 1 & \kappa_{12} & \sigma_{31} \\ \kappa_{21} & 1 & \kappa_{23} \\ \sigma_{31} & \kappa_{32} & 1 \end{vmatrix} \geq 0. \tag{8}$$

...

Alternatively, one may choose to work with the composition law for the vector parameter instead

$$\mathbf{c}_1 = \langle -\mathbf{c}_2, \langle -\mathbf{c}_3, \mathbf{c}\rangle\rangle, \quad \mathbf{c}_2 = \langle -\mathbf{c}_3, \langle \mathbf{c}, -\mathbf{c}_1\rangle\rangle, \quad \mathbf{c}_3 = \langle \mathbf{c}, \langle -\mathbf{c}_1, -\mathbf{c}_2\rangle\rangle.$$

Multiplying with $\hat{\mathbf{c}}_k^\times$ and considering dot product with $\hat{\mathbf{n}}$ we express each scalar parameters in a liner fractional manner, in particular

$$\begin{aligned}
\tau_1^\pm &= \frac{(\kappa_{23} - \rho_2\rho_3)\tau\tau_2^\pm - \tilde{\rho}_1\tau_2^\pm}{(\rho_1\tilde{\rho}_1 + \rho_2\tilde{\rho}_2)\tau\tau_2^\pm + (\kappa_{23}\rho_1 - \kappa_{13}\rho_2)\tau_2^\pm + (\rho_1\rho_3 - \kappa_{13})\tau - \tilde{\rho}_2} \\
\tau_2^\pm &= \frac{(\kappa_{12} - \rho_1\rho_2)\tau\tau_2^\pm - \tilde{\rho}_3\tau_2^\pm}{(\rho_2\tilde{\rho}_2 + \rho_3\tilde{\rho}_3)\tau\tau_2^\pm + (\kappa_{12}\rho_3 - \kappa_{13}\rho_2)\tau_2^\pm + (\rho_1\rho_3 - \kappa_{13})\tau - \tilde{\rho}_2}
\end{aligned}$$

with the notations

$$\rho_k = (\hat{\mathbf{c}}_k, \hat{\mathbf{n}}), \quad \tilde{\rho}_1 = (\hat{\mathbf{c}}_2 \times \hat{\mathbf{c}}_3, \hat{\mathbf{n}}), \quad \tilde{\rho}_2 = (\hat{\mathbf{c}}_3 \times \hat{\mathbf{c}}_1, \hat{\mathbf{n}}), \quad \tilde{\rho}_3 = (\hat{\mathbf{c}}_1 \times \hat{\mathbf{e}}_2, \hat{\mathbf{n}}) \cdot$$

Half Turns

If we rotate by an angle $\varphi = \pi$

$$\mathcal{R}(\hat{\mathbf{n}}, \pi) = 2\,\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}^t - \mathcal{I} = \mathcal{O}(\hat{\mathbf{n}})$$

we end up with $\tau \to \infty$. Then, by *l'Hôpital*'s rule

$$\tau_1^{\pm} = \frac{(\kappa_{23} - \rho_2\rho_3)\,\tau_2^{\pm}}{(\rho_1\tilde{\rho}_1 + \rho_2\tilde{\rho}_2)\,\tau_2^{\pm} + \rho_1\rho_3 - \kappa_{13}}, \qquad \tau_3^{\pm} = \frac{(\kappa_{12} - \rho_1\rho_2)\,\tau_2^{\pm}}{(\rho_2\tilde{\rho}_2 + \rho_3\tilde{\rho}_3)\,\tau_2^{\pm} + \rho_1\rho_3 - \kappa_{13}}.$$

Each of the τ_k's can diverge itself, which can be dealt with by lifting up back to the universal cover using

$$\zeta_0^2 = 1 - \boldsymbol{\zeta}^2 = \frac{1}{1 + \mathbf{c}^2}, \qquad \boldsymbol{\zeta} = \zeta_0 \mathbf{c}.$$

then $\tau \to \infty$ corresponds to $\zeta_0 \to 0$ while $\boldsymbol{\zeta}$ remains finite. Alternatively, one may use geometric conditions, such as

$$\mathcal{O}(\hat{\mathbf{n}}) = \mathcal{R}(w\hat{\mathbf{c}}_3)\mathcal{R}(v\hat{\mathbf{c}}_2)\mathcal{O}(\hat{\mathbf{c}}_1) \iff 2\kappa_{12}\rho_1\rho_3 = \rho_2\rho_3 + \kappa_{12}\kappa_{13}$$

$$\mathcal{O}(\hat{\mathbf{n}}) = \mathcal{O}(\hat{\mathbf{c}}_3)\mathcal{R}(v\hat{\mathbf{c}}_2)\mathcal{R}(u\hat{\mathbf{c}}_1) \iff 2\kappa_{23}\rho_1\rho_3 = \rho_1\rho_2 + \kappa_{13}\kappa_{23}$$

