# Natural States and Symmetry Properties of Two-Dimensional Ciarlet-Mooney-Rivlin Nonlinear Constitutive Models

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# Motivations

Finding closed-form solutions in compressible hyperelasticity quite challenging: very few such solutions in the literature.

Particular classes of exact solutions obtained for several models using Lie groups.

Lie symmetries widely used in the analysis of contemporary elasticity models.
In particular, classification of Lie point symmetries for 1D and 2D nonlocal elastodynamics in [Bluman, Anco, Cheviakov, Springer, 2010; Bower. *Applied Mechanics of Solids*, 2009].

Invariant solutions for radial motions of compressible hyperelastic spheres & cylinders [Capriz & Mariano, 2007].

Similarity solutions for the motion of hyperelastic solids in [Cheviakov, GEM package, *Computer Physics Communications*, 2007].

From a more formal viewpoint, Lie symmetries constitute a guide in the Lagrangian and Hamiltonian formalisms in continuum mechanics, especially for complex materials endowed with a microstructure [England & Spencer, 2005].

For models admitting a variational formulation, one-to-one correspondence between variational Lie symmetries and local conservation laws through Noether's theorem.
For non-variational models, this relation generally does not hold [Ericksen, 2000].

# Outline of the presentation

- Nonlinear dynamic equations for isotropic homogeneous hyperelastic materials considered in the Lagrangian formulation.

- Derive explicit criterion of existence of a natural state of a given constitutive law. Used to derive natural state conditions for some common constitutive relations.

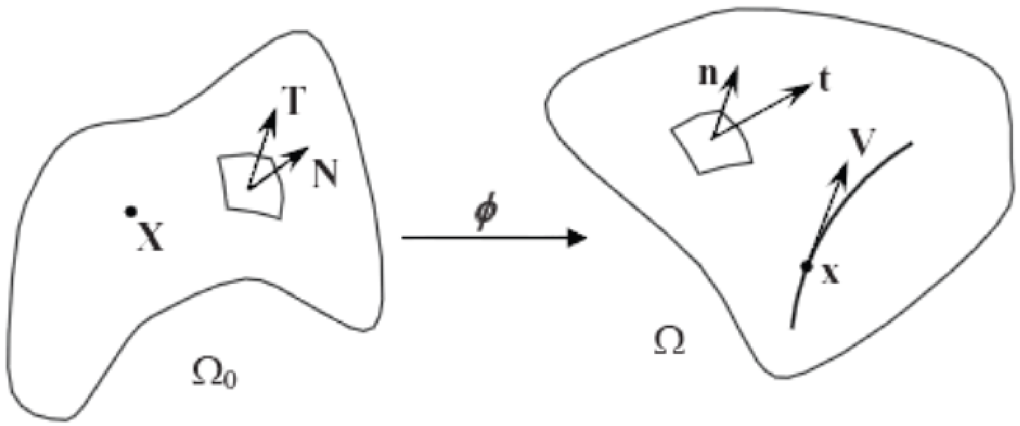
- Equivalence transformations computed for 2D planar motions of Ciarlet-Mooney-Rivlin solids; yields reduction of the number of parameters in the constitutive law.

- Find special value of traveling wave speed for which nonlinear Ciarlet-Mooney-Rivlin equations admit an additional infinite set of point symmetries.

- Classification of point symmetries in dynamical case & for traveling wave coordinates.

- Perspectives.

# Finite strain kinematics




Material and Eulerian coordinates.

$\mathbf{x} = \phi(\mathbf{X}, t) \rightarrow \mathbf{F} = \nabla_X \mathbf{x}$ Transformation gradient (tangent mapping)

$\mathbf{v} = \frac{d\mathbf{x}}{dt}$ velocity

Use cartesian coordinates with flat space: $g^{ij} = \delta^{ij} \rightarrow F^i_j := \frac{\partial \phi^i}{\partial X^j} = F_{ij}$

Orientation preserving condition: $J := \det(\mathbf{F}) > 0$

Continuity equation: $\rho = \rho_0 / J$

# Boundary value problem in hyperelasticity

Material (Lagrangian) format:

$\rho_0 \mathbf{x}_{tt} = \mathrm{Div}_X \mathbf{P} + \rho_0 \mathbf{R}$   cons. linear momentum

$\mathbf{R} = \mathbf{R}(\mathbf{X}, t)$  body forces per unit mass

$\mathbf{F}.\mathbf{P}^T = \mathbf{P}.\mathbf{F}^T$  cons. angular momentum

$\mathbf{P} = \rho_0 \partial_F W(\mathbf{F})$  hyperelastic model

$W(\mathbf{F})$  strain energy density per unit volume

Physical (Eulerian) format:

$\rho_0 \mathbf{v}^e{}_t = \mathrm{div}_x \boldsymbol{\sigma} + \rho \mathbf{r}$   cons. linear momentum

$\mathbf{v}^e(\mathbf{x}, t) = \mathbf{v}(\mathbf{X}, t) = d\mathbf{x}(\mathbf{X}, t) / dt$

$\mathbf{r} = \mathbf{r}(\mathbf{x}, t)$  body forces per actual unit mass

$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$  cons. angular momentum

$\mathbf{t} = \boldsymbol{\sigma}.\mathbf{n}$ traction vector, $\boldsymbol{\sigma}$ Cauchy stress $\rightarrow \mathbf{P} := J\boldsymbol{\sigma}.\mathbf{F}^{-T}$ first Piola-Kirchhoff stress

$\rightarrow \mathbf{T} = \mathbf{P}.\mathbf{N}$  nominal traction (force on reference surface element)

# Boundary value problem in hyperelasticity

For zero forcing, strong form obeys extremum principle:

$$V[\mathbf{x}] := \int_{\Omega_0} \rho_0 \left( W(\mathbf{F}) + V_R(\mathbf{x}) \right) dV$$ potential energy

$$K[\mathbf{x}] := \frac{1}{2} \int_{\Omega_0} \rho_0 \left\| \mathbf{x}_t \right\|^2 dV$$ kinetic energy

$H = K + V$ Hamiltonian

$L = K - V$ Lagrangian $\rightarrow$ $L = \int_{\Omega_0} l(\mathbf{x}, \mathbf{x}_t, \mathbf{F}) dV$

$$\rightarrow \frac{\partial}{\partial t} D_{x_t} l(\mathbf{x}, \mathbf{x}_t, \mathbf{F}) = D_x l(\mathbf{x}, \mathbf{x}_t, \mathbf{F})$$ Euler-Lagrange equ.= momentum equ.

in Lagrangian format.

# Hyperelastic constitutive models

polar decomposition $\mathbf{F} = \mathrm{R}.\mathbf{U} = \mathbf{V}.\mathrm{R} \rightarrow \mathbf{B} := \mathbf{F}.\mathbf{F}^{T} = \mathbf{V}^{2},\ \mathbf{C} := \mathbf{F}^{T}.\mathbf{F} = \mathbf{U}^{2} \rightarrow \lambda_1, \lambda_2, \lambda_3$ eigenvalues of $\mathbf{U}, \mathbf{V}$

$\rightarrow \mathrm{I}_1 := \mathrm{Tr}(\mathbf{B}) = \lambda_1^{2} + \lambda_2^{2} + \lambda_3^{2},\ \mathrm{I}_2 := \lambda_1^{2}\lambda_2^{2} + \lambda_2^{2}\lambda_3^{2} + \lambda_3^{2}\lambda_1^{2},\ \mathrm{I}_1 := \mathrm{Det}(\mathbf{B}) = \lambda_1^{2}\lambda_2^{2}\lambda_3^{2}$

$\bar{\mathrm{I}}_1 := \mathrm{J}^{-2/3}\mathrm{I}_1,\ \bar{\mathrm{I}}_2 := \mathrm{J}^{-4/3}\mathrm{I}_2,\ \bar{\mathrm{I}}_3 := \mathrm{J}$ reduced invariants to isolate volume changes

$\rightarrow \mathrm{W} = \Phi(\lambda_1, \lambda_2, \lambda_3) = \mathrm{U}(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3) = \overline{\mathrm{U}}(\bar{\mathrm{I}}_1,\ \bar{\mathrm{I}}_2,\ \bar{\mathrm{I}}_3)$ for isotropic materials

