# Analytic Representation of a Class of Axisymmetric Solutions to the Willmore Equation

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# Abstract


• The so-called Willmore functional assigns to each surface in the three dimensional Euclidean space its total squared mean curvature. The surfaces providing local extrema to this functional are referred to as the Willmore surfaces. The mean and Gaussian curvatures of these surfaces obey the corresponding Euler-Lagrange equation, which is usually called the Willmore equation.

• The present work is concerned with a particular class of axially symmetric solutions to the Willmore equation, which are solutions of an intermediate integral arising due to the additional scaling invariance of the rotationally-invariant solutions of the considered equation. An analytic representation of the foregoing class of solutions is given in terms of Jacobi elliptic functions and integrals.

# Overview

1. Willmore Surfaces
   - Willmore Functional
   - Willmore Equation
   - Related Functionals and Equations
2. Symmetry Groups of the Willmore Equation
   - Willmore Equation in Monge Representation
   - The Group of Special Conformal Transformations in $\mathbb{R}^3$
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3. Axisymmetric Solutions of the Willmore Equation
   - Reduced Equation
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• The so-called Willmore functional (energy)

$$\mathcal{W} = \int_{\mathcal{S}} H^2 \mathrm{d}A \tag{1}$$

assigns to each surface $\mathcal{S}$ in the three-dimensional Euclidean space $\mathbb{R}^3$ its total squared mean curvature $\mathcal{W}$. Here, $H$ is the local mean curvature of the surface $\mathcal{S}$, $\mathrm{d}A$ is the area element on the surface $\mathcal{S}$.

• This functional has drawn much attention after [Willmore, 1965] where T. J. Willmore proposed to study the surfaces providing extremum to the functional (1), which are usually referred to as the Willmore surfaces.

• This interest is related to the so-called Willmore conjecture [Willmore, 1965] concerning the global problem of minimizing of (1) among the class of immersed tori: *the integral of the square of the mean curvature of a torus immersed in $\mathbb{R}^3$ is at least $2\pi^2$*, which have been proved recently [Marques & Neves, 2012].

• The Willmore surfaces are of great importance for the conformal geometry because of the invariance of Willmore functional (energy) under the 10-parameter group of special conformal transformations in $\mathbb{R}^3$.

• The Euler-Lagrange equation associated with the Willmore functional, which is further referred to as the Willmore equation, has the form

$$\Delta H + 2(H^2 - K)H = 0 \tag{2}$$

Here $\Delta$ is the Laplace-Beltrami operator on the surface $\mathcal{S}$ and $K$ is the Gaussian curvature of $\mathcal{S}$.

• Actually, according to [Thomsen, 1923], Schadow was the first who derived this equation in 1922 as the Euler-Lagrange equation for the variational problem

$$\int_{\mathcal{S}} \left( \frac{1}{R_1} - \frac{1}{R_2} \right)^2 \mathrm{d}A \tag{3}$$

where $1/R_1$ and $1/R_2$ are the two principal curvatures of the surface $\mathcal{S}$. (This variational problem was studied by Thomsen in connection with the conformal geometry). In fact, the Lagrangian densities of the functionals (1) and (3) are proportional up to the divergence term $2K$ and that is why they lead to the same Euler-Lagrange equation.

• Sometimes called bending energy, the Willmore energy appears naturally in some physical contexts. For instance, it had been proposed in 1812 by Poisson and later, in 1821, by Sophie Germain to describe elastic shells.