The Case of Two Axes

A similar approach for the case $\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}_2)\,\mathcal{R}(\mathbf{c}_1)$ leads to

$$(\hat{\mathbf{c}}_2, \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1) = (\hat{\mathbf{c}}_2,\ \hat{\mathbf{c}}_1)$$

which gives the necessary and sufficient condition for the decomposition

$$\sigma_{21} = \kappa_{21}.$$

Then we multiply $\mathbf{c} = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle$ on the left by $\hat{\mathbf{n}}^{\times}$ and consider dot products with $\hat{\mathbf{c}}_1$ and $\hat{\mathbf{c}}_1$ respectively to obtain

$$\tau_1 = \frac{\tilde{\rho}_3}{\kappa_{12}\rho_1 - \rho_2}, \qquad \tau_2 = \frac{\tilde{\rho}_3}{\kappa_{12}\rho_2 - \rho_1}. \tag{9}$$

Gimbal Lock

In the critical points of the map $\mathbb{RP}^3 \cong \mathbb{S}^3/\mathbb{Z}_2 \to \mathbb{T}^3 \cong (\mathbb{RP}^1)^3$

$$\hat{\mathbf{c}}_3 = \pm\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1$$

one cannot determine τ_1 and τ_3 independently, since

$$\mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}(\tau_1\hat{\mathbf{c}}_1) = \mathcal{R}(\mp\tau_3\mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_1)\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c})\mathcal{R}(\mp\tau_3\hat{\mathbf{c}}_1)$$

where we use the equality $\langle\mathbf{a},\mathbf{b}\rangle = \langle\mathcal{R}(\mathbf{a})\mathbf{b},\mathbf{a}\rangle = \langle\mathbf{b},\mathcal{R}(-\mathbf{b})\mathbf{a}\rangle$. Then

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}(\langle\tau_1\hat{\mathbf{c}}_1, \pm\tau_3\hat{\mathbf{c}}_1\rangle) = \mathcal{R}(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}(\tau_1'\hat{\mathbf{c}}_1)$$

and the solution is readily given as

$$\tau_2 = \frac{\tilde{\rho}_3}{\kappa_{12}\rho_2 - \rho_1}, \qquad \tau_1' = \frac{\tau_1 \pm \tau_3}{1 \mp \tau_1\tau_3} = \frac{\tilde{\rho}_3}{\kappa_{12}\rho_1 - \rho_2}$$

which gives for the *Euler* angles

$$\psi_1 \pm \psi_2 = 2\arctan\left(\frac{\tilde{\rho}_3}{\kappa_{12}\rho_1 - \rho_2}\right).$$

The Identity Transformation

The identity transformation $\mathcal{I}$ has triple degenerate fixes axis $\Longrightarrow$ the solutions need to be obtained in a different manner!
Apart from the trivial solution $\tau_k = 0$, as long as $\omega \neq 0$, we have

$$\tau_1 = \frac{\omega}{\kappa_{12}\kappa_{13} - \kappa_{23}}, \qquad \tau_2 = \frac{\omega}{\kappa_{12}\kappa_{23} - \kappa_{13}}, \qquad \tau_3 = \frac{\omega}{\kappa_{13}\kappa_{23} - \kappa_{12}}.$$

The case $\omega = 0$ has nontrivial solution only if $\hat{\mathbf{c}}_1 = \pm\hat{\mathbf{c}}_3$ and it is

$$\psi_2 = 0, \quad \psi_1 \pm \psi_3 = 0 \qquad (\mathcal{R}\mathcal{R}^{-1} = \mathcal{R}^{-1}\mathcal{R} = \mathcal{I})$$

which is a particular example of a *gimbal lock*.