Ciarlet Mooney-Rivlin materials: $\mathrm{W} = \mathrm{a}(\mathrm{I}_1 - 3) + \mathrm{b}(\mathrm{I}_2 - 3) - \mathrm{c}\mathrm{I}_3 - \frac{1}{2}\mathrm{d}\log \mathrm{I}_3,\ \ \mathrm{a} > 0, \mathrm{b}, \mathrm{c}, \mathrm{d} \geq 0$

General form of const. law for isotropic materials:

$$\mathbf{S} = \mathbf{F}^{-1}.\mathbf{P} = 2\rho_0 \left\{\alpha_0 \mathbf{I} + \alpha_1 \mathbf{C} + \alpha^{2}\mathbf{C}^{2}\right\},\ \alpha_{0,1,2} = \alpha_{0,1,2}(\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3)$$

For incompressible materials: $\mathbf{P} = -\mathrm{p}\mathbf{F}^{-T} + \rho_0 \partial_{F} \mathrm{W}(\mathbf{F}),\ \ \mathrm{p} = \mathrm{p}(\mathbf{X}, \mathrm{t})$ hydrostatic pressure

Neo-Hookean and Mooney-Rivlin constitutive models

| Type | neo-Hookean | Mooney-Rivlin |
|---|---|---|
| Standard [21] | $W = aI_1$, <br> $a > 0$. | $W = aI_1 + bI_2$, <br> $a, b > 0$ |
| Generalized [23] | $W = a\bar{I}_1 + c(J-1)^2$, <br> $a, c > 0$. | $W = a\bar{I}_1 + b\bar{I}_2 + c(J-1)^2$ <br> $a, b, c > 0$ |
| Generalized (Ciarlet) "compressible" [20] | $W = aI_1 + \Gamma(J)$, <br> $\Gamma(q) = cq^2 - d\log q, \quad a, c, d > 0$ | $W = aI_1 + bI_2 + \Gamma(J)$ <br> $\Gamma(q) = cq^2 - d\log q, \quad a, b, c, d > 0$ |

# Constitutive relations and natural states

Natural state $\mathbf{F}=\mathbf{I}\Rightarrow\boldsymbol{\sigma}=\mathbf{0}$ vanishing self-equilibrated stress (no external load)

Neo-Hookean material fails to have natural states: $W=aI_1\rightarrow\mathbf{P}=2\rho_0\mathbf{F}\rightarrow\boldsymbol{\sigma}=2\rho_0\mathbf{I}$

Mooney-Rivlin materials (c=0=d) do not have natural states.

$\longrightarrow$ Natural state requires external forces $\longrightarrow$ Stress tensors cannot be linear in **F**.

<u>Theorem</u>: a hyperelastic material with const. law given by $W=U(I_1,I_2,I_3)$

has zero residual stress iff $\dfrac{\partial U}{\partial I_1}+2\dfrac{\partial U}{\partial I_2}+\dfrac{\partial U}{\partial I_3}=0$ when $\mathbf{F}=\mathbf{I}$

Alternative: for $W=\Phi(\lambda_1,\lambda_2,\lambda_3)\rightarrow t_k=\dfrac{\partial\Phi}{\partial\lambda_k}=0$ when $\lambda_1=\lambda_2=\lambda_3=1$: Biot stress vanishes

| Type | Strain energy density function $W$ | Natural state condition |
|---|---|---|
| Hadamard material (incl. neo-Hookean, Mooney-Rivlin) | $c_1(I_1-3)+c_2(I_2-3)+H(I_3)$ | $a+2b+H'(1)=0$ |
| Generalized Blatz-Ko rubber model | $\frac{\mu}{2}f\left(I_1-1-\frac{1}{\nu}+\frac{1}{q}I_3^{-q}\right)$ $+\frac{\mu}{2}(1-f)\left(\frac{I_2}{I_3}-1-\frac{1}{\nu}+\frac{1}{q}I_3^{q}\right)$ | All admissible $\mu,\nu,f$; $q\equiv\frac{\nu}{1-2\nu}$ |
| Generalized Mooney-Rivlin material | $a(\bar{I}_1-3)+b(\bar{I}_2-3)+c(J-1)^2$ | All admissible $a,b,c$ |
| Ogden material | $\sum_{i=1}^{M}a_i(\lambda_1^{\alpha_i}+\lambda_2^{\alpha_i}+\lambda_3^{\alpha_i})$ $+\sum_{j=1}^{N}b_j((\lambda_1\lambda_2)^{\beta_j}+(\lambda_2\lambda_3)^{\beta_j}+(\lambda_3\lambda_1)^{\beta_j})$ $+H(\lambda_1\lambda_2\lambda_3)$ | All admissible $a_i,b_j,\alpha_i,\beta_j$; $H'(1)=0$ |