• In mathematical biology it appears in the Helfrich model as one of the terms that contribute to the energy of cell membranes:

$$\mathcal{F}_b = \int_{\mathcal{S}} \left[\frac{1}{2}k_c(2H + c_0)^2 + k_G K\right] \mathrm{d}A + \lambda \int_{\mathcal{S}} \mathrm{d}A + p \int \mathrm{d}V.$$

Here $k_c$ and $k_G$ are real constants representing the bending and Gaussian rigidity of the membrane, $c_0$ is the spontaneous curvature, $\lambda$ is the tensile stress, $p$ is the pressure, $V$ is the enclosed volume. The corresponding Euler-Lagrange equation reads

$$\Delta H + (2H + c_0)\left(H^2 - \frac{c_0}{2}H - K\right) - \frac{\lambda}{k_c}H = -\frac{p}{2k_c}\,\cdot \tag{4}$$

• In 2D string theory and 2D gravity based on the Polyakov integral over surfaces [Polyakov, 1981], the Willmore functional (1) is known as the Polyakov's extrinsic action.

• Let $(x^1, x^2, x^3)$ be a fixed right-handed rectangular Cartesian coordinate system in the 3-dimensional Euclidean space $\mathbb{R}^3$ in which a surface $\mathcal{S}$ is immersed, and let this surface be given in Monge representations, i.e. by the equation

$$\mathcal{S} : x^3 = w(x^1, x^2), \quad (x^1, x^2) \in \Omega \subset \mathbb{R}^2 \tag{5}$$

where $w : \mathbb{R}^2 \to \mathbb{R}$ is a single-valued and smooth function possessing as many derivatives as may be required on the domain $\Omega$. Let us take $x^1, x^2$ to serve as Gaussian coordinates on the surface $\mathcal{S}$.

• Then the components of the first $g_{\alpha\beta}$, second $b_{\alpha\beta}$ fundamental tensor, and the alternating tensor $\varepsilon^{\alpha\beta}$ of $\mathcal{S}$ are given by the expressions

$$g_{\alpha\beta} = \delta_{\alpha\beta} + w_\alpha w_\beta, \quad b_{\alpha\beta} = g^{-1/2} w_{\alpha\beta}, \quad \varepsilon^{\alpha\beta} = g^{-1/2} e^{\alpha\beta} \tag{6}$$

$$g = \det(g_{\alpha\beta}) = 1 + (w_1)^2 + (w_2)^2 \tag{7}$$

$\delta_{\alpha\beta}$ is the Kronecker delta symbol, $e^{\alpha\beta}$ is the alternating symbol and $w_{\alpha_1 \dots \alpha_k}$ $(k = 1, 2, ...)$ denote the $k$-th order partial derivatives of the function $w$ with respect to the variables $x^1$ and $x^2$.

- The mean $H$ and Gaussian $K$ curvatures of the surface $\mathcal{S}$ are given as follows

$$H = \frac{1}{2} g^{\alpha\beta} b_{\alpha\beta} = \frac{1}{2} g^{-3/2} \left( \delta^{\alpha\beta} w_{\alpha\beta} + e^{\alpha\mu} e^{\beta\nu} w_{\alpha\beta} w_\mu w_\nu \right) \tag{8}$$

$$K = \frac{1}{2} \varepsilon^{\alpha\mu} \varepsilon^{\beta\nu} b_{\alpha\beta} b_{\mu\nu} = \frac{1}{2} g^{-2} e^{\alpha\mu} e^{\beta\nu} w_{\alpha\beta} w_{\mu\nu} \tag{9}$$

where

$$g^{\alpha\beta} = g^{-1} \left( \delta^{\alpha\beta} + e^{\alpha\mu} e^{\beta\nu} w_\mu w_\nu \right) \tag{10}$$

are the contravariant components of the first fundamental tensor.

- The Willmore functional (1) reads

$$\mathcal{W} = \int \int_\Omega L \, \mathrm{d}x^1 \mathrm{d}x^2, \quad L = \frac{1}{4} g^{-5/2} \left( \delta^{\alpha\beta} w_{\alpha\beta} + e^{\alpha\mu} e^{\beta\nu} w_{\alpha\beta} w_\mu w_\nu \right)^2 \tag{11}$$

- Here and in what follows, Greek indices have the range 1, 2, and the usual summation convention over a repeated index is employed.