Transition to Moving Frames

If $\mathbf{c}'_k$ and $\mathbf{c}_k$ denote the moving and the fixed axes respectively then

$$\hat{\mathbf{c}}'_1 = \hat{\mathbf{c}}_1, \qquad \hat{\mathbf{c}}'_2 = \mathcal{R}(\mathbf{c}'_1)\,\hat{\mathbf{c}}_2, \qquad \hat{\mathbf{c}}'_3 = \mathcal{R}(\mathbf{c})\,\hat{\mathbf{c}}_3$$

and thus $\kappa'_{12} = \kappa_{12}$, $\kappa'_{23} = \kappa_{23}$. Premultiplication with $\mathcal{R}(-\mathbf{c}'_3)$ yields

$$\langle -\mathbf{c}'_3, \mathbf{c}\,\rangle = \langle \mathbf{c}'_2, \mathbf{c}'_1\,\rangle \implies \langle -\mathcal{R}(\mathbf{c})\,\mathbf{c}_3, \mathbf{c}\,\rangle = \langle \mathcal{R}(\mathbf{c}_1)\,\mathbf{c}_2, \mathbf{c}_1\,\rangle.$$

Applying $\langle \mathbf{a}, \mathbf{b}\rangle = \langle \mathcal{R}(\mathbf{a})\mathbf{b}, \mathbf{a}\rangle = \langle \mathbf{b}, \mathcal{R}(-\mathbf{b})\mathbf{a}\rangle$ we obtain

$$\langle \mathbf{c}, -\mathbf{c}_3\rangle = \langle \mathbf{c}_1, \mathbf{c}_2\rangle \implies \mathbf{c} = \langle \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\rangle.$$

Therefore, as long as $\mathcal{R}(\mathbf{c})$ is decomposable in $\{\mathbf{c}_k\}$ (or $\{\mathbf{c}'_k\}$), we have

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}'_3)\mathcal{R}(\mathbf{c}'_2)\mathcal{R}(\mathbf{c}'_1) = \mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2)\mathcal{R}(\mathbf{c}_3).$$

Similarly, if $\sigma'_{21} = \kappa'_{21}$ (or $\sigma_{12} = \kappa_{12}$), then

$$\mathcal{R}(\mathbf{c}) = \mathcal{R}(\mathbf{c}'_2)\mathcal{R}(\mathbf{c}'_1) = \mathcal{R}(\mathbf{c}_1)\mathcal{R}(\mathbf{c}_2).$$

Possible Extensions

How can we generalize?

1. extend to the dual groups $SO^+(2,1)$, $SU(1,1) \cong SL(2,\mathbb{R})$ via conjugation
2. extend to $SO^+(3,1) \cong SO(3,\mathbb{C})$ via complexification
3. extend to product groups $SO(4) \cong SO(3) \times SO(3)$ and $SO(2,2) \cong SO(2,1) \times SO(2,1)$
4. Higher dimensional generalizations.

Split Quaternions

We may choose a basis in $\mathfrak{su}(1,1)$ in the form

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

and expand each element ζ of $\mathrm{SU}(1,1)$ as

$$\zeta = (\zeta_0, \boldsymbol{\zeta}) = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \qquad \det \zeta = \zeta_0^2 - \zeta_1^2 - \zeta_2^2 + \zeta_3^2 = 1.$$

Alternatively, one may work in the $\mathfrak{sl}(2,\mathbb{R})$ basis

$$\tilde{\mathbf{e}}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{\mathbf{e}}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \tilde{\mathbf{e}}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and all results are transferable by the explicit isomorphism

$$\mathcal{U} : \mathcal{H}^2 \to \Delta, \qquad z \to \mathcal{U}(z) := \mathrm{i}\,\frac{z - \mathrm{i}}{z + \mathrm{i}}$$

The Spin Cover

Vectors $\mathbf{x} \in \mathbb{R}^{2,1}$ can be mapped to $\mathfrak{su}(1,1)$ as

$$\mathbf{x} \to \mathbf{X} = \begin{pmatrix} \mathrm{i}x_3 & x_1 + \mathrm{i}x_2 \\ x_1 - \mathrm{i}x_2 & -\mathrm{i}x_3 \end{pmatrix}$$

and the pseudo-*Euclidean* scalar square is given by

$$\mathbf{x} \cdot \mathbf{x} = -\det \mathbf{X} = x_1^2 + x_2^2 - x_3^2 .$$

Then we consider the norm-preserving SU(1,1) action in its algebra

$$\mathrm{Ad}_{\zeta} : \mathbf{X} \to \zeta\, \mathbf{X}\, \zeta^{-1}.$$

The *Cartesian* coordinates of $\mathbf{X}$ are transformed by the matrix

$$\mathcal{R}_h(\zeta) = \mathcal{I} - 2\,\boldsymbol{\zeta} \otimes (\eta\, \boldsymbol{\zeta})^t + 2\zeta_0\, \boldsymbol{\zeta}^{\curlywedge}.$$

where $\mathcal{P}_{\boldsymbol{\zeta}} = \boldsymbol{\zeta} \otimes (\eta\, \boldsymbol{\zeta})^t$ stands for $\mathcal{P}_{\boldsymbol{\zeta}}{}^i_j = \eta_{jk} \zeta^i \zeta^k$ and $\boldsymbol{\zeta}^{\curlywedge}{}^i_j = \eta^{ik} \varepsilon_{klj}\, \zeta^l$.

