Locality of the Conservation Laws for the Soliton Equations Related to Caudrey-Beals-Coifman system

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1. Introduction

This talk is about the nonlinear evolution equations (NLEEs) of soliton type that is, equations admitting **Lax representation** $[L, A] = 0$, where L, A are linear operators on ∂_x, ∂_t depending also on some set of functions $q = (q_\alpha(x,t))$, $1 \le \alpha \le s$ (called 'potentials') and a spectral parameter λ. The equation $[L, A] = 0$ is then equivalent to a system of the type $(q_\alpha)_t = f_\alpha(q, q_x, \dots)$.
Hierarchy of NLEEs related to $L\psi = 0$ (auxiliary linear problem) – the evolution equations obtain changing A.
Integration. Most of the schemes share the property: the Lax representation permits to pass from the original evolution defined by $[L, A] = 0$ to the evolution of some spectral data related to the problem $L\psi = 0$: (Faddeev, Takhtadjian 1987; Gerdjikov, Vilasi, Yanovski 2008). The following auxiliary problem and its generalizations are perhaps the best known auxiliary problems:

$$L\psi = (\mathrm{i}\partial_x + q(x) - \lambda J)\,\psi = 0 \qquad (1)$$

When q is 2×2 off-diagonal matrix and $J = \mathrm{diag}\,(1, -1)$ this is the classi-

cal **Zakharov-Shabat problem**. Generalizations followed immediately:
a) Considering the system on some other Lie algebra with higher rank than sl(2) but J remains to be real. In this case we speak about **Generalized Zakharov-Shabat system (GZS system)**.(Zakharov, Manakov, Novikov, Pitaevski 1981, Gerdjikov, Kulish 1981; Gerdjikov 1986).
b) Considering the system on the algebra sl(n) with J complex. (Caudrey 1982, Beals and Coifman 1984, 1985; Beals, Sattinger 1991; Zhou 1989).
c) The final generalization is obtained when $q(x)$ *and* J *belong to a fixed simple Lie algebra* $\mathfrak{g}$ *in some finite dimensional irreducible representation*, (Gerdjikov, Yanovski, 1994). *The element* J *should be regular, that is* $\ker \mathrm{ad}_J$ *(*$\mathrm{ad}_J(X) \equiv [J, X]$, $X \in \mathfrak{g}$*) is a Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}$. $q(x)$ *belongs to the orthogonal complement* $\mathfrak{h}^{\perp} = \bar{\mathfrak{g}}$ *of* $\mathfrak{h}$ *with respect to the Killing form:* $\langle X, Y \rangle = \mathrm{tr}\,(\mathrm{ad}_X \mathrm{ad}_Y)$; $X, Y \in \mathfrak{g}$. Thus $q(x) = \sum_{\alpha\in\Delta} q_\alpha E_\alpha$ where E_α are the root vectors; Δ is the root system of $\mathfrak{g}$. The scalar functions $q_\alpha(x)$ are defined on $\mathbb{R}$, are *complex valued, smooth and tend to zero as* $x \to \pm\infty$. We shall call the above auxiliary problem **Caudrey-Beals-Coifman system (CBC system)**.

Complex J case becomes necessary to study also because because if one has Mikhailov type reductions (Mikhailov 1979,1981; Lombardo and Mikhailov 2002) we just cannot limit ourselves with real J, both in the case of caanonical and pole gauge, for example (Grahhovski 2002, Grahovski 2002,2003) (canonical gauge); (Gerdjikov, Mikhailov and Valchev 2010; Valchev 2011, Gerdjikov, Grahovski, Mikhailov and Valchev, 2011; Yanovski 2011 JMP) (pole gauge).

The present talk is about the integrals of motion of the NLEEs related to the CBC system. The fact that for the CBC system there are r local series (r is the rank of $\mathfrak{g}$) of conservation laws is well known. (Drinfeld, and Sokolov, 1984). In the Λ-operator approach to the NLEEs related to the Zakharov-Shabat system this result is reproduced, though the question of locality remained neglected. Let us note, that the derivation of these formulae depends both on the so-called Generalized Fourier Expansions and on the analyticity properties of some other expressions. Then since all these change significantly for the CBC system one must see how to change the proof. We intend to address these issues in the present note.

2. Fundamental analytical solutions for the CBC system (FAS)

2.1 General properties

The fundamental solutions to the CBC system and their analytic properties play crucial role in the theory of the NLEEs related to it. It turns out that it is better to study not the CBC system, but the system

$$\mathrm{i}\partial_x m + q(x)m - \lambda J m + \lambda m J = 0 \quad \lim_{x\to -\infty} m = \mathbf{1}_V \tag{2}$$

If ψ is a solution to the CBC problem, $m(x,\lambda) = \psi(x,\lambda)\exp \mathrm{i}\lambda J x$ satisfies (2) so the two problems are equivalent. We have

Theorem 0.1 *Suppose that for fixed λ the bounded fundamental solution $m(x,\lambda)$, satisfying the equation (1) exists. Suppose that λ does not belong to the bunch of straight lines $\Sigma = \cup_{\alpha\in\Delta} l_\alpha$ where*

$$l_\alpha = \{\lambda : \mathrm{Im}(\lambda\alpha(J)) = 0\} \tag{3}$$

Then the solution $m(x,\lambda)$ is unique. (In the above Im *denotes the imaginary part).*

Consider now **the set of lines** $\Sigma = \cup_{\alpha\in\Delta} l_\alpha$. The connected components of $\mathbb{C}\setminus\Sigma$ are open sectors in the λ-plain. In every such sector either $\mathrm{Im}[\lambda(\gamma_1-\gamma_2)(J)]$, $\gamma_1,\gamma_2\in\Gamma$ – the set of weights, is identically zero or it has the same sign. We denote these sectors by Ω_ν and order them anti-clockwise. Clearly ν takes values from 1 to some even number $2M$ and