- The application of the Euler operator

$$E=\frac{\partial}{\partial\, w}-D_\mu\frac{\partial}{\partial\, w_\mu}+D_\mu D_\nu\frac{\partial}{\partial\, w_{\mu\nu}}-\cdots \tag{12}$$

$$D_\alpha=\frac{\partial}{\partial\, x^\alpha}+w_\alpha\frac{\partial}{\partial\, w}+w_{\alpha\mu}\frac{\partial}{\partial\, w_\mu}+w_{\alpha\mu\nu}\frac{\partial}{\partial\, w_{\mu\nu}}+w_{\alpha\mu\nu\sigma}\frac{\partial}{\partial\, w_{\mu\nu\sigma}}+\cdots$$

on the Lagrangian density $L$ of the Willmore functional leads, after taking into account

$$\Delta=g^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha\partial x^\beta}+g^{-1/2}\frac{\partial}{\partial x^\alpha}\left(g^{1/2}g^{\alpha\beta}\right)\frac{\partial}{\partial x^\beta}$$

to the Willmore equation (2), which takes the form

$$\mathcal{E}\equiv(1/2)g^{-1/2}g^{\alpha\beta}g^{\mu\nu}w_{\alpha\beta\mu\nu}+\Phi\left(x_1,x_2,w,w_1,\ldots,w_{222}\right)=0 \tag{13}$$

where $\Phi\left(x_1,x_2,w,w_1,\ldots,w_{222}\right)$ is a differential function of the independent and dependent variables and the derivatives of the dependent variable up to third order.

translations

$$\mathbf{v}_1 = \frac{\partial}{\partial x^1}, \quad \mathbf{v}_2 = \frac{\partial}{\partial x^2}, \quad \mathbf{v}_3 = \frac{\partial}{\partial w}$$

rotations

$$\mathbf{v}_4 = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}, \quad \mathbf{v}_5 = x^1 \frac{\partial}{\partial w} - w \frac{\partial}{\partial x^1}, \quad \mathbf{v}_6 = x^2 \frac{\partial}{\partial w} - w \frac{\partial}{\partial x^2}$$

dilatation

$$\mathbf{v}_7 = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + w \frac{\partial}{\partial w}$$

inversions

$$\mathbf{v}_8 = \left[(x^1)^2 - (x^2)^2 - w^2\right] \frac{\partial}{\partial x^1} + 2x^1 x^2 \frac{\partial}{\partial x^2} + 2x^1 w \frac{\partial}{\partial w}$$

$$\mathbf{v}_9 = 2x^2 x^1 \frac{\partial}{\partial x^1} + \left[(x^2)^2 - (x^1)^2 - w^2\right] \frac{\partial}{\partial x^2} + 2x^2 w \frac{\partial}{\partial w}$$

$$\mathbf{v}_{10} = 2x^1 w \frac{\partial}{\partial x^1} + 2x^2 w \frac{\partial}{\partial x^2} + \left[w^2 - (x^2)^2 - (x^1)^2\right] \frac{\partial}{\partial w}$$

• The following Propositions clarify the invariance properties of the Willmore equation relative to one-parameter Lie groups of point transformations of $\mathbb{R}^3$. The coordinates $(x^1, x^2, w)$ on $\mathbb{R}^3$ represent the involved independent and dependent variables $x^1, x^2$ and $w$, respectively. The results are obtained using Lie infinitesimal technique.

**Proposition 1.** The 10-parameter Lie group $G_{SCT}$ of special conformal transformations in $\mathbb{R}^3$ (whose basic generators are $\mathbf{v}_j$, $j = 1 \dots 10$) is the largest group of point (geometric) transformations of the involved independent and dependent variables that a generic equation of form (13) could admit.