Versions of *Rodrigues'* Formula

We have three versions of *Rodrigues* formula:

1. For *hyperbolic* transformations $\operatorname{Tr}\mathcal{R}_h(\zeta) > 3$, or $\boldsymbol{\zeta}^2 > 0$ (*space-like*) the substitution is $\zeta_0 = \cosh\dfrac{\varphi}{2}$, $\boldsymbol{\zeta} = \sinh\dfrac{\varphi}{2}\,\hat{\mathbf{n}}$ and thus

$$\mathcal{R}_h(\hat{\mathbf{n}},\varphi) = \cosh\varphi\,\mathcal{I} + (1-\cosh\varphi)\,\hat{\mathbf{n}}\otimes(\eta\,\hat{\mathbf{n}})^t + \sinh\varphi\,\hat{\mathbf{n}}^{\curlywedge}.$$

2. For *elliptic* ones $\operatorname{Tr}\mathcal{R}_h(\zeta) < 3$, $\boldsymbol{\zeta}^2 < 0$ (*time-like*) we have $\zeta_0 = \cos\dfrac{\varphi}{2}$, $\boldsymbol{\zeta} = \sin\dfrac{\varphi}{2}\,\hat{\mathbf{n}}$ and

$$\mathcal{R}_h(\hat{\mathbf{n}},\varphi) = \cos\varphi\,\mathcal{I} - (1-\cos\varphi)\,\hat{\mathbf{n}}\otimes(\eta\,\hat{\mathbf{n}})^t + \sin\varphi\,\hat{\mathbf{n}}^{\curlywedge}.$$

3. In the *parabolic* case $\operatorname{Tr}\mathcal{R}_h(\zeta) = 3$, i.e., $\boldsymbol{\zeta}^2 = 0$ (*light-like* or *null*)

$$\mathcal{R}_h(\hat{\mathbf{n}},\varphi) = \mathcal{I} + \varphi\,\hat{\mathbf{n}}^{\curlywedge} - \frac{\varphi^2}{2}\,\hat{\mathbf{n}}\otimes(\eta\,\hat{\mathbf{n}})^t, \qquad \hat{\mathbf{n}}_E^2 = 1.$$

The Hyperbolic Vector Parameter

Similarly to the *Euclidean* case $\mathbf{c} = \dfrac{\boldsymbol{\zeta}}{\zeta_0}$ with the two-valued inverse

$$\mathbf{c}^2 = \frac{\zeta_0^2 - 1}{\zeta_0^2} \implies \zeta_0^{\pm} = \pm(1 - \mathbf{c}^2)^{-\frac{1}{2}}, \qquad \boldsymbol{\zeta}^{\pm} = \zeta_0^{\pm}\mathbf{c}.$$

Moreover, if we write $\mathbf{c} = \tau\hat{\mathbf{n}}$ and $\epsilon = \hat{\mathbf{n}}^2$, we have three cases

1. $\tau = \tanh\dfrac{\varphi}{2}$ in the *hyperbolic* (space-like) case $\epsilon > 0$
2. $\tau = \tan\dfrac{\varphi}{2}$ in the *elliptic* (time-like) case $\epsilon < 0$
3. $\tau = \dfrac{\varphi}{2}$ in the *parabolic* (light-like) case $\epsilon = 0$.

Composition Law

The split-quaternion multiplication law

$$\zeta\,\xi = (\zeta_0\,\xi_0 + \boldsymbol{\zeta}\cdot\boldsymbol{\xi},\ \zeta_0\,\boldsymbol{\xi} + \xi_0\,\boldsymbol{\zeta} + \boldsymbol{\zeta} \curlywedge \boldsymbol{\xi}).$$

determines the composition law for the vector parameters

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \frac{\mathbf{c}_2 + \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_2 \cdot \mathbf{c}_1}.$$

In the case $\mathbf{c}_1 \parallel \mathbf{c}_2$ the above reduces to addition formula for tanh, tan and usual scalars resp. in the *hyperbolic* , *elliptic* and *parabolic* case. For composition is associative and non-commutative

$$\mathbf{c} = \frac{\mathbf{c}_3 + \mathbf{c}_2 + \mathbf{c}_1 + (\mathbf{c}_3\cdot\mathbf{c}_2)\,\mathbf{c}_1 + \mathbf{c}_3 \curlywedge \mathbf{c}_2 + \mathbf{c}_3 \curlywedge \mathbf{c}_1 + \mathbf{c}_2 \curlywedge \mathbf{c}_1 + (\mathbf{c}_3 \curlywedge \mathbf{c}_2) \curlywedge \mathbf{c}_1}{1 + \mathbf{c}_3\cdot\mathbf{c}_2 + \mathbf{c}_3\cdot\mathbf{c}_1 + \mathbf{c}_2\cdot\mathbf{c}_1 + (\mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1)}$$

and may be proved to constitute a representation of $SO^+(2,1)$.