# Point symmetries and equivalence transformations

Consider PDE system $R^{\sigma}\left(x,u,\partial u,\ldots,\partial^{k}u\right)=0,\ \ \sigma=1\ldots N \rightarrow$ derivatives of order at most $k$

$n$ independent variables $z=\left(z^{1},\ldots,z^{n}\right)$

$m$ dependent var. $u(z)=\left(u^{1}(z),\ldots,u^{m}(z)\right)$

One-parameter Lie group of transformations of variables (z,u): one-to-one transformation acting in the m+n dimensional space (z,u) of the form:

$$z^{*i}=f^{i}\left(z,u;\varepsilon\right)=z^{i}+\varepsilon\xi^{i}\left(z,u\right)+O\left(\varepsilon^{2}\right),\ i=1\ldots n$$

$$u^{*\mu}=g^{\mu}\left(z,u;\varepsilon\right)=z^{\mu}+\varepsilon\eta^{\mu}\left(z,u\right)+O\left(\varepsilon^{2}\right),\ \mu=1\ldots m$$

components define infinitesimal generator $\mathbf{Y}=\xi^{i}\left(z,u\right)\dfrac{\partial}{\partial x^{i}}+\eta^{\mu}\left(z,u\right)\dfrac{\partial}{\partial u^{\mu}}$

(z,u) transform -> transform of partial derivatives of u(z) given by prolongation formulas.

One-parameter Lie group of point transformations = group of point symmetries of a PDE system iff its prolongation leaves invariant the solution manifold of the PDE in the space including u, z and all required partial derivatives of u(z).

$\longrightarrow$ If u(z) is a solution, then $u^{*}(z^{*})$ is also solution of the same system.

Symmetry components found in algorithmic way from the determining equations:

$Y^{(k)}R^{\alpha}\left(x,u,\ldots,\partial^{k}u\right)=0,\ \alpha=1\ldots N$ on solutions

# Point symmetries and equivalence transformations (2)

Notion of equivalence transformations closely related to local symmetries.

Equivalence transformations preserve differential structure of equations / modify constitutive and/or parameters of a DE model.

Equivalence transformations used to reduce number of parameters of a given const. model.

<u>Interest</u>: analyses involving classifications; construct exact solutions for new sets of constitutive functions/parameters, from known exact solutions for given constitutive functions/parameters.

Consider family of PDE systems $R^{\sigma}\left(x,u,\partial u,\ldots,\partial^{k}u\right)=0,\ k=1..N$

with n independent variables $z=\left(z^{1},\ldots,z^{n}\right)$ m dependent variables $u(z)=\left(u^{1}(z),\ldots,u^{m}(z)\right)$

involving set of constitutive functions and/or parameters $K=\left(K_{1},\ldots,K_{L}\right)$

One-parameter Lie group of equivalence transformations = Lie group of transformations

$$\overline{z}^{i}=f^{i}\left(z,u;\varepsilon\right),\ i=1\ldots n$$

$$\overline{u}^{\mu}=g^{\mu}\left(z,u;\varepsilon\right),\ \mu=1\ldots m$$

$$\overline{K}^{l}=G_{l}\left(z,u;\varepsilon\right),\ l=1\ldots L$$

maps a given PDE syst. Into a new one with new constitutive functions.