**Proposition 2.** In Monge representation, the Willmore equation admits all the transformations of the group $G_{SCT}$.

**Remark.** All vector fields $\mathbf{v}_j$, $j = 1, \dots, 10$ are variational symmetries of the Willmore equation, i.e., infinitesimal divergence symmetries of the Willmore functional. Hence, Noether's theorem implies the existence of ten linearly independent conservation laws that hold on the smooth solutions of the Willmore equation.

• Once a group $G$ is found to be a symmetry group of a given differential equation, it is possible to look for the so-called group-invariant ($G$-invariant) solutions of this equation – the solutions, which are invariant under the transformations of the symmetry group $G$.

• The main advantage that one can gain when looking for this kind of particular solutions of the given differential equation consists in the fact that each group-invariant solution is determined by a reduced equation obtained by a symmetry reduction of the original one and involving less independent variables than the latter.

• Let $G(\mathbf{v})$ be a one parameter group generated by a vector field $\mathbf{v}$ belonging to the Lie algebra $L_{SCT}$, that is $\mathbf{v}$ is a linear combination of the vector fields $\mathbf{v}_j$, $j = 1 \ldots 10$,

$$\mathbf{v} = \sum_{j=1}^{10} c_j \mathbf{v}_j \tag{14}$$

where $c_j$, $j = 1 \ldots 10$, are real numbers – the components of the vector field $\mathbf{v}$ with respect to the basic vector fields $\mathbf{v}_j$.

• Then, $G(\mathbf{v})$ is a symmetry group of the Willmore equation and so one can look for the $G(\mathbf{v})$-invariant solutions of this equation. For that purpose, first one should find a complete set of functionally independent invariants of the group $G(\mathbf{v})$. In the present case this is a set of two functionally independent functions $I_\alpha\left(x^1, x^2, w\right)$ such that

$$\mathbf{v}I_\alpha = 0$$

the vector field $\mathbf{v}$ being regarded here as an operator acting on the functions $\zeta : \mathbb{R}^3 \to \mathbb{R}$. Then, if the necessary condition for the existence of group invariant solutions is satisfied, which in the present case reads

$$\operatorname{rank}\left(\frac{\partial I_\alpha}{\partial w}\right) = 1 \tag{15}$$

assuming that $\partial I_1/\partial w \neq 0$, one can seek the $G(\mathbf{v})$-invariant solutions in the form

$$U = U(s), \quad U = I_1, \quad s = I_2. \tag{16}$$

• The $G(\mathbf{v}_4)$-invariant solutions of the Willmore equation are sought in the form

$$w = w(r), \qquad r = \sqrt{(x^1)^2 + (x^2)^2}.$$

Note that $r$ and $w$ are two functionally independent invariants of the operator $\mathbf{v}_4$ generating the one-parameter group of rotations admitted by the equation considered. After such a symmetry reduction, the Willmore equation (2) takes the form

$$\begin{aligned}
\mathcal{R} \quad \equiv \quad & (2\,r^3 + 4\,r^3\,w_r^2 + 2\,r^3\,w_r^4)w_{rrrr} \\
& +(4\,r^2 + 8\,r^2\,w_r^2 + 4\,r^2\,w_r^4 - 20\,r^3\,w_r w_{rr} - 20\,r^3\,w_r^3 w_{rr})w_{rrr} \\
& -5r^2\,(3\,w_r + 3\,w_r^3 + r\,w_{rr} - 6\,r\,w_r^2 w_{rr})w_{rr}^2 \\
& +(r\,w_r^6 - 2\,r - 3\,r\,w_r^2)w_{rr} + 2\,w_r + 7w_r^3 + 9\,w_r^5 + 5\,w_r^7 + w_r^9 = 0
\end{aligned}$$

where

$$w_r = \frac{\mathrm{d}w}{\mathrm{d}r},\ w_{rr} = \frac{\mathrm{d}^2 w}{\mathrm{d}r^2},\ w_{rrr} = \frac{\mathrm{d}^3 w}{\mathrm{d}r^3},\ w_{rrrr} = \frac{\mathrm{d}^4 w}{\mathrm{d}r^4}.$$

