A Construction of a Recursion Operator for Some Solutions of the Einstein Field Equations

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Outline

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For example, there are researches such as the following.

- "A geometrical setting for the Lax representation" (1982)
- "A new characterization of completely integrable systems" (1984)
- "When do recursion operators generate new conservation laws?" (1992)
- "Hamiltonian dynamics" (2001)

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Since a recursion operator is constructed based on the local coordinate system (q, p), is not uniquely determined.

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Purpose

We consider geodesic flows on the pseudo-Riemann metric and Kerr-Newman metric as concrete examples, and we construct recursion operators.

Moreover, we get constants of motion with the recursion operator.

Lemma

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We put $T = \begin{pmatrix} A & \\ & A \end{pmatrix}$, $A = \begin{pmatrix} q^1 & & \\ & \ddots & \\ & & q^n \end{pmatrix}$ $(q^k = x^k,\ p_k = x^{n+k})$.

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Then we have the following Lemmas.

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Lemma.1 $\mathcal{N}_T = 0.$

$$\left((\mathcal{N}_T)_{ij}{}^h = T_i^k \frac{\partial T^h{}_j}{\partial x^k} - T_j^k \frac{\partial T^h{}_i}{\partial x^k} + T_k^h \frac{\partial T^k{}_i}{\partial x^j} - T_k^h \frac{\partial T^k{}_j}{\partial x^i}. \right)$$

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Lemma.2 If $\Delta = -\dfrac{\partial}{\partial p_h}$ $(h = 1, \cdots, n)$, then $\mathcal{L}_\Delta T = 0$.

$$\left((\mathcal{L}_\Delta T)^i{}_j = \Delta^k \frac{\partial T^i{}_j}{\partial x^k} - \frac{\partial \Delta^i}{\partial x^k} T^k_j + T^i_k \frac{\partial \Delta^k}{\partial x^j}. \right)$$

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We construct the vector field Δ for the geodesic flow on the Minkowski metric.

$$\Delta = -p_1 \frac{\partial}{\partial q_1} + \sum_{k=2}^{4} p_k \frac{\partial}{\partial q_k},$$

where a matrix g_{ij} and a equation of geodesic flow are

$$g_{ij} = g^{ij} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \frac{d^2 q^\kappa}{dt^2} + \Gamma^{\kappa}_{\mu\nu} \frac{dq^\mu}{dt} \frac{dq^\mu}{dt} = 0, \quad \left(p_\kappa = \frac{dq^\kappa}{dt} \right).$$

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Generating function

$$W = \sqrt{\sum_{k=2}^{4} a_k^2 - 2E}\ q_1 + \sum_{k=2}^{4} a_k q_k.$$

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Canonical coordinate system

$$Q_1 = E, \quad Q_k = a_k = p_k, \quad P_1 = -\frac{\partial W}{\partial Q_1} = \frac{q_1}{p_1}, \quad P_k = -\frac{\partial W}{\partial Q_k} = -\frac{q_1 p_k}{p_1} - q_k.$$

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The relationship between (P, Q) and the original coordinate system (p, q)

$$p_1 = \sqrt{\sum_{k=2}^{4} Q_k^2 - 2Q_1}, \quad q_1 = P_1 \sqrt{\sum_{k=2}^{4} Q_k^2 - 2Q_1},$$
$$p_k = Q_k, \quad q_k = -P_k - Q_k P_1.$$

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If we take Tr(T), Tr(T^2), Tr(T^3) and Tr(T^4), it is possible to obtain the constants of motion.

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$$T = \sum_{i,j=1}^{4} \left(\left({}^t A\right)^i{}_j \frac{\partial}{\partial p_i} \otimes dp_j + B^i{}_j \frac{\partial}{\partial q_i} \otimes dp_j + A^i{}_j \frac{\partial}{\partial q_i} \otimes dq_j \right),$$

where $A = \begin{pmatrix} H & & & \\ \frac{p_2}{p_1}(p_2 - H) & p_2 & & \\ \frac{p_3}{p_1}(p_3 - H) & & p_3 & \\ \frac{p_4}{p_1}(p_4 - H) & & & p_4 \end{pmatrix}$, $B = \frac{q_1}{p_1}({}^t A - A)$.