Euler Decomposition

$$\mathcal{R}_h(\tau\hat{\mathbf{n}}) = \mathcal{R}_h(\tau_3\hat{\mathbf{c}}_3)\mathcal{R}_h(\tau_2\hat{\mathbf{c}}_2)\mathcal{R}_h(\tau_1\hat{\mathbf{c}}_1)$$

leads to a quadratic equation for τ_2 in the form ($\epsilon_k = \mathbf{c}_k^2$)

$$\left[\epsilon_2(\sigma_{31}+\kappa_{31}) - 2\kappa_{12}\kappa_{23}\right]\tau_2^2 - 2\omega\tau_2 + \kappa_{31} - \sigma_{31} = 0$$

which has solutions

$$\tau_2^{\pm} = \frac{\omega \pm \sqrt{\Delta}}{\epsilon_2(\sigma_{31}+\kappa_{31}) - 2\kappa_{12}\kappa_{23}}$$

as long as $\Delta = \omega^2 + \left[\epsilon_2(\sigma_{31}+\kappa_{31}) - 2\kappa_{12}\kappa_{23}\right](\sigma_{31}-\kappa_{31}) \geq 0$.
For the remaining scalar parameters the procedure, described above gives

$$\tau_1^{\pm} = \frac{(\epsilon\kappa_{23} - \rho_2\rho_3)\tau\tau_2^{\pm} + \tilde{\rho}^1\tau_2^{\pm}}{(\rho_1\tilde{\rho}^1 + \rho_2\tilde{\rho}^2)\tau\tau_2^{\pm} + (\kappa_{23}\rho_1 - \kappa_{13}\rho_2)\tau_2^{\pm} + (\rho_1\rho_3 - \epsilon\kappa_{13})\tau + \tilde{\rho}^2},$$

$$\tau_3^{\pm} = \frac{(\epsilon\kappa_{12} - \rho_1\rho_2)\tau\tau_2^{\pm} + \tilde{\rho}^3\tau_2^{\pm}}{(\rho_2\tilde{\rho}^2 + \rho_3\tilde{\rho}^3)\tau\tau_2^{\pm} + (\kappa_{12}\rho_3 - \kappa_{13}\rho_2)\tau_2^{\pm} + (\rho_1\rho_3 - \epsilon\kappa_{13})\tau + \tilde{\rho}^2}.$$

...

In the case of two axes we have

$$\tau_1 = \frac{\tilde{\rho}^3}{\epsilon_1 \rho_2 - \kappa_{12} \rho_1}, \qquad \tau_2 = \frac{\tilde{\rho}^3}{\epsilon_2 \rho_1 - \kappa_{12} \rho_2}.$$

if the condition

$$\sigma_{21} = \hat{\mathbf{c}}_2 \cdot \mathcal{R}_h(\tau \hat{\mathbf{n}})\, \hat{\mathbf{c}}_1 = \hat{\mathbf{c}}_2 \cdot \hat{\mathbf{c}}_1 = \kappa_{21}$$

is fulfilled and whenever

$$\hat{\mathbf{c}}_3 = \pm \mathcal{R}_h(\mathbf{c})\, \hat{\mathbf{c}}_1$$

the solution is degenerate

$$\tau_2 = \frac{\tilde{\rho}^3}{\epsilon_2 \rho_1 - \kappa_{12} \rho_2}, \qquad \tau_1' = \frac{\tilde{\rho}^3}{\epsilon_1 \rho_2 - \kappa_{12} \rho_1}. \tag{10}$$

where

$$\tau_1' = \frac{\tau_1 \pm \tau_3}{1 \pm \epsilon_1 \tau_1 \tau_3} \implies \psi' = \psi_1 \pm \psi_3. \tag{11}$$