Consider the following normal system of two ordinary differential equations

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}r} &= v\\ \frac{\mathrm{d}v}{\mathrm{d}r} &= \pm\frac{1}{r}\left(v^2+1\right)\sqrt{v^2+2\,\omega\sqrt{v^2+1}}\end{aligned} \tag{17}$$

which is equivalent to the single second-order equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}r^2} = \pm\frac{1}{r}\left[\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2+1\right]\sqrt{\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2+2\,\omega\sqrt{\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2+1}}\,. \tag{18}$$

Substituting (18) into the expression $\mathcal{R}$ one obtains $\mathcal{R}=0$ and thus shows that each solution of system (17) or equation (18) is a solution of the reduced Willmore equation $\mathcal{R}=0$. In this way, we have obtained a special class of axisymmetric solutions to the Willmore equation, i.e., a special class of axially symmetric Willmore surfaces.

Substitutions

$$u = \sqrt{v^2 + 1}, \quad \rho = \ln r$$

transform system (17) to the following one

$$\frac{\mathrm{d}w}{\mathrm{d}\rho} = e^{\rho}\sqrt{u^2 - 1} \tag{19}$$

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\rho}\right)^2 = u^2\left(u^2 + 2\omega u - 1\right)\left(u^2 - 1\right). \tag{20}$$

In terms of a new variable $t$, relation (20) may be written in the form

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 = P\left(u\right), \quad P\left(u\right) = \left(u^2 + 2\omega u - 1\right)\left(u^2 - 1\right) \tag{21}$$

$$\left(\frac{\mathrm{d}\rho}{\mathrm{d}t}\right)^2 = \frac{1}{u^2} \tag{22}$$

Using the standard procedure [Whittaker and Watson, 1922], one can express the solution $u(t)$ of equation (21) in the form

$$u\left(t\right)=\frac{\operatorname{sn}^2\left(\lambda t,k\right)\left(\sqrt{\omega^2+1}-\omega+1\right)-2\sqrt{\omega^2+1}}{\operatorname{sn}^2\left(\lambda t,k\right)\left(\sqrt{\omega^2+1}+\omega+1\right)-2\sqrt{\omega^2+1}} \tag{23}$$

where

$$\lambda=\sqrt[4]{\omega^2+1},\quad k=\frac{1}{\sqrt{2}}\sqrt{1+\frac{1}{\sqrt{\omega^2+1}}}\,.$$

Consequently, using expressions (22) and (23), one can write down the solution $\rho\left(t\right)$ of equation (22) in the form

$$\rho\left(t\right)=\lambda\left(\sqrt{\omega^2+1}-\omega\right)t+\frac{2\omega\Pi\left(\frac{\omega+\sqrt{\omega^2+1}+1}{2\sqrt{\omega^2+1}},\operatorname{am}\left(\lambda t,k\right),k\right)}{\sqrt{\omega^2+1}+\omega+1}\,. \tag{24}$$

Finally, rewriting Eq. (19) in the form

$$\frac{\mathrm{d}w}{\mathrm{d}t} = e^{\rho(t)}\frac{\sqrt{u\left(t\right)^2-1}}{u\left(t\right)}$$

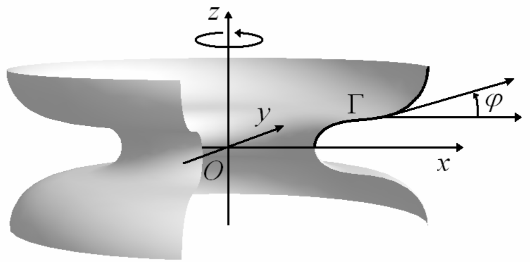
we obtain

$$w\left(t\right) = \int \frac{e^{\rho(t)}}{u\left(t\right)}\sqrt{u\left(t\right)^2-1}\mathrm{d}t + c \tag{25}$$

where $c$ is a constant.