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Constants of motion

$$\mathrm{Tr}(T^\ell) = \frac{1}{2^{\ell-1}}\left(-p_1^2 + p_2^2 + p_3^2 + p_4^2\right)^\ell + 2\left(p_2^\ell + p_3^\ell + p_4^\ell\right), \quad (\ell = 1, 2, 3, 4).$$

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We regard T as a matrix:

$$T = \begin{pmatrix} \mathcal{S} & \\ & \mathcal{S} \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} Q_1 & & & \\ & Q_2 & & \\ & & Q_3 & \\ & & & Q_4 \end{pmatrix}.$$

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And we define K_i, ω_1 and Γ as follows:

$$K_i := Q_i\, P_i,\ \ \omega_1 := \sum_{i=1}^{4} dK_i \wedge d\alpha_i\ (\alpha_i = Q_i),\ \ \Gamma := \sum_{i=1}^{4} K_i \frac{\partial}{\partial P_i}.$$

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At this time, ω_1 is a symplectic form and satisfies the following:

$$\omega_1 = \mathcal{L}_\Gamma \omega.$$

Now, we constract another vector field by using $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$,

$$\boldsymbol{\Delta}_{h+1} := [\boldsymbol{\Delta}_h, \boldsymbol{\Gamma}], \quad (\boldsymbol{\Delta}_0 = \boldsymbol{\Delta} = -\frac{\partial}{\partial P_1}).$$

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$$\boldsymbol{\Delta}_2 = [\boldsymbol{\Delta}_1, \boldsymbol{\Gamma}] = -Q_1^2 \frac{\partial}{\partial P_1}, \quad \boldsymbol{\Delta}_3 = [\boldsymbol{\Delta}_1, \boldsymbol{\Gamma}] = -Q_1^3 \frac{\partial}{\partial P_1}.$$

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And we define the following Poisson bracket $\{\ ,\ \}_1$ by using the symplectic form ω_1:

$$\{f, g\}_1 := \sum_{i,j=1} \left(\mathcal{S}^{-1}\right)^i_j \left(\frac{\partial f}{\partial P_j} \frac{\partial g}{\partial Q_i} - \frac{\partial f}{\partial Q_i} \frac{\partial g}{\partial P_j} \right).$$

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Thus, we get
$$\Delta_k := \{H_k, \cdot\ \} = \{H_{k+1}, \cdot\ \}_1$$

where
$$H_1 = \frac{1}{2} Q_1^2, \quad H_2 = \frac{1}{3} Q_1^3, \quad H_3 = \frac{1}{4} Q_1^4.$$

We choose a vector field $\boldsymbol{\Delta_1}$ and a Hamiltonian function $\boldsymbol{H_1}$.

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T_1 and T are the same, so T_1 is a recursion operator not only on Δ_1 but also original Δ.

We choose a vector field Δ_1 and a Hamiltonian function H_1.

vector field $\Delta_1 = -Q_1 \frac{\partial}{\partial P_1}$ Hamiltonian function $H_1 = \frac{1}{2} Q_1^2$

In this case, a recursion operator corresponding to the Δ_1 is described as follows:

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In the same way, by Δ_2 and H_2, we have that T_2 coincide with T. Similarly, we have that T_3 coincide with T.

Solution of the Einstein field equations

Purpose

We consider geodesic flow on Kerr-Newman metric, and we construct a recursion operator. And we get constants of motion with the recursion operator.

Einstein field equations (1915-1916)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}.$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$: Einstein tensor,

Λ : Cosmological term, κ : Constant.

$\Longrightarrow$ Field equation (Einstein's field equations of General Relativity(EFE))

$\Longrightarrow$ Several exact solutions are given.

Exact solutions of Einstein field equation

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We consider recursion operators using some solutions of the Einstein equation.

The Schwarzschild metric is the simplest solution among the four solution in the Einstein field equations. Also the Kerr-Newman metric is the most complex solution in this.

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Now, we consider the Schwarzschild metric and the Kerr-Newman metric.

Kerr-Newman metric

$$ds^2 = -\frac{1}{\rho^2}\left(\kappa - a^2\sin^2\theta\right)dt^2 + \frac{2a\sin^2\theta}{\rho^2}\left(Q^2 - 2Mr\right)dt\,d\phi$$
$$+\frac{\rho^2}{\kappa}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left\{\left(r^2+a^2\right)^2 - a^2\kappa\sin^2\theta\right\}d\phi^2.$$

$$\kappa := r^2 - 2rM + a^2 + Q^2, \quad \rho^2 := r^2 + a^2\cos^2\theta.$$

M : the mass of the black hole,

J : angular momentum, Q : electric charge

$$t \in (-\infty, \infty),\ r \in (2M, \infty),\ \theta \in (0, \pi),\ \phi \in (0, 2\pi). \quad \left(a^2 + Q^2 \leq M^2\right)$$

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Kerr metric $(Q = 0)$

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right)dt^2 - \frac{4aMr\sin^2\theta}{\rho^2}dtd\phi$$
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Schwarzschild metric $(Q = 0,\ J = 0)$