# Point symmetries and equivalence transformations (3)

Equivalence transformations computed through sol. of determining equ.

$$Y(k)R^{\alpha}\left(x,u,\partial u,\ldots\partial^{k}u\right)=0,\ \alpha=1\ldots N$$

Prolongation of inifnitesimal generator for the one-parameter Lie group of point transformations

$$z^{*}=f(z,u;\varepsilon),\quad u^{*}=g(z,u;\varepsilon)\ \text{global form}$$

$$Y=\xi^{i}(z,u)\frac{\partial}{\partial x^{i}}+\eta^{\mu}(z,u)\frac{\partial}{\partial u^{\mu}}$$

$$\rightarrow \text{local form}\ z^{*i}=z^{i}+\varepsilon\xi^{i}(x,u)+O\left(\varepsilon^{2}\right),\ i=1..n$$

$$u^{*\mu}=u^{\mu}+\varepsilon\eta^{i}(x,u)+O\left(\varepsilon^{2}\right),\ \mu=1..m$$

Need to specify variables on which the const. Functions / Parameters depend.

<u>Ex.1</u>: if $K_1$=a, $K_2$=b are 2 constant parameters of Mooney-Rivlin model,

Corresponding equivalence transformations will be of the form $\tilde{a}=G_1(a,b;\varepsilon),\ \tilde{b}=G_2(a,b;\varepsilon)$

<u>Ex.2</u>: for $K_1$=Q(x), x independent var., equivalence transf. for Q(x) will be $\tilde{Q}(\tilde{x})=G_1(x,Q(x);\varepsilon)$

# Equivalence transformations for 2D Ciarlet-Mooney-Rivlin model

Strain energy density $\quad W = a(I_1 - 3) + b(I_2 - 3) - cI_3 - \frac{1}{2} d \log I_3, \quad a > 0, b, c, d \geq 0$

<u>Theorem</u>: 2D Ciarlet-Mooney-Rivlin model admits following equivalence transformations

$$
\begin{aligned}
\tilde{t} &= e^{\varepsilon_2} t + \varepsilon_1, \\
\tilde{X}^1 &= e^{\varepsilon_3}\left(X^1 \cos\varepsilon_7 - X^2 \sin\varepsilon_7\right) + \varepsilon_4, \\
\tilde{X}^2 &= e^{\varepsilon_3}\left(X^1 \sin\varepsilon_7 + X^2 \sin\varepsilon_7\right) + \varepsilon_5, \\
\tilde{x}^1 &= e^{2\varepsilon_2} x^1 + f^1(t), \\
\tilde{x}^2 &= e^{2\varepsilon_2} x^2 + f^2(t), \\
\tilde{\rho}_0 &= e^{\varepsilon_6} \rho_0,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{a} &= a + \varepsilon_8 G(a,b,c,d) + O(\varepsilon^2), \\
\tilde{b} &= b - \varepsilon_8 G(a,b,c,d) + O(\varepsilon^2), \\
\tilde{c} &= c + \varepsilon_8 G(a,b,c,d) + O(\varepsilon^2), \\
\tilde{d} &= d.
\end{aligned}
$$

$$
\begin{aligned}
\tilde{R}^1 &= R^1 + \frac{d^2 f^1(t)}{dt^2}, & \tilde{R}^2 &= R^2 + \frac{d^2 f^2(t)}{dt^2}, \\
\tilde{a} &= -b + e^{2\varepsilon_3 - 2\varepsilon_2}(a+b), & \tilde{b} &= b, \\
\tilde{c} &= -b + e^{4\varepsilon_3 - 6\varepsilon_2}(b+c), & \tilde{d} &= e^{2\varepsilon_2} d,
\end{aligned}
$$

Includes scalings, translations and rotations of material coordinates (parameters $\varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_7$ )

Galilean transformations, transformations of eulerian coord. and time (parameters $\varepsilon_1, \varepsilon_2$ , arbitrary functions $f_1(t)$, $f_2(t)$)

Scaling of body density in ref. configuration ( $\varepsilon_6$ ),
transformations of parameters a,b,c,d of const. model (parameters $\varepsilon_2, \varepsilon_3$ )