# Axisymmetric Equilidrium Shapes

## Sketch of a Surface of Revolution




Sketch of a surface of revolution obtained by revolving around the $z$-axis a plane curve $\Gamma$ laying in the $xOz$-plane, which is defined by the graph $(x, z(x))$ of a function $z = z(x)$. Here, $\varphi$ is the (tangent) slope angel.
Suppose that a part of an axisymmetrically deformed SWCNT admits graph parametrization. This means that it may be thought of as a surface of revolution obtained by revolving around the $z$-axis a plane curve $\Gamma$ laying in the $xOz$-plane, which is determined by the graph $(x, z(x))$ of a function $z = z(x)$.

# Axisymmetric Equilidrium Shapes

## Shape Equation

For each such surface the general shape equation (4) reduces to the following nonlinear third-order ordinary differential equation

$$
\begin{aligned}
\cos^3\varphi \frac{\mathrm{d}^3\varphi}{\mathrm{d}x^3} \quad = \quad & 4\sin\varphi\cos^2\varphi\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2}\frac{\mathrm{d}\varphi}{\mathrm{d}x} - \cos\varphi\left(\sin^2\varphi - \frac{1}{2}\cos^2\varphi\right)\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^3 \\
& + \frac{7\sin\varphi\cos^2\varphi}{2x}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}x}\right)^2 - \frac{2\cos^3\varphi}{x}\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} \\
& + \left(\frac{\lambda}{k_c} + \frac{c_0^2}{2} - \frac{2c_0\sin\varphi}{x} - \frac{\sin^2\varphi - 2\cos^2\varphi}{2x^2}\right)\cos\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}x} \\
& + \left(\frac{\lambda}{k_c} + \frac{c_0^2}{2} - \frac{\sin^2\varphi + 2\cos^2\varphi}{2x^2}\right)\frac{\sin\varphi}{x} - \frac{p}{k_c}
\end{aligned}
\tag{26}
$$

(derived in [Hu & Ou-Yang, 1993]) where $\varphi$ is the angle between the $x$-axis and the tangent vector to the profile curve $\Gamma$, i.e., the tangent (slope) angel, considered as a function of the variable $x$.

[Naito at al., 1995] discovered that the shape equation (26) has the following class of exact solutions

$$\sin\varphi = ax + b + dx^{-1}, \tag{27}$$

provided that $a$, $b$ and $d$ are real constants, which meet the conditions

$$\frac{p}{k_c} - 2a^2c_0 - 2a\left(\frac{c_0^2}{2} + \frac{\lambda}{k_c}\right) = 0, \tag{28}$$

$$b\left(2ac_0 + \frac{c_0^2}{2} + \frac{\lambda}{k_c}\right) = 0, \tag{29}$$

$$b\left(b^2 - 4ad - 4c_0d - 2\right) = 0, \tag{30}$$

and

$$d\left(b^2 - 4ad - 2c_0d\right) = 0. \tag{31}$$

# Exact Solutions of the Shape Equation

Six types of solutions of form (27) to Eq. (26) can be distinguished on the ground of conditions (28) – (31) depending on the values of $c_0$, $\lambda$ and $p$.
Case A. If $c_0 = 0$, $\lambda = 0$, $p = 0$, then the solutions to Eq. (26) of the form (27) are $\sin\varphi = ax$, $\sin\varphi = ax \pm \sqrt{2}$ and $\sin\varphi = dx^{-1}$, the respective surfaces being spheres, Clifford tori and catenoids.
Case B. If $c_0 = 0$, $\lambda \neq 0$, $p = 0$, then the solutions of the considered type reduces to $\sin\varphi = dx^{-1}$ (catenoids).
Case C. If $c_0 = 0$, $\lambda \neq 0$, $p \neq 0$ and $p = 2a\lambda$, then only one branch of the regarded solutions remains, namely $\sin\varphi = ax$ (spheres).
Case D. If $c_0 \neq 0$, $\lambda = 0$, $p = 0$, then one arrives at the whole family of Delaunay surfaces corresponding to the solutions of the form