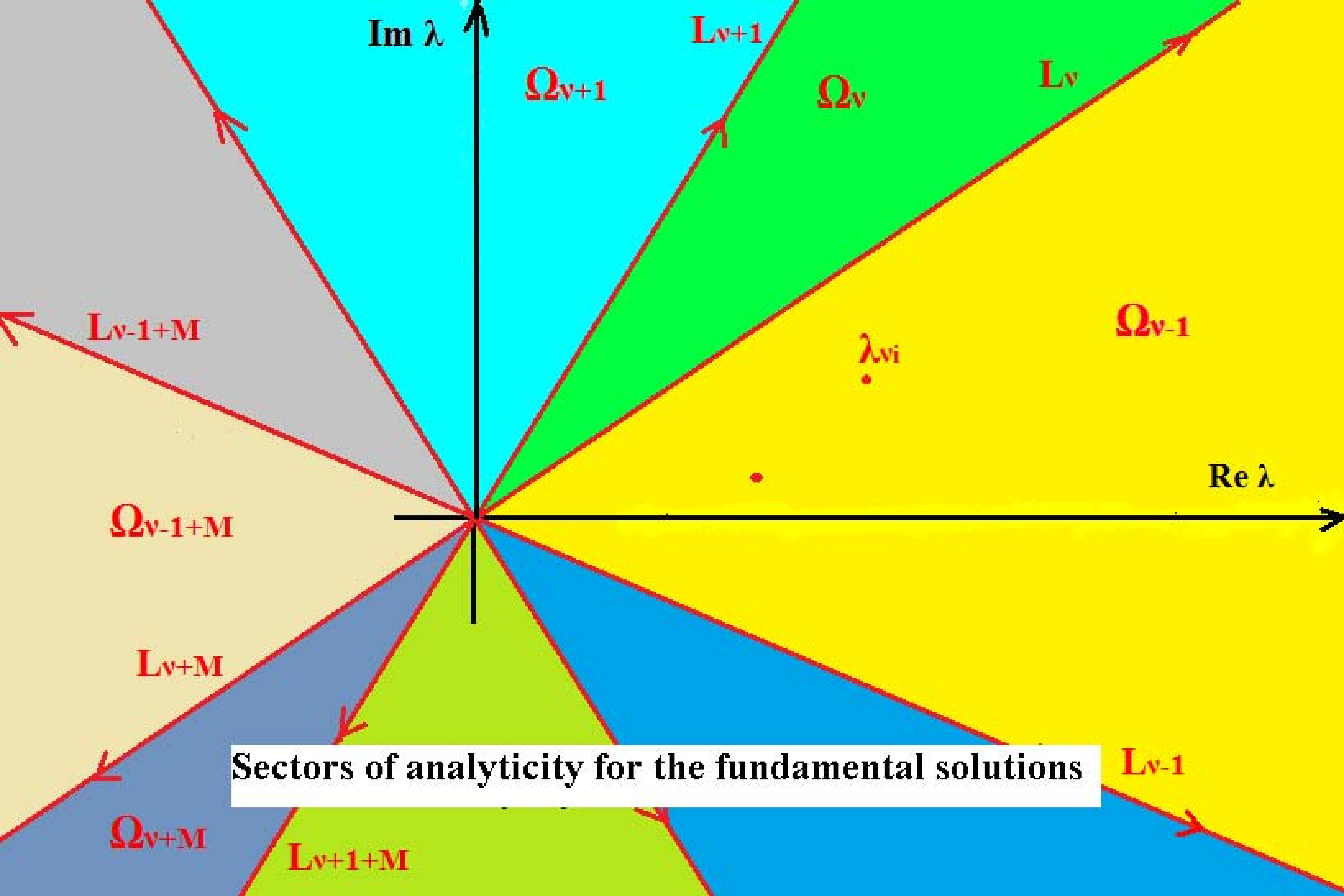
$$\mathbb{C}\setminus\Sigma = \bigcup_{\nu=1}^{2M}\Omega_\nu, \quad \Omega_\nu\bigcap\Omega_\mu=\emptyset, \qquad \nu\neq\mu \tag{4}$$

The **boundary of the sector** Ω_ν **consists of two rays** – L_ν **and** $L_{\nu+1}$ **(**L_ν **comes before** $L_{\nu+1}$ **when we turn anti-clockwise) so that** $\bar\Omega_\nu\cap\bar\Omega_{\nu-1}=L_\nu$. Of course, we understand the number ν modulo $2M$. In the ν-th sector we introduce the ordering :

$$\begin{aligned} \alpha \geq_\nu \beta \quad &\text{iff} \quad \mathrm{Im}\lambda(\alpha-\beta)(J)\geq 0 \\ \alpha >_\nu \beta \quad &\text{iff} \quad \mathrm{Im}\lambda(\alpha-\beta)(J) > 0 \end{aligned} \tag{5}$$

In each sector we have the sets of positive and negative roots $\Delta_\nu^\pm$ with respect to the ν-ordering and we define $\delta_\nu^\pm = \Delta_\nu^\pm\cap\delta_\nu$ where

$$\delta_\nu = \{\alpha\in\Delta : \mathrm{Im}(\lambda\alpha(J)) = 0 \text{ for } \lambda\in L_\nu\} \tag{6}$$



Sectors of analyticity for the fundamental solutions

Then a system of integral equations can be written in every sector Ω_ν, $\nu = 1, 2, \ldots, 2M$.:

$$\begin{aligned} &\langle \alpha | m | \beta \rangle = \langle \alpha | \beta \rangle + \mathrm{i} \int\limits_{-\infty}^{x} \langle \alpha | q(y) m(y) | \beta \rangle e^{-\mathrm{i}\lambda(\alpha-\beta)(J)(x-y)} \mathrm{d}y \\ &\text{for} \quad \alpha - \beta \leq_\nu 0, \quad \alpha, \beta \in \Gamma \\ &\langle \alpha | m \beta | \rangle = \mathrm{i} \int\limits_{+\infty}^{x} \langle \alpha | q(y) m(y) | \beta \rangle e^{-\mathrm{i}\lambda(\alpha-\beta)(J)(x-y)} \mathrm{d}y \\ &\text{for} \quad \alpha - \beta >_\nu 0, \quad \alpha, \beta \in \Gamma \end{aligned} \tag{7}$$

where Γ is the set of weights in a given finite dimensional irreducible representation of $\mathfrak{g}$.