$SO^+(3,1) \cong SO(3,\mathbb{C})$

Consider the representation of 4-vectors

$$x \in \mathbb{R}^{3,1} \longrightarrow X = \begin{pmatrix} \mathrm{i}(x_0 + x_1) & x_2 + \mathrm{i}x_3 \\ \mathrm{i}x_3 - x_2 & \mathrm{i}(x_0 - x_1) \end{pmatrix}$$

$$x \cdot x = \det X = -x_0^2 + x_1^2 + x_2^2 + x_3^2$$

with the norm-preserving isomorphism

$$X \longrightarrow \tilde{X} = \zeta\, X\, \zeta^{\dagger}, \qquad \zeta \in SL(2,\mathbb{C}).$$

Then the *Cartesian* coordinates are transformed with the matrix

$$\Lambda(\zeta) = (\zeta_0^2 - \boldsymbol{\zeta}^2)\mathcal{I} + 2\boldsymbol{\zeta} \otimes \boldsymbol{\zeta}^t + 2\zeta_0\boldsymbol{\zeta}^{\times} \in SO^+(3,1).$$

Parametrization

The complex vector parameter

$$\mathbf{c} = \frac{\zeta}{\zeta_0} = \boldsymbol{\alpha} + \mathrm{i}\boldsymbol{\beta}, \qquad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^3$$

can be used directly in the representation

$$\Lambda(\mathbf{c}) = \frac{1}{|1 + \mathbf{c}^2|} \begin{pmatrix} 1 + |\,\mathbf{c}\,|^2 & \bar{\mathbf{c}} - \mathbf{c} + \bar{\mathbf{c}} \times \mathbf{c} \\ \mathbf{c} - \bar{\mathbf{c}} + \bar{\mathbf{c}} \times \mathbf{c} & 1 - |\,\mathbf{c}\,|^2 + \mathbf{c}\,\bar{\mathbf{c}} + \bar{\mathbf{c}}\mathbf{c} + (\mathbf{c} + \bar{\mathbf{c}})^{\times} \end{pmatrix}$$

or to form the *Cayley* representation

$$\Lambda = \frac{\mathcal{I} + \Theta}{\mathcal{I} - \Theta}, \qquad \Theta = \begin{pmatrix} 0 & -\mathrm{i}\boldsymbol{\beta} \\ \mathrm{i}\boldsymbol{\beta} & \boldsymbol{\alpha}^{\times} \end{pmatrix} \tag{12}$$

via the so-called "tensor parameter" Θ.

Invariant Axes

The characteristic equation of Θ

$$\Theta^4 - \mathrm{tr}^2(\Theta)\,\Theta^2 + \det\Theta = 0$$

allows for a (double) vanishing roots only for $\det\Theta = -(\boldsymbol{\alpha}\cdot\boldsymbol{\beta})^2 = 0$ in which case the corresponding eigenvector is given by

$$\boldsymbol{\xi} = \begin{pmatrix} 0 \\ \boldsymbol{\alpha} \end{pmatrix}$$

and if $\mathrm{tr}^2(\Theta) = \boldsymbol{\beta}^2 - \boldsymbol{\alpha}^2$ is also zero, i.e., $\mathbf{c}^2 = 0$, we have one more eigenvector for the (quadruple) vanishing root in the form

$$\tilde{\boldsymbol{\xi}} = \begin{pmatrix} -\mathrm{i}\boldsymbol{\alpha}^2 \\ \boldsymbol{\alpha}\times\boldsymbol{\beta} \end{pmatrix}.$$

Euler Decomposition

Just as in the *Euclidean* case we have

$$\tau_1^\pm = \frac{(\kappa_{23}-\rho_2\rho_3)\tau\tau_2^\pm - \tilde{\rho}_1\tau_2^\pm}{(\rho_1\tilde{\rho}_1+\rho_2\tilde{\rho}_2)\tau\tau_2^\pm + (\kappa_{23}\rho_1-\kappa_{13}\rho_2)\tau_2^\pm + (\rho_1\rho_3-\kappa_{13})\tau - \tilde{\rho}_2}$$

$$\tau_2^\pm = \frac{(\kappa_{12}-\rho_1\rho_2)\tau\tau_2^\pm - \tilde{\rho}_3\tau_2^\pm}{(\rho_2\tilde{\rho}_2+\rho_3\tilde{\rho}_3)\tau\tau_2^\pm + (\kappa_{12}\rho_3-\kappa_{13}\rho_2)\tau_2^\pm + (\rho_1\rho_3-\kappa_{13})\tau - \tilde{\rho}_2}$$

but since there are generally no invariant eigenvectors, τ_2 should be defined in another way - one possibility is to use explicitly the unaltered $\mathbf{c} = \langle \mathbf{c}_3, \mathbf{c}_2, \mathbf{c}_1 \rangle$ which provides a cubic equation in the form