$$\sin\varphi = -\frac{1}{2}c_0 x + \frac{d}{x}\,\cdot \tag{32}$$

Case E. If $c_0 \neq 0$, $\lambda \neq 0$, $p = 0$ and

$$\frac{\lambda}{k_c} = -\frac{1}{2}c_0\,(2a + c_0)\,,$$

one gets only solutions of the form $\sin\varphi = ax$ (spheres).

# Exact Solutions of the Shape Equation

Case F. If $c_0 \neq 0$, $\lambda \neq 0$, $p \neq 0$, then four different types of solutions of form (27) to Eq. (26) are encountered: (a) $\sin\varphi = ax$ (spheres) if

$$\frac{p}{k_c} = 2a\left(\frac{\lambda}{k_c} + ac_0 + \frac{c_0^2}{2}\right); \tag{33}$$

(b) $\sin\varphi = ax \pm \sqrt{2}$ (Clifford tori) if

$$\frac{p}{k_c} = -2a^2c_0, \quad \frac{\lambda}{k_c} = -\frac{1}{2}c_0\left(4a + c_0\right); \tag{34}$$

(c) solutions of the form (32) (Delaunay surfaces) if

$$p + c_0\lambda = 0; \tag{35}$$

(d) solutions of the form

$$\sin\varphi = -\frac{1}{4}c_0\left(b^2 + 2\right)x + b - \frac{1}{c_0 x}, \tag{36}$$

which take place provided that

$$\frac{p}{k_c} = -\frac{1}{8}c_0^3\left(b^2 + 2\right)^2, \qquad \frac{\lambda}{k_c} = \frac{1}{2}c_0^2\left(b^2 + 1\right). \tag{37}$$

# Parametric Equations of the Unduloid-Like Surfaces

Below, we derive the parametric equations of the surfaces corresponding to the solutions of form (36) to Eq. (26).

First, it is clear that the variable $x$ must be strictly positive or negative, otherwise the right-hand side of Eq. (27) is both undefined and its absolute value is greater than one, which is in contradiction with the sin-function appearing in the left-hand side of this relation.

Next, according to the meaning of the tangent angle

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \tan\varphi \tag{38}$$

which for the foregoing class of solutions (36) implies

$$\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 = \frac{\left[b - \frac{1}{c_0 x} - \frac{1}{4}c_0\left(b^2+2\right)x\right]^2}{1 - \left[b - \frac{1}{c_0 x} - \frac{1}{4}c_0\left(b^2+2\right)x\right]^2}\,. \tag{39}$$

# Parametric Equations of the Unduloid-Like Surfaces

In terms of an appropriate new variable $t$, relation (39) may be written in the form

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = -\frac{1}{u^2}Q_1(x)Q_2(x) \tag{40}$$

$$\left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2 = \frac{1}{4u^2}\left(Q_1(x)+Q_2(x)\right)^2 \tag{41}$$

where

$$u = -\frac{4}{c_0\left(2+b^2\right)^{3/4}}$$

$$Q_1(x) = x^2 - \frac{4\,(b+1)}{c_0\,(b^2+2)}x + \frac{4}{c_0^2\,(b^2+2)} \tag{42}$$

$$Q_2(x) = x^2 - \frac{4\,(b-1)}{c_0\,(b^2+2)}x + \frac{4}{c_0^2\,(b^2+2)}\,. \tag{43}$$