# Equivalence transformations for 2D Ciarlet-Mooney-Rivlin model

2D motions:

$$\begin{cases} \rho_0 x^1{}_{,tt} - \dfrac{\partial P^{11}}{\partial X^1} - \dfrac{\partial P^{12}}{\partial X^2} - \rho_0 R^1 = 0, \;\; \rho_0 x^2{}_{,tt} - \dfrac{\partial P^{21}}{\partial X^1} - \dfrac{\partial P^{22}}{\partial X^2} - \rho_0 R^2 = 0 \\ \rho_0 = \rho_0\left(X^1, X^2\right), \; P^{ij} = \rho_0\left(X^1, X^2\right)\dfrac{\partial W}{\partial F_{ij}}, \;\; i, j = 1, 2 \end{cases}$$

$$F = \begin{bmatrix} F_{11} & F_{12} & 0 \\ F_{21} & F_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Th.: Ciarlet-Mooney-Rivlin model in 2D depends only on 3 constitutive parameters

$\longrightarrow$ 2D first Piola-Kirchhoff stress tensor takes the form

$$\mathbf{P} = \rho_0\left(A\mathbf{F}_2 + BJ\mathbf{C}_2 - \frac{d}{J}\mathbf{C}_2\right) \qquad \mathbf{F}_2 = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \rightarrow \mathbf{C}_2 = \begin{bmatrix} F_2^2 & -F_1^2 \\ -F_2^1 & F_1^1 \end{bmatrix} \text{ cofactor matrix}$$

$$A = a + b \geq 0, \; B = b + c \geq 0, \; d \geq 0$$

Previous Theorems can be used for direct analysis of 2D models.

# Equivalence transformations for 2D Ciarlet-Mooney-Rivlin model

Used to present equivalence and symmetry classification in a more compact form:

Set of equivalence transformations of 2D Ciarlet-Mooney-Rivlin model given by

$$
\begin{aligned}
\tilde{t} &= e^{2\varepsilon_2} t + \varepsilon_1, & & \\
\tilde{X}^1 &= e^{2\varepsilon_2+\varepsilon_3}\left(X^1\cos\varepsilon_7 - X^2\sin\varepsilon_7\right) + \varepsilon_4, & \tilde{X}^2 &= e^{2\varepsilon_2+\varepsilon_3}\left(X^1\sin\varepsilon_7 + X^2\sin\varepsilon_7\right) + \varepsilon_5, \\
\tilde{x}^1 &= e^{4\varepsilon_2} x^1 + f^1(t), & \tilde{x}^2 &= e^{4\varepsilon_2} x^2 + f^2(t), \\
\tilde{\rho}_0 &= e^{\varepsilon_6}\rho_0, & & \\
\tilde{R}^1 &= R^1 + \frac{d^2 f^1(t)}{dt^2}, & \tilde{R}^2 &= R^2 + \frac{d^2 f^2(t)}{dt^2}, \\
\tilde{A} &= e^{2\varepsilon_2+2\varepsilon_3} A, & \tilde{B} &= e^{4\varepsilon_3} B, \\
\tilde{d} &= e^{4\varepsilon_2} d, & &
\end{aligned}
$$

$\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$ arbitrary constants.

$f^1(t), f^2(t)$ arbitrary functions.

Action of equivalence transformations on essential parameters A, B, d is pure scalings.

# Symmetry classification for time-dependent 2D Ciarlet-Mooney-Rivlin models

Classify symmetries w.r. to constants A, B>0, d and types of density functions $\rho_0\left(X^1, X^2\right) = Cte$
Restrict to zero external body forces. Given modulo the previous equivalence transformations