$$(\kappa_{13}\rho_2-\kappa_{12}\rho_3-\kappa_{23}\rho_1-\epsilon\omega)\tau_1\tau_2\tau_3+\begin{pmatrix} \epsilon\kappa_{23}-\tilde{\rho}_1 \\ \epsilon\kappa_{13}+\tilde{\rho}_2 \\ \epsilon\kappa_{12}-\tilde{\rho}_3 \end{pmatrix}^t \begin{pmatrix} \tau_2\tau_3 \\ \tau_1\tau_3 \\ \tau_1\tau_2 \end{pmatrix}+\begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{pmatrix}^t \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix}=\epsilon.$$

Then one root should be dropped out, as it violates the composition law.

Wigner Rotation in $2+1$ dimensions

Let $\mathbf{c}_2 \parallel \mathbf{e}_z$ (pure rotation) and $\mathbf{c}_1$ belong to the XY-plane (pure boost)

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle = \tau_1(\cos\gamma - \tau_2 \sin\gamma)\,\hat{\mathbf{e}}_x + \tau_1(\sin\gamma + \tau_2\cos\gamma)\,\hat{\mathbf{e}}_y + \tau_2\,\hat{\mathbf{e}}_z.$$

If $\mathbf{c}_{1,2}$ correspond to pure boosts, they can be mapped to $\mathbb{C}$

$$z_k = x_k + \mathrm{i} y_k \ \rightarrow\ \mathbf{c}_k = (x_k, y_k, 0)^t, \qquad k = 1, 2$$

and the rotational z-component of their composition is

$$\tilde{\tau} = \langle \mathbf{c}_2, \mathbf{c}_1 \rangle_z = \frac{x_1 y_2 - x_2 y_1}{1 + x_1 x_2 + y_1 y_2} = -\frac{\Im(1 + z_1 \bar{z}_2)}{\Re(1 + z_1 \bar{z}_2)}$$

so the *Wigner* angle of rotation is given by

$$\theta = -2 \arg\left(1 + z_1 \bar{z}_2\right).$$

Scattering Theory

The scattering process

$$\begin{aligned} \Psi(k,x) &\sim e^{ikx} + r(k)\, e^{-ikx}, \qquad x \to -\infty \\ \Psi(k,x) &\sim t(k)\, e^{ikx}, \qquad x \to \infty \end{aligned}$$

can be described by its *monodromy matrix*

$$\mathcal{M} = \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} \in \mathrm{SU}(1,1), \qquad \alpha = \frac{1}{\bar t}, \quad \beta = -\frac{\bar r}{\bar t} \tag{13}$$

and the *Wigner* angle of the left/right scattering process is given by

$$\theta^{\pm} = \pm 2 \arg(1 + r_1 \bar r_2)$$

while for the remaining (scattering) factor we have $\tilde\tau = \arctan(1 + r_1 \bar r_2)$

$$\mathbf{c}_{\pm}{}^2 = \left| \frac{r_1 + r_2}{1 + r_1 \bar r_2} \right|^2 \implies \zeta_0 = (1 + \mathbf{c}_{\pm}^2)^{-\frac{1}{2}} = \frac{|1 + r_1 \bar r_2|}{\sqrt{|1 + r_1 \bar r_2|^2 - |r_1 + r_2|^2}}$$

$$\zeta_{\pm} = \frac{\Re(1 + r_1 \bar r_2)}{\sqrt{|1 + r_1 \bar r_2|^2 - |r_1 + r_2|^2}} \begin{pmatrix} -\Re(r_1 + r_2) \pm \tilde\tau\, \Im(r_1 + r_2) \\ \Im(r_1 + r_2) \pm \tilde\tau\, \Re(r_1 + r_2) \\ 0 \end{pmatrix}.$$

The $3+1$ Dimensional Case

For two purely imaginary boosts, the real (rotational) part of the composition is

$$\Re\langle \mathbf{c}_2, \mathbf{c}_1\rangle = \frac{\boldsymbol{\beta}_1 \times \boldsymbol{\beta}_2}{1 + \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2}, \qquad \mathbf{c}_k = \mathrm{i}\boldsymbol{\beta}_k$$

while for the imaginary (boost) contribution we have

$$\Im\langle \mathbf{c}_2, \mathbf{c}_1\rangle = \frac{\boldsymbol{\beta}_1 + \boldsymbol{\beta}_2}{1 + \boldsymbol{\beta}_1 \cdot \boldsymbol{\beta}_2}.$$