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vector field

$$\Delta = -\left(1 - \frac{2M}{q_2}\right)^{-1} p_1 \frac{\partial}{\partial q_1} + \left(1 - \frac{2M}{q_2}\right)^{-1} p_2 \frac{\partial}{\partial q_2} + q_2^{-2} p_3 \frac{\partial}{\partial q_3} + \frac{p_4}{q_2^2 \sin^2 q_3} \frac{\partial}{\partial q_4}$$
$$+ \left\{ -\frac{M}{q_2^2}\left(1 - \frac{2M}{q_2}\right)^{-2} p_1^2 - \frac{M}{q_2^2} p_2^2 + q_2^{-3} p_3^2 + \frac{p_4^2}{q_2^3 \sin^2 q_3} \right\} \frac{\partial}{\partial p_2} + \frac{p_4^2 \cos q_3}{q_2^2 \sin^3 q_3} \frac{\partial}{\partial p_3}.$$

$$t = q_1 \in (-\infty, \infty),\ r = q_2 \in (2M, \infty),\ \theta = q_3 \in (0, \pi),\ \phi = q_4 \in (0, 2\pi).$$

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Next, we consider the Hamilton-Jacobi equation by this Hamiltonian function.
The Hamiltonian function does not include q_1 and q_4.
Thus, we put p_1 and p_4 are constant.

Hamilton-Jacobi equation

$$W = \sum_{k=1}^{4} W_k(q_k), \quad p_1 = \frac{dW_1}{dq_1} = \alpha, \quad p_4 = \frac{dW_4}{dq_4} = \beta.$$

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Generating function

$$\begin{aligned} W &= \alpha q_1 + \int \frac{dW_2}{dq_2} dq_2 + \int \frac{dW_3}{dq_3} dq_3 + \beta q_4 \\ &= \alpha q_1 + W_2 + W_3 + \beta q_4. \end{aligned}$$

It is difficult to describe W_2 and W_3 by elementary function.

We determine the canonical coordinate system (P, Q) using the generating function W.

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Canonical coordinate system

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Constants of motion

$$\mathrm{Tr}(T^\ell) = 2 \sum_{i=1}^{4} Q_i^\ell = 2 \left(E^\ell + K^\ell + \alpha^\ell + \beta^\ell \right), \quad (\ell = 1, 2, 3, 4).$$

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$$g_{ij} = \begin{pmatrix} -\left(\kappa - a^2\sin^2\theta\right)\rho^{-2} & & & a\sin^2\theta\left(Q^2-2Mr\right)\rho^{-2} \\ & \rho^2\kappa^{-1} & & \\ & & \rho^2 & \\ a\sin^2\theta\left(Q^2-2Mr\right)\rho^{-2} & & & \sin^2\theta\left\{\left(r^2+a^2\right)^2 - a^2\kappa\sin^2\theta\right\}\rho^{-2} \end{pmatrix}.$$

Vector field

$$\Delta = \sum_{i=1}^{4} \left(X_i \frac{\partial}{\partial p_i} + Y_i \frac{\partial}{\partial q_i} \right).$$

We omitted details of this equation because it is too complicated.

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$$H = \frac{1}{2}\left[\left\{\frac{a^2}{\rho^2}\sin^2 q_3 - \frac{(q_2^2 + a^2)^2}{\kappa\rho^2}\right\} p_1^2 + \frac{\kappa}{\rho^2} p_2^2 + \frac{1}{\rho^2} p_3^2 + \left\{\frac{a^2}{\kappa\rho^2} - \frac{1}{\rho^2 \sin^2 q_3}\right\} p_4^2 + 2\left\{\frac{a}{\rho^2} - \frac{a(q_2^2 + a^2)}{\kappa\rho^2}\right\} p_1 p_4\right].$$

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$$\mathrm{Tr}(T^{\ell}) = 2 \sum_{i=1}^{4} Q_i^{\ell} = 2 \left(E^{\ell} + K^{\ell} + \alpha^{\ell} + \beta^{\ell} \right), \quad (\ell = 1, 2, 3, 4).$$

At this time, a recursion operator T and the constants of motion $\mathrm{Tr}(T^{\ell})$ are ...

Recursion operator T, Constants of motion $\mathrm{Tr}(T^{\ell})$

Recursion operator

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We were able to construct a recursion operator, determined by the geodesic flow from Kerr-Newman metric.

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We were able to construct a recursion operator, determined by the geodesic flow from Kerr-Newman metric.

Thus, we get it to be integrable system. And we get that it has a constants of motion.

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Thank you for your attention!