Without going into details which could be found in (Gerdjikov and Yanovski 1994), we only state that for large classes of potentials $q(x) = \sum_{\alpha\in\Delta} q_\alpha(x) E_\alpha$ for $\lambda \in \Omega_\nu$ there exist fundamental solution $m_\nu(x, \lambda)$ which is analytic in Ω_ν except for finite number of poles. The set Σ coincides with the continuous spectrum of the problem and the points where the solutions m_ν have poles – give the discrete spectrum. For example if one defines $\|q\|_1 = \sum_{\alpha\in\Delta} \int\limits_{-\infty}^{+\infty} |q_\alpha(x)|\mathrm{d}x$ then this is a norm on the space of integrable potentials $L^1(\bar{\mathfrak{g}}, \mathbb{R})$ and it can be shown that **if** $\|q\|_1 < 1$ **there is no discrete spectrum**.

For potentials belonging to $L^1(\bar{\mathfrak{g}}, \mathbb{R})$ the spectrum could be much more complicated, involving both discrete and continuous spectrum. (Beals and Coifman 1985,1987; Zhou 1989).

One should mention also that **for fixed** ν **the solution** $m_\nu(x,\lambda)$ **allows continuous extension to the closure** $\overline{\Omega}_\nu$ **of the sector** Ω_ν **and if** q **has integrable derivatives up to the** n**-th order then** $m(x,\lambda) = \mathbf{1}_V + \sum_{i=1}^{n} a_i(x)\lambda^{-i} + o(\lambda^{-(n+1)})$ **when** $|\lambda| \to \infty$**, uniformly in** $x \in \mathbb{R}$**, where the coefficients** $a_i(x)$ **are calculated through** q **and its x-derivatives**.

Consequently, for small potentials in any fixed Ω_ν we have unique solution $m_\nu(x,\lambda)$ with the stated properties, anaytic in Ω_ν and allowing continuation to its boundary – the rays L_ν and $L_{\nu+1}$.

Knowing the fundamental solutions $m_\nu(x,\lambda)$ one can construct the fundamental solutions (FAS) $\chi_\nu(x,\lambda)$ of the CBC system setting $\chi_\nu(x,\lambda) = m_\nu(x,\lambda)e^{iJx\lambda}$ and using them to build up the spectral theory of the operator L in any faithful representation of $\mathfrak{g}$. This includes of course completeness relations. The details of this can be found in (Gerdjikov and Yanovski 1994). Below we shall just recall the main results but first we need a few words about the Inverse Scattering Method for the CBC system.

2.2 Elements of Inverse Scattering Method

Let us introduce some notation. First, we write an inverse putting 'hat' over the corresponding symbol. Next, if we have a function $f(\lambda)$ that is section analytic on the sectors Ω_ν and in each sector Ω_ν it allows extension by continuity to the boundary of the sector, then for λ on the ray L_ν we shall denote the restriction of f in the sector Ω_ν by $f_\nu(\lambda)$, and for λ belonging to the ray L_ν the limit from Ω_ν by $f_\nu^+(\lambda)$ and from $\Omega_{\nu-1}$

by $f_\nu^-(\lambda)$. Next, let

$$\begin{aligned} &\delta_\nu^\pm = \Delta_\nu^\pm \cap \delta_\nu, \quad \delta_\nu = \{\alpha \in \Delta : \mathrm{Im}(\lambda\alpha(J)) = 0 \text{ for } \lambda \in L_\nu\} \\ &\mathfrak{g}_\nu\text{– the semisimple algebra with root system } \delta_\nu \\ &\pi_\nu\text{– the system of simple roots for } \mathfrak{g}_\nu \end{aligned} \tag{8}$$

Now we are ready to describe the jumps of the solutions $m(x,\lambda)$ on the rays L_ν. For $\lambda \in L_\nu$ they are given by:

$$m_\nu^+(x,\lambda) = m_\nu^-(x,\lambda) e^{-\mathrm{i}J\lambda x} g_\nu(\lambda) e^{\mathrm{i}J\lambda x} \tag{9}$$

$$g_\nu(\lambda) = \hat{S}_\nu^-(\lambda) S_\nu^+(\lambda) = \hat{D}_\nu^-(\lambda)\hat{T}_\nu^+(\lambda) T_\nu^-(\lambda) D_\nu^+(\lambda) \tag{10}$$

Here $S_\nu^\pm(\lambda)$, $T_\nu^\pm(\lambda)$, $D_\nu^\pm(\lambda)$ are defined by the asymptotic of $m_\nu^\pm(x,\lambda)$ when $x \to \pm\infty$:

$$\begin{aligned} S_\nu^\pm(\lambda) &= \lim_{x\to-\infty} \left(e^{\mathrm{i}\lambda Jx} m_\nu^\pm(x,\lambda) e^{-\mathrm{i}\lambda Jx}\right) = \lim_{x\to-\infty} e^{\mathrm{i}J\lambda x}\chi_\nu^\pm(x,\lambda) \\ T_\nu^\mp(\lambda) D_\nu^\pm(\lambda) &= \lim_{x\to+\infty} \left(e^{\mathrm{i}\lambda Jx} m_\nu^\pm(x,\lambda) e^{-\mathrm{i}\lambda Jx}\right) = \lim_{x\to+\infty} e^{\mathrm{i}J\lambda x}\chi_\nu^\pm(x,\lambda) \end{aligned} \tag{11}$$

One can write $S_\nu^\pm, T_\nu^\pm, D_\nu^\pm$ also into the form

$$S_\nu^\pm(\lambda) = \exp \sum_{\alpha\in\delta_\nu^+} s_{\nu,\alpha}^\pm(\lambda) E_{\pm\alpha}, \quad T_\nu^\pm(\lambda) = \exp \sum_{\alpha\in\delta_\nu^+} t_{\nu,\alpha}^\pm(\lambda) E_{\pm\alpha} \tag{12}$$

$$D_{\nu,\alpha}^\pm(\lambda) = \exp \sum_{\alpha\in\pi_\nu} d_{\nu,\alpha}^\pm(\lambda) H_\alpha \tag{13}$$

In other words $S_\nu^\pm, T_\nu^\pm, D_\nu^\pm$ belong to the subgroup G_ν with algebra $\mathfrak{g}_\nu$.