# Parametric Equations of the Unduloid-Like Surfaces

It should be noticed that the roots of the polynomial $Q(x) = Q_1(x)Q_2(x)$ read

$$\alpha = \frac{2\,\mathrm{sign}\,(b)}{c_0\sqrt{b^2+2}}\,\frac{h-1}{h+1}, \qquad \beta = \frac{2\,\mathrm{sign}\,(b)}{c_0\sqrt{b^2+2}}\,\frac{h+1}{h-1} \tag{44}$$

$$\gamma = \frac{4b}{c_0\left(b^2+2\right)} - \frac{\alpha+\beta}{2} + i\,\frac{2\sqrt{2|b|+1}}{c_0\left(b^2+2\right)}$$

$$\delta = \frac{4b}{c_0\left(b^2+2\right)} - \frac{\alpha+\beta}{2} - i\,\frac{2\sqrt{2|b|+1}}{c_0\left(\epsilon^2+2\right)}$$

where

$$h = \sqrt{\frac{1+|b|+\sqrt{2+b^2}}{1+|b|-\sqrt{2+b^2}}}\,. \tag{45}$$

Hence, Eq. (40) has real-valued solutions if and only if at least tow of these roots are real and different. Evidently, the roots $\gamma$ and $\delta$ can not be real, but $\alpha$ and $\beta$ are real provided that $|b| > 1/2$ as follows be relations (44) and (45).

# Parametric Equations of the Unduloid-Like Surfaces

Now, using the standard procedure for handling elliptic integrals (see [Whittaker and Watson, 1922, 22.7]), one can express the solution $x(t)$ of equation (40) in the form

$$x(t) = \frac{2\,\mathrm{sign}\,(b)}{c_0\sqrt{b^2+2}}\left(1 - \frac{2h}{h + \mathrm{cn}(t,k)}\right) \tag{46}$$

where

$$k = \sqrt{\frac{1}{2} - \frac{3}{4\sqrt{2+b^2}}}\,.$$

Consequently, using expressions (42) and (43), one can write down the solution $z(t)$ of equation (41) in the form

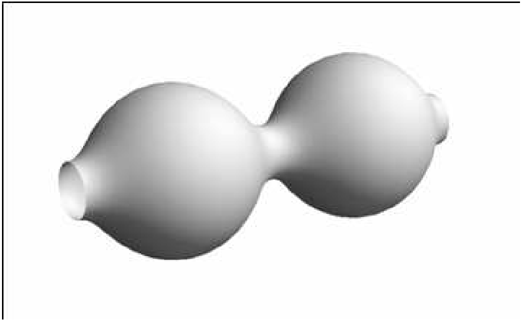
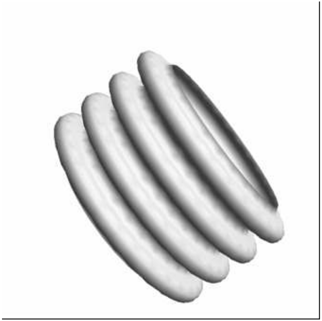
$$z\,(t) = \frac{1}{u}\int\left[x^2(t) - \frac{4\,b\,x(t)}{c_0\,(b^2+2)} + \frac{4}{c_0^2\,(b^2+2)}\right]\mathrm{d}t. \tag{47}$$

# Parametric Equations of the Unduloid-Like Surfaces

Finally, performing the integration in the right-hand-side of Eq. (47), one obtains

$$z(t) = u\left[\mathrm{E}(\mathrm{am}(t,k),k) - \frac{\mathrm{sn}(t,k)\,\mathrm{dn}(t,k)}{h+\mathrm{cn}(t,k)} - \frac{t}{2}\right]. \tag{48}$$

Thus, for each couple of values of the parameters $c_0$ and $b$, (46) and (48) are the sought parametric equations of the contour of an axially symmetric unduloid-like surface corresponding to the respective solution of the membrane shape equation (26) of form (36).

(a)  (b) 

Unduloid-like surfaces obtained using the parametric equations (46) and (48) for: (a) $p/k_c = 1.75$, (b) $p/k_c = 12.1$.

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