| Case | **Point symmetries** |
|---|---|
| General | $Y_1 = \frac{\partial}{\partial t}, Y_2 = \frac{\partial}{\partial X^1}, Y_3 = \frac{\partial}{\partial X^2}, Y_4 = \frac{\partial}{\partial x^1}, Y_5 = \frac{\partial}{\partial x^2}, Y_6 = t\frac{\partial}{\partial x^1}, Y_7 = t\frac{\partial}{\partial x^2}$, $Y_8 = X^2\frac{\partial}{\partial X^1} - X^1\frac{\partial}{\partial X^2}, Y_9 = x^2\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^2}, Y_{10} = t\frac{\partial}{\partial t} + X^1\frac{\partial}{\partial X^1} + X^2\frac{\partial}{\partial X^2} + x^1\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^2}$ |
| $A = 0$, $B, d$ arbitrary | $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}$, $Y_{11} = f_1(X^2)\frac{\partial}{\partial X^1}, Y_{12} = \left(\frac{\partial}{\partial X_2} f_2(X^1, X^2)\right)\frac{\partial}{\partial X^1} - \left(\frac{\partial}{\partial X_1} f_2(X^1, X^2)\right)\frac{\partial}{\partial X^2}$, $f_1(X^2), f_2(X^1, X^2)$ are arbitrary functions |
| $A = d = 0$ $B$ arbitrary | $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}$, $Y_{13} = t\frac{\partial}{\partial t} + X^1\frac{\partial}{\partial X^1}$ |
| $A = B = 0$ $d$ arbitrary | $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}$, $Y_{14} = X^1\frac{\partial}{\partial X^1}$ |

symmetry generators $Y_1, Y_2, Y_3, Y_4, Y_5$ correspond to translations in dependent and independent variables; generators $Y_6, Y_7$ correspond to Galilean transformations

$$x^1 \to x^1 + \varepsilon t, \quad x^2 \to x^2 + \varepsilon t;$$

generators $Y_8, Y_9$ correspond to rotations in material and Eulerian coordinates respectively; generators $Y_{10}, Y_{13}, Y_{14}$ correspond to scaling transformations.

Each of the generators $Y_{11}, Y_{12}$ corresponds to an infinite number of symmetries through an arbitrary function.

# Symmetry classification of 2D Ciarlet-Mooney-Rivlin model in traveling wave coordinates

Assume that medium moves w.r. to observer at constant speed s>0 in direction $X^1$.

With this ansatz, one has $x^i\left(t, X^1, X^2\right) = w^i\left(z, X^2\right),\ z = X^1 - st,\ i = 1,2$

Consider body density in ref. configuration of the form $\rho_0 = \rho_0\left(X^2\right)$

Assume no-forcing.

Transformation of partial derivatives in PDE's: $\rho_0 \dfrac{\partial^2 x^i\left(X^1 - st, X^2\right)}{\partial t^2} = -s^2 \dfrac{\partial^2 w^i\left(z, X^2\right)}{\partial z^2}$

# Full symmetry classification of PDE system in travelling wave coordinates modulo previous equivalence transformations

Point symmetry classification for the two-dimensional Ciarlet-Mooney-Rivlin models in traveling-wave coordinates, given by (30), (34), (38), with zero forcing and $\rho_0 = \text{const} > 0$.