Consider the sets

$$\mathcal{T}_S = \bigcup_{\nu=1}^{2M} \{s_{\nu,\alpha}^\pm(\lambda) : \alpha \in \Delta_\nu^+, \lambda \in L_\nu\}, \mathcal{T}_T = \bigcup_{\nu=1}^{2M} \{t_{\nu,\alpha}^\pm(\lambda) : \alpha \in \Delta_\nu^+, \lambda \in L_\nu\}$$

The factors $D_\nu^\pm(\lambda)$ could be recovered from each of these sets. Each of the above sets could be chosen as a set of minimal scattering data. The Inverse scattering Method then consists of recovering the potential from the scattering data. It is important that the relations (9) could be regarded as Riemann-Hilbert problem related to the bunch of rays L_ν with canonical normalization at $\lambda = \infty$. The possibility to solve that problem underlies the Inverse Scattering techniques and in particular the so-called dressing method for finding the potential and therefore the so-called soliton solutions. These solutions correspond to the situation when we have only discrete spectrum.

Another sets of scattering data which is related to the expansions over the so-called adjoint solutions or Generalized Exponents in the Λ-operator approach or AKNS method. It has been introduced by (Ablowitz, Kaup, Newell and Segur, 1974), see (Gerdjikov, Vilasi, and Yanovski 2008) for comprehensive bibliography

3. Λ-operator approach to the soliton equations associated with the linear CBC system

3.1 Expansions over adjoint solutions

We define in each Ω_ν analytic solutions $\chi_\nu(x,\lambda)$ of (1) and then we set

$$e^\nu_\alpha(x,\lambda) = \pi_0(\chi_\nu(x,\lambda)E_\alpha\chi_\nu^{-1}(x,\lambda)), \quad \lambda \in \Omega_\nu \tag{14}$$

According to our agreement for $\lambda \in L_\nu$ we shall write $m^+_\nu(x,\lambda)$ and $\chi^+_\nu(x,\lambda)$ if the solution is extended from the sector Ω_ν and $m^-_\nu(x,\lambda)$

$(\chi_\nu^-(x,\lambda))$ if it is extended from $\Omega_{\nu-1}$. Analogously, we shall write $e_\alpha^{(-;\nu)}(x,\lambda)$ if the solution is extended from the sector $\Omega_{\nu-1}$ and $e_\alpha^{(+;\nu)}(x,\lambda)$ if the solution is extended from the sector Ω_ν. In other words, for $\lambda \in L_\nu$

$$\begin{aligned} e_\alpha^{(-;\nu)}(x,\lambda) &= \pi_0(\chi_{\nu-1}(x,\lambda) E_\alpha \chi_{\nu-1}^{-1}(x,\lambda)) \\ e_\alpha^{(+;\nu)}(x,\lambda) &= \pi_0(\chi_\nu(x,\lambda) E_\alpha \chi_\nu^{-1}(x,\lambda)) \end{aligned} \tag{15}$$

Suppose that we have a L^1-integrable function $h : \mathbb{R} \mapsto \bar{\mathfrak{g}}$. Then the completeness relation in adjoint representation (we remind that there is no discrete spectrum) can be cast into the following form :

$$h(x) = \frac{1}{2\pi} \sum_{\nu=1}^{2M} \int_{L_\nu} \{ (\sum_{\alpha \in \delta_\nu^+} e_\alpha^{(+;\nu)}(x) \langle\langle e_{-\alpha}^{(+;\nu)}, [J,h] \rangle\rangle - e_{-\alpha}^{(-;\nu)}(x) \langle\langle e_\alpha^{(-;\nu)}, [J,h] \rangle\rangle \}) \mathrm{d}\lambda \tag{16}$$

$$h(x) = -\frac{1}{2\pi} \sum_{\nu=1}^{2M} \int_{L_\nu} \{ \sum_{\alpha \in \delta_\nu^+} (e_{-\alpha}^{(+;\nu)}(y) \langle\langle e_\alpha^{(+;\nu)}, [J,h] \rangle\rangle - e_\alpha^{(-;\nu)}(y) \langle\langle e_{-\alpha}^{(-;\nu)}, [J,h] \rangle\rangle \} \mathrm{d}\lambda \tag{17}$$

In the above we used the following notation: for two functions $f(x), g(x)$ with values in $\mathfrak{g}$ we put

$$\langle\langle f, g\rangle\rangle = \int\limits_{-\infty}^{+\infty} \langle f(x), g(x)\rangle \mathrm{d}x$$

It can be shown that the expansions (16),(17) converge in the same sense as the classical Fourier expansions for $h(x)$.

Using Theorem 3.2 from (Gerdjikov and Yanovski 1994) one can see that

$$(\Lambda_- - \lambda)e_{\alpha}^{(+;\nu)} = 0, \qquad (\Lambda_- - \lambda)e_{-\alpha}^{(-;\nu)} = 0, \quad \alpha \in \delta_\nu^+ \tag{18}$$

$$(\Lambda_+ - \lambda)e_{-\alpha}^{(+;\nu)} = 0, \qquad (\Lambda_+ - \lambda)e_{\alpha}^{(-;\nu)} = 0, \quad \alpha \in \delta_\nu^+ \tag{19}$$

where the operators $\Lambda_\pm$ are given by

$$\begin{aligned} &\Lambda_\pm(X(x)) = \\ &\mathrm{ad}\,_J^{-1}\left(\mathrm{i}\partial_x X + \pi_0[q, X] + \mathrm{i}\mathrm{ad}\,_q \int\limits_{\pm\infty}^{x} (\mathrm{id} - \pi_0)[q(y), X(y)]\mathrm{d}y\right) \end{aligned} \tag{20}$$

The above operators are the famous Generating, Recursion or Λ-operators related to the CBC system, see (Gerdjikov, Vilasi and Yanovski 2008, Gerdjikov and Yanovski 1994). We see that the expansions in the above are in fact the spectral decompositions for the operators Λ_- and Λ_+. This is the reason the expansions we had are sometimes called the **Generalized Fourier Expansions and the functions $e_\alpha^{\pm;\nu}(x,\lambda)$ are called Generalized Exponents**. When one expands over the Generalized Exponents the potential $q(x)$ one gets as coefficients the minimal scattering data. Let us briefly outline this construction. Suppose that $B \in \mathfrak{h}$, $\delta q(x)$ is a variation of the potential function. Then we have the following expansions for the function $\mathrm{ad}\,_J^{-1}[B,q]$:

$$\mathrm{ad}\,_J^{-1}[B,q](x) = \frac{\mathrm{i}}{2\pi}\sum_{\nu=1}^{2M}\int_{L_\nu}\sum_{\alpha\in\delta_\nu^+}\left(\rho^+_{\nu;B,-\alpha}e^{(+;\nu)}_{\alpha} - \rho^-_{\nu;B,\alpha}e^{(-;\nu)}_{-\alpha}\right)\mathrm{d}\lambda \quad (21)$$

$$\mathrm{ad}\,_J^{-1}[B,q](x) = \frac{\mathrm{i}}{2\pi}\sum_{\nu=1}^{2M}\int_{L_\nu}\sum_{\alpha\in\delta_\nu^+}\left(\sigma^+_{\nu;B,\alpha}e^{(+;\nu)}_{-\alpha} - \sigma^-_{\nu;B,-\alpha}e^{(-;\nu)}_{\alpha}\right)\mathrm{d}\lambda \quad (22)$$

where

$$\rho^{\pm}_{\nu;B,\mp\alpha} \equiv \mathrm{i} \int\limits_{-\infty}^{+\infty} \langle [q,B], e^{(\pm;\nu)}_{\mp\alpha} \rangle \mathrm{d}x = \langle \hat{S}^{\pm}_{\nu} B S^{\pm}_{\nu}, E_{\mp\alpha} \rangle \tag{23}$$

$$\sigma^{\pm}_{\nu;B,\pm\alpha} \equiv -\mathrm{i} \int\limits_{-\infty}^{+\infty} \langle [q,B], e^{(\pm;\nu)}_{\pm\alpha} \rangle \mathrm{d}x = \langle \hat{D}^{\pm}_{\nu} \hat{T}^{\mp}_{\nu} B T^{\mp}_{\nu} D^{\pm}_{\nu}, E_{\pm\alpha} \rangle \tag{24}$$

The expansions for $\mathrm{ad}_J^{-1}\delta q$ run as follows

$$\mathrm{ad}_J^{-1}\delta q(x) = \frac{\mathrm{i}}{2\pi} \sum_{\nu=1}^{2M} \int_{L_\nu} \sum_{\alpha\in\delta^+_\nu} \left(\delta\rho^{+}_{\nu;-\alpha} e^{(+;\nu)}_{\alpha} - \delta\rho^{-}_{\nu;\alpha} e^{(-;\nu)}_{-\alpha} \right) \mathrm{d}\lambda \tag{25}$$

$$\mathrm{ad}_J^{-1}\delta q(x) = \frac{\mathrm{i}}{2\pi} \sum_{\nu=1}^{2M} \int_{L_\nu} \sum_{\alpha\in\delta^+_\nu} \left(\delta\sigma^{+}_{\nu;\alpha} e^{(+;\nu)}_{-\alpha} - \delta\sigma^{-}_{\nu;-\alpha} e^{(-;\nu)}_{\alpha} \right) \mathrm{d}\lambda \tag{26}$$

where

$$\delta\rho^{\pm}_{\nu;\mp\alpha}(\lambda) \equiv -\mathrm{i}\int\limits_{-\infty}^{+\infty} \langle \delta q, e^{(\pm;\nu)}_{\mp\alpha}\rangle \mathrm{d}x = \langle \hat{S}^{\pm}_{\nu}\delta S^{\pm}_{\nu}, E_{\mp\alpha}\rangle(\lambda) \tag{27}$$

$$\delta\sigma^{\pm}_{\nu;\pm\alpha}(\lambda) \equiv \mathrm{i}\int\limits_{-\infty}^{+\infty} \langle \delta q, e^{(\pm;\nu)}_{\pm\alpha}\rangle \mathrm{d}x = \langle \hat{D}^{\pm}_{\nu}\hat{T}^{\mp}_{\nu}\delta(T^{\mp}_{\nu}D^{\pm}_{\nu}), E_{\pm\alpha}\rangle(\lambda) \tag{28}$$

Let us remark that for $q(x)$ with values in $\bar{\mathfrak{g}} = \mathfrak{h}^{\perp}$ both $\mathrm{ad}_J^{-1}[B,q](x)$ and $\mathrm{ad}_J^{-1}\delta q(x)$ are well defined. Let us introduce now another set of scattering data:

$$\mathcal{T}_{\rho,B} = \bigcup_{\nu=1}^{2M}\{\rho^{+}_{\nu;B,-\alpha}(\lambda), \rho^{-}_{\nu;B,\alpha}(\lambda); \alpha\in\Delta^{+}_{\nu}, \lambda\in L_{\nu}\} \tag{29}$$

$$\mathcal{T}_{\sigma,B} = \bigcup_{\nu=1}^{2M}\{\sigma^{+}_{\nu;B,\alpha}(\lambda), \sigma^{-}_{\nu;B,-\alpha}(\lambda); \alpha\in\Delta^{+}_{\nu}, \lambda\in L_{\nu}\} \tag{30}$$

The formulae (21) and (25) show that the mapping from the potential function $q(x)$ to either of the scattering data sets $\mathcal{T}_{\rho}$,$\mathcal{T}_{\rho}$ can be regarded as generalized Fourier transform. Then (23), (24) play the role of the inverse Fourier transform formulae; the functions $e_\alpha^{(\pm;\nu)}$ play the role as the exponents in the Fourier transform and the operators $\Lambda_\pm$ then play the role the operator $\mathrm{i}\partial_x$ plays for the usual Fourier transform.

3.2 The NLEEs related to the CBC system

The NLEEs associated with the linear problem L (and integrable through some kind of inverse scattering technique for L) are most easily found if we adopt the approach based on the expansions (27), (28), (21). Indeed, consider the equations having the form:

$$\sum_{k=1}^{r} f_k(\Lambda_\pm)\mathrm{ad}\,_J^{-1}[H_k, q] + \mathrm{i}\,\mathrm{ad}\,_J^{-1} q_t = 0 \tag{31}$$

where $f(\lambda) = \sum_{k=1}^{r} f_k(\lambda)H_k$. Here $\{H_k\}_{k=1}^{r}$ is the part of the Cartan-Weil basis giving a basis in $\mathfrak{h}$ and $f_k(\lambda)$ are polynomials in λ and λ^{-1}. $f(\lambda)$ is known as the dispersion law of the corresponding NLEE. Expanding through the generalized exponents it could be written also in the following two equivalent forms:

$$\mathrm{i}\frac{\mathrm{d}\rho^{\pm}_{\nu;\mp\alpha}}{\mathrm{d}t} + \sum_{k=1}^{r} f_k(\lambda)\rho^{\pm}_{\nu;H_k,\mp\alpha}(\lambda,t) = 0, \quad 1 \leq \nu \leq 2M \tag{32}$$

$$\mathrm{i}\frac{\mathrm{d}\sigma^{\pm}_{\nu;\pm\alpha}}{\mathrm{d}t} + \sum_{k=1}^{r} f_k(\lambda)\sigma^{\pm}_{\nu;H_k,\pm\alpha}(\lambda,t) = 0, \quad 1 \leq \nu \leq 2M \tag{33}$$

where of course for the functions with index ν the argument λ belongs to L_ν. Naturally, this is in accordance with the fact that the equations we are speaking about have a Lax representation of the type $[L, A] = 0$. In terms of the scattering data factors $S^{\pm}_{\nu}, T^{\pm}_{\nu}, D^{\pm}_{\nu}$ these equations is written as

$$\mathrm{i}\frac{\mathrm{d}S^{\pm}_{\nu}}{\mathrm{d}t} + [f(\lambda), S^{\pm}_{\nu}] = 0, \quad \mathrm{i}\frac{\mathrm{d}T^{\pm}_{\nu}}{\mathrm{d}t} + [f(\lambda), T^{\pm}_{\nu}] = 0, \quad \mathrm{i}\frac{\mathrm{d}D^{\pm}_{\nu}}{\mathrm{d}t} = 0 \tag{34}$$

So the method of solving the Cauchy problem for corresponding NLEEs consists of the following steps: i) first finding the scattering data corresponding to $q(x, t = 0)$; ii) finding the evolution of the scattering data using (34); iii) finally, by some inverse scattering technique (for example Cauchy-Riemann problem) finding the function $q(x, t)$.

The equations for $D_\nu^\pm$ show that they are not changed by the evolution and then naturally they are related with the conservation laws for the given NLEE.

4. CBC system – the conservation laws

4.1 Properties of the adjoint solutions $h_{\nu,H}$

Below we present the formulae for the conservation laws obtained through the theory of recursion operators. Their advantage is that they are compact in contrast with the formulae obtained via another approaches which are constructed with recurrent procedures and give us the possibility to describe which of them trivialize if we have reductions.

The conservation laws are closely related to the functions $h_{\nu,H}(x,\lambda) = \chi_\nu H \hat{\chi}_\nu(x,\lambda)$, $H \in \mathfrak{h}$, $x \in \mathbb{R}$, $\lambda \in \Omega_\nu$ and more precisely of their projections $h^a_{\nu,H}(x,\lambda) = \pi_0 h_{\nu,H}(x,\lambda)$ as well as for the corresponding extensions $h^\pm_{\nu,H}(x,\lambda)$, $h^{\pm a}_{\nu,H}(x,\lambda)$ of these functions to the rays L_ν. Here of course $\chi_\nu(x,\lambda)$ is a FAS to the CBC system analytic in the sector Ω_ν.

The results that follow are important ingredients for our construction. We start with the asymptotic behavior of the "h"-solutions (functions of the type $h_{\nu,H} = \chi_\nu H \chi_\nu^{-1}$):

Proposition 0.1 *Let $H \in \mathfrak{h}$ be an arbitrary element from the Cartan subalgebra. Consider $h_{\nu,H}(x,\lambda) = \chi_\nu(x,\lambda) H \chi_\nu^{-1}(x,\lambda)$ where $\chi_\nu(x,\lambda)$ is the fundamental solution to $L\chi = 0$ in some sector of analyticity Ω_ν. Suppose the derivatives of the potential $q(x)$ up to the order N belong to the class $L^1(\mathbb{R})$. Suppose also that S_ν is proper open sub-sector of Ω_ν, that is, S_ν is an open sub-sector of Ω_ν and its closure belongs to Ω_ν. Then for λ tending to infinity but remaining in S_ν we have the following asymptotic formulae which hold uniformly in x:*

$$h_{\nu,H} = H + \sum_{k=1}^{N} \lambda^{-k} (\Lambda_-^{k-1} q_H + \mathrm{i}\mathrm{I}_-(\Lambda_-^{k-1}) q_H) + \mathrm{o}(\lambda^{-N}) \qquad (35)$$