If we use quaternion representation the above may be written as

$$\boldsymbol{\xi} = \frac{\Im\bar{\zeta}_1\zeta_2}{1 + \Re\bar{\zeta}_1\zeta_2}, \qquad \mathbf{c}_k \to \zeta_k, \quad \boldsymbol{\alpha} \to \boldsymbol{\xi}$$

Thomas Precession

In the $2+1$ case consider *Wigner* rotation with $z_1 = z$ and $z_2 = z + \delta z$

$$\dot{\tau}_W = \Im \frac{\bar{z}\,\dot{z}}{1+|z|^2} \quad \Longrightarrow \mathrm{d}\tau_W = \Im \frac{\bar{z}\,\mathrm{d}z}{1+|z|^2} \tag{14}$$

which can be seen as a connection of the non-compact *Hopf* fibration

$$\mathrm{SU}(1,1) \longrightarrow \mathbb{S}^1 \longrightarrow \Delta.$$

Similarly, in the $3+1$ case we have

$$\dot{\boldsymbol{\alpha}} = \frac{\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}}}{1+\boldsymbol{\beta}^2} \quad \Longrightarrow \quad \mathrm{d}\,\boldsymbol{\xi} = \Im \frac{\bar{\boldsymbol{\zeta}}\,\mathrm{d}\,\boldsymbol{\zeta}}{1+|\boldsymbol{\zeta}|^2}$$

that has an interpretation as a connection for the fibre bundle

$$\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{SU}(2) \longrightarrow \mathcal{H}^3.$$

Recommended Readings

- Aravind P., *The Wigner Angle as an Anholonomy in Rapidity Space*, Am. J. Phys. **65** (1997) 634-636.
- Ben-Manahem A., *Wigner's Rotation Revisited*, Am. J. Phys. **53** (1985) 62-66.
- Han D., Kim Y., Son D., *Thomas Precession, Wigner Rotations and Gauge Transformations*, Class. Quantum Grav., **4** (1987) 1777-1783.
- Hasebe K., *The Split-Algebras and Non-compact Hopf Maps*, J. Math. Phys., **51** (2010).
- Rhodes A., Semon M., *Relativistic Velocity Space, Wigner Rotation and Thomas Precession*, Am. J. Phys. **72** (2004) 943-961
- Shankar R., Mathur H., *Thomas Precession, Berry Potential and the Meron* Phys. Rev. Lett. **73** (1994)

The Compact Case

The z component of the composition of two vector parameters $\mathbf{c}_{1,2}$ in the XY plane

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle_z = \frac{\mathbf{c}_2 \times \mathbf{c}_1}{1 - \mathbf{c}_1 \cdot \mathbf{c}_2}.$$

can be written as

$$\langle \mathbf{c}_2, \mathbf{c}_1 \rangle_z = \frac{x_2\, y_1 - y_2\, x_1}{1 - x_1\, x_2 - y_1\, y_2} = \frac{\Im(z_1 \bar{z}_2)}{1 - \Re(z_1 \bar{z}_2)}$$

if we use coordinates $\mathbf{c}_k \to z_k = x_k + \mathrm{i} y_k \in \bar{\mathbb{C}} \cong \mathbb{S}^2$. Then $\theta = 2\arg(1 + z_1 \bar{z}_2)$ and for the *Thomas* precession we have

$$\tau' = \Im \frac{z\, \bar{z}\,'}{1 - |z|^2} \quad \Longrightarrow \quad \mathrm{d}\tau = \Im \frac{z\, \mathrm{d}\bar{z}}{1 - |z|^2}.$$

which has an interpretation as a connection of the *Hopf* fibration

$$\mathbb{S}^3 \longrightarrow \mathbb{S}^1 \longrightarrow \mathbb{S}^2.$$

The case $SO(4) \cong SO(3) \times SO(3)$ is globally trivial.

Higher Dimensional Generalizations

There is a possible generalization to the principal G-bundle of higher dimensional *Möbius* groups over *Lobachevsky* spaces

$$\mathcal{L}_n \cong \mathrm{SO}^+(n,1)/\mathrm{SO}(n).$$

Since the compact analogue of these homogeneous spaces are spheres

$$\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$$

in both cases the fibre is isomorphic to $\mathrm{SO}(n)$ and it is possible to define *Wigner* rotation and *Thomas* precession in an analogous way, but the explicit description is going to be different, since the nice algebraic structure of complex and hypercomplex numbers is not available in arbitrary dimension.

Thank You!

THANKS FOR YOUR PATIENCE!