| Case number | Case | **Point symmetries** |
|---|---|---|
| 1 | General | $Y_1 = \frac{\partial}{\partial z}, Y_2 = \frac{\partial}{\partial w^1}, Y_3 = \frac{\partial}{\partial w^2}, Y_4 = w^2\frac{\partial}{\partial w^1} - w^1\frac{\partial}{\partial w^2}$ |
| 2 | $\rho_0(X^2) = (X^2+q_1)^{q_2}$, $q_1, q_2 = \text{const}$, $q_2 \neq 0$, $A, B, d, s$ arbitrary | $Y_1, Y_2, Y_3, Y_4$, $Y_5 = z\frac{\partial}{\partial z} + (X^2+q_1)\frac{\partial}{\partial X^2} + w^1\frac{\partial}{\partial w^1} + w^2\frac{\partial}{\partial w^2}$ |
| 3a | $\rho_0(X^2) = \exp(q_1 X^2)$, $q_1 = \text{const} \neq 0$, $A, B, d, s$ arbitrary | $Y_1, Y_2, Y_3, Y_4$, $Y_6 = \frac{\partial}{\partial X^2}$ |
| 3b | $\rho_0(X^2) = \exp(q_1 X^2)$, $q_1 = \text{const} \neq 0$, $A, d$ arbitrary, $B = 0$, $\boxed{s^2 = A}$ | $Y_1, Y_2, Y_3, Y_4$, $Y^{\infty}_{(1)} = -\left(\frac{1}{q_1}\frac{d}{dz}f_1(z)\right)\frac{\partial}{\partial X^2} + f_1(z)\frac{\partial}{\partial z}$ |
| 4a | $\rho_0(X^2) > 0$ arbitrary, $A, B$ arbitrary, $d = 0$, $\boxed{s^2 = A}$ | $Y_1, Y_2, Y_3, Y_4$, $Y_7 = z\frac{\partial}{\partial z} + w^1\frac{\partial}{\partial w^1} + w^2\frac{\partial}{\partial w^2}$, $Y_8 = \left(\rho_0 \int \frac{1}{\rho_0} dX^2\right)\frac{\partial}{\partial X^2}$, $Y^{\infty}_{(2)} = f_2(z)\rho_0\frac{\partial}{\partial X^2}$, $f_2(z)$ is an arbitrary function |
| 4b | $\rho_0(X^2) > 0$ arbitrary, $A, d$ arbitrary, $B = 0$, $\boxed{s^2 = A}$ | $Y_1, Y_2, Y_3, Y_4$, $Y_9 = z\frac{\partial}{\partial z}$ |
| 5a | $\rho_0 = \text{const}$ $A, B, d, s$ arbitrary | $Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y_6$, $Y_{10} = X^2\frac{\partial}{\partial z} - \frac{Az}{A-s^2}\frac{\partial}{\partial X^2}$ |
| 5b | $\rho_0 = \text{const}$, $\boxed{s^2 = A}$, $A, B, d$ arbitrary | $Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0)$, $Y^{\infty}_{(3)} = f_3(z)\frac{\partial}{\partial X^2}$, $f_3(z)$ is an arbitrary function |
| 5c | $\rho_0 = \text{const}$, $\boxed{s^2 = A}$, $A, d$ arbitrary, $B = 0$ | $Y_1, Y_2, Y_3, Y_4, Y_5(q_1 = 0), Y_9, Y^{\infty}_{(3)}$, |

symmetry generators $Y_1, Y_2, Y_3$ correspond to translations in dependent and independent variables; generator $Y_4$ corresponds to rotations in Eulerian coordinates; generators $Y_5, Y_7, Y_9, Y_{10}$ correspond to scaling transformations. Additional symmetries $Y_7, Y_8, Y_9, Y_{10}$ and additional infinite families of symmetries $Y^{\infty}_{(1)}$, $Y^{\infty}_{(2)}$, $Y^{\infty}_{(3)}$ arise for the special value (39) of the translation speed.

# Full symmetry classification of PDE system in travelling wave coordinates modulo previous equivalence transformations

General non-linear Ciarlet-Mooney-Rivlin model has special wave speed for which equations

admit additional symmetries including infinite set of point symmetries, with generators $Y_1^{(\infty)}, Y_2^{(\infty)}, Y_3^{(\infty)}$

Special wave speed equal to the constant wave speed $s^* = \sqrt{a+b}$

of linear neo-Hookean version of dynamic equations

$$\begin{cases} x^1{}_{,tt} = A\left(\dfrac{\partial^2 x^1}{\partial\left(X^1\right)^2} + \dfrac{\partial^2 x^1}{\partial\left(X^2\right)^2}\right) \\ x^2{}_{,tt} = A\left(\dfrac{\partial^2 x^2}{\partial\left(X^1\right)^2} + \dfrac{\partial^2 x^2}{\partial\left(X^2\right)^2}\right) \end{cases}$$

$$A = a + b \geq 0,\ B = b + c \geq 0,\ d \geq 0$$

# Conclusion - Perspectives

• Existence of natural states for hyperelastic materials analyzed: question of high importance for consistency formulation of BVPs for numerical computations.

• Classical neo-Hookean and Mooney-Rivlin models do not have a natural state / Hadamard materials may admit a natural state according to range of parameters.

• Point symmetries of the 2D Ciarlet-Mooney-Rivlin model in full dynamical setting classified with symbolic software. Such symmetries important to construct exact solutions and conservation laws.

• Computation of symmetry structure of elasticity models in non-planar 2D reductions (axial symmetry) & in 3D.

• Extension to anisotropic constitutive behavior: case of one and two families of fibers.