where we have used the notation

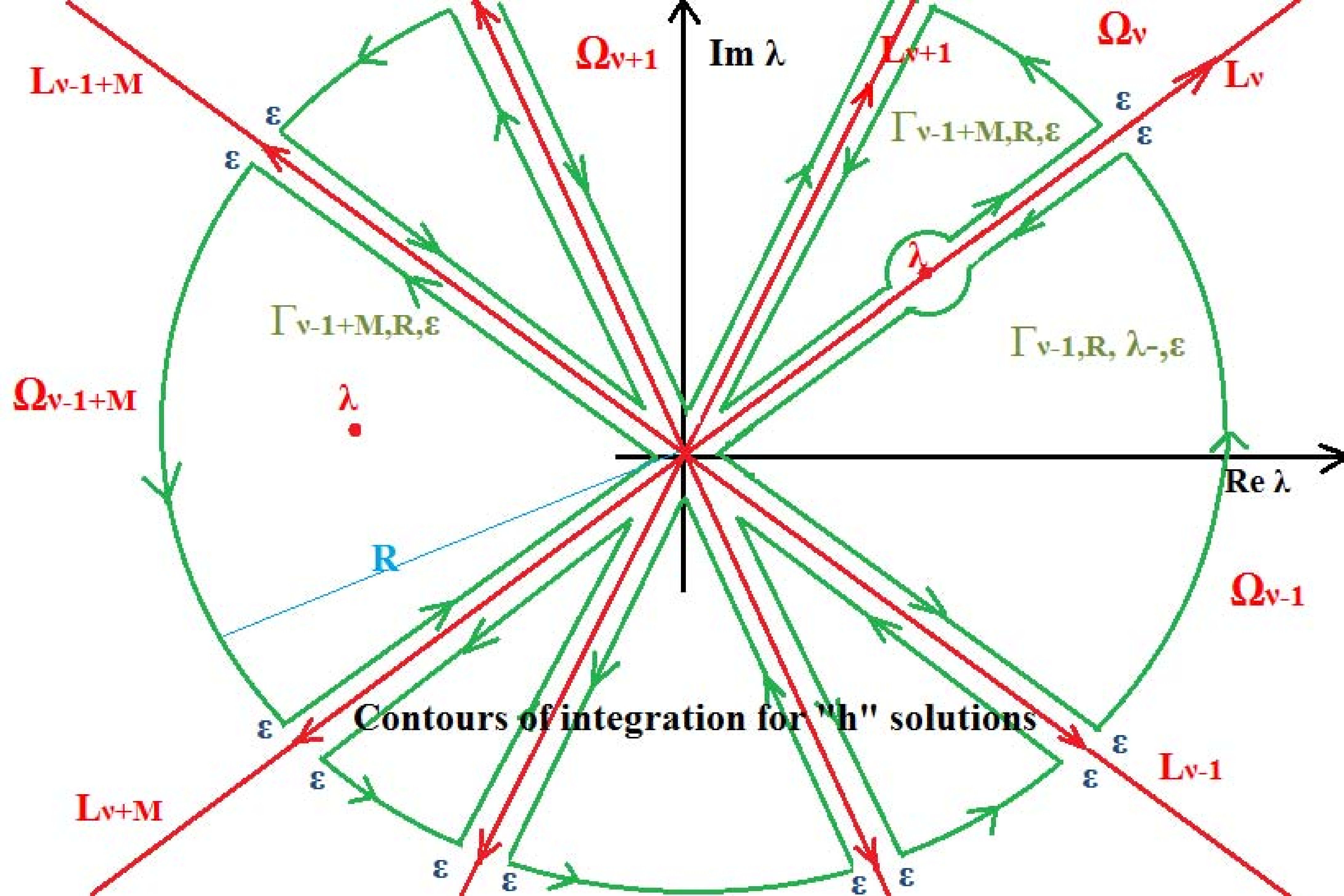
$$q_H \equiv \mathrm{ad}_J^{-1}[q, H], \quad \mathrm{I}_{\pm} f \equiv \int_{\pm\infty}^{x} (\mathrm{id} - \pi_0)\mathrm{ad}_{q(y)} f(y)\mathrm{d}y. \qquad (36)$$

In the above $f(x)$ is differentiable, absolutely integrable on the line, taking values in $\bar{\mathfrak{g}}$ and Λ_- is the operator we introduced in (20). In what follows up to the end of this subsection we shall assume that the potential $q(x)$ is a Schwartz-type function. Then the asymptotic formula can be written for arbitrary N and all the expressions we write make sense. Next result is about the integral representation of the adjoint solutions:

Proposition 0.2 *The following integral representations hold:*

1. *For $\lambda \in \Omega_\nu$ we have:*

$$h^a_{\nu,H}(x, \lambda) = \frac{\mathrm{i}}{2\pi} \sum_{\eta=1}^{2M} \int_{L_\eta} \frac{h^{-a}_{\eta,H}(x, \mu) - h^{+a}_{\eta,H}(x, \mu)}{\mu - \lambda} \mathrm{d}\mu \qquad (37)$$



Contours of integration for "h" solutions

2. For λ on the ray L_ν and $\lambda \neq 0$ we have

$$\frac{1}{2}(h^{-a}_{\nu,H}(x,\lambda) - h^{+a}_{\nu,H}(x,\lambda)) = \frac{\mathrm{i}}{2\pi}\sum_{\eta=1}^{2M} \mathrm{p.v.}\int_{L_\eta} \frac{h^{-a}_{\eta,H}(x,\mu) - h^{+a}_{\eta,H}(x,\mu)}{\mu - \lambda} d\mu \tag{38}$$

Of course the principle value is necessary to be taken only on the ray L_ν but in the way the things are now written (38) becomes more symmetric.

4.2 Conservation laws. General description

The idea of finding the conservation laws is to use $D^{\pm}_{\nu}(\lambda)$ (or their 'logarithms') are generating functions for the conservation laws. More precisely, we have the following situation. Consider the quantities

$$D^{\pm}_{\nu,j}(\lambda) = \langle \omega_{\nu,j} | D^{\pm}_{\nu}(\lambda) | \omega_{\nu,j} \rangle, \quad j = 1, 2, \ldots r = \operatorname{rank} \mathfrak{g} \tag{39}$$

where $\omega_{\nu,j}$ are the highest weights for the representations of the algebra $\mathfrak{g}$ with the ordering in Ω_ν. We define also

$$d^{\pm}_{\nu,j}(\lambda) = \log D^{\pm}_{\nu,j}(\lambda), \lambda \in L_\nu \tag{40}$$

It can be seen that $D^{\pm}_{\nu,j}(\lambda)$, which originally were defined only on the ray L_ν, are extensions of functions $\bar{D}_{\eta,j}(\lambda)$ which are analytic in the sectors Ω_η, $\eta = \nu, \nu+1$. In fact for $\lambda \in \Omega_\nu$

$$\bar{D}_{\nu,j}(\lambda) = \lim_{x\to+\infty} \langle \omega_{\nu,j}| e^{\mathrm{i}\lambda J x} m_\nu(\lambda, x) e^{-\mathrm{i}\lambda J x} |\omega_{\nu,j}\rangle = \lim_{x\to+\infty} \langle \omega_{\nu,j}| m_\nu(\lambda.x) |\omega_{\nu,j}\rangle$$
$$j = 1, 2, \dots r = \operatorname{rank} \mathfrak{g}, \quad \nu = 1, 2, \dots 2M \tag{41}$$

Thus for $\lambda \in L_\nu$ we have $\bar{D}_{\nu+1,j}(\lambda) = D^{+}_{\nu,j}(\lambda)$, $\bar{D}_{\nu,j}(\lambda) = D^{-}_{\nu,j}(\lambda)$. In addition, $\lim_{\lambda\to\infty} \bar{D}_{\nu,j} = 1$. For $j = 1, 2, \dots r = \operatorname{rank} \mathfrak{g}$, $\nu = 1, 2, \dots 2M$ define

$$\bar{d}_{\nu,j}(\lambda) = \log \bar{D}_{\nu,j}(\lambda) \tag{42}$$

The functions $\bar{d}_{\nu,j}(\lambda)$ are also analytic in Ω_ν and $\lim_{\lambda\to\infty} \bar{d}_{\nu,j}(\lambda) = 0$. If one considers them in any sub-sector S_ν of Ω_ν one has the asymptotic expansions

$$\bar{d}_{\nu,j}(\lambda) = \sum_{s=1}^{\infty} d_{\nu,j,s} \lambda^{-s}, \quad \lambda \in S_\nu, \quad |\lambda| >> 1 \tag{43}$$

and $d_{\nu,j,s}$ are then the required conservation laws. It turns out that it is easier to work with some linear combinations of the functions $\bar{d}_{\nu,j}(\lambda)$. In order to see that let us remark that since $D^{\pm}_{\nu}(\lambda) = \exp\left(\sum_{\alpha\in\pi_\nu} d^{\pm}_{\nu,\alpha}(\lambda) H_\alpha\right)$ we have

$$d^{\pm}_{\nu,j}(\lambda) = \log\langle\omega_{\nu,j}|D^{\pm}_{\nu}(\lambda)|\omega_{\nu,j}\rangle = \sum_{\alpha\in\pi_\nu} \omega_{\nu,j}(H_\alpha) d^{\pm}_{\nu,\alpha}(\lambda) \tag{44}$$

and the sets $d^{\pm}_{\nu,j}(\lambda)$ and $d^{\pm}_{\nu,\alpha}(\lambda)$ are expressed one through another as linear combinations with constant coefficients. The above means that the coefficients $d^{\pm}_{\nu,\alpha}(\lambda)$ also allow analytic continuation, that is, these functions are extensions of functions $d_{\eta,\alpha}(\lambda)$ analytic in Ω_η, $\eta = \nu, \nu+1$ and more generally, for $H \in \mathfrak{h}$ the functions $d^{\pm}_{\nu,H}(\lambda) = \sum\limits_{\alpha\in\pi_\nu} d^{\pm}_{\nu,\alpha}(\lambda)\langle H_\alpha, H\rangle$ allow analytic extensions $d_{\nu,H}$ to the sectors Ω_ν, and have the asymptotic:

$$d^{\pm}_{\nu,H}(\lambda) = \sum_{k=1}^{\infty} d^{\pm}_{H,k}\lambda^{-k}, \quad |\lambda| >> 1 \tag{45}$$

(One can prove that actually the coefficients in the asymptotic expansion do not depend on the sector). We shall denote the extensions in the sector Ω_η by $d_{\eta,H}(\lambda)$. Naturally, in a similar way as above, the factors $D^{\pm}_{\nu}(\lambda)$ allow analytic continuations denoted by $D_\nu(\lambda)$. The first way to obtain the coefficients $d_{\nu,\alpha,s}$ is first to use the Wronskian relations

$$\left(\mathrm{i}\hat{\chi}_\nu^\pm \frac{d\chi_\nu^\pm}{d\lambda}(x,\lambda) - Jx\right)\Big|_{-\infty}^{+\infty} = \int\limits_{-\infty}^{+\infty} \left(\hat{\chi}_\nu^\pm J \chi_\nu^\pm(x,\lambda) - J\right) \mathrm{d}x, \qquad \lambda \in L_\nu \quad (46)$$

in order to relate the derivatives with respect to λ of the $d_{\eta,H}(\lambda)$ and the functions $h^a_{\nu,H}$. The next step is to use the integral representations of the functions $h^a_{\nu,H} = \pi_0 \chi_\nu H \chi_\nu^{-1}$, $H \in \mathfrak{h}$ in order to obtain the relation

$$h^a_{\nu,H} = (\Lambda_\pm - \lambda)^{-1} \mathrm{ad}_J^{-1}[H,q], \lambda \in \Omega_\nu \quad (47)$$

and finally to use the asymptotic formulae for $h^a_{\nu,H}$. We cannot go into more details in such a short note so we shall present the final results:

$$d_{H,s} = \frac{1}{s} \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{x} \langle [J,q], \Lambda_\pm^s \mathrm{ad}_J^{-1}[H,q]\rangle \mathrm{d}y \mathrm{d}x, \qquad s = 1, 2, \ldots \quad (48)$$

The second way of obtaining the conservation laws is similar, but we use another type of Wronskian relations:

$$\mathrm{i}\langle \hat{\chi}_\nu^{\pm}\delta\chi_\nu^{\pm}, H\rangle|_{-\infty}^{+\infty} = -\int\limits_{-\infty}^{+\infty}\langle \hat{\chi}_\nu^{\pm}\delta\chi_\nu^{\pm}, H\rangle \mathrm{d}x \tag{49}$$

this time relating the variations of $d_{\eta,H}(\lambda)$ when we make a variation δq of the potential and the functions $h_{\nu,H}^a$. Then we again use the asymptotic of the functions $h_{\nu,H}^a$ and we get another formula

$$\delta d_{H,s} = -\mathrm{i}\int\limits_{-\infty}^{+\infty}\langle \delta q, \Lambda_{\pm}^{s-1}\mathrm{ad}_J^{-1}[H,q]\rangle \mathrm{d}x, \quad s = 1,2,\ldots \tag{50}$$

The last equations are more popular in another form. Let us identify the space $\mathcal{S}_J$ consisting of Schwartz-type functions on the line with values in $\mathfrak{h}^{\perp} = \bar{\mathfrak{g}}$ and its dual $\mathcal{S}_J^*$ through the bilinear form: $(X,Y) \mapsto \langle\langle X, Y\rangle\rangle$. In other words, we shall consider the elements from $\mathcal{S}_J^*$ as generalized functions (distributions) with test functions from $\mathcal{S}_J$. In fact the functionals we shall have will be regular, that is locally integrable functions over $\mathbb{R}$, and even most of them will belong to $\mathcal{S}_J$. Taking into account all this for the differentials of the conservation laws we get

$$\mathrm{d}d_{H,s} = -\mathrm{i}\Lambda_{\pm}^{s-1}\mathrm{ad}_{J}^{-1}[H,q] \tag{51}$$

and hence

$$\mathrm{d}d_{H,s} = \Lambda_{\pm}\mathrm{d}d_{H,s-1}, \quad s = 2,3,\ldots \tag{52}$$

The above relations in the case $\mathfrak{g} = \mathrm{sl}\,(2,\mathbb{C})$ are called *Lenart relations*, see (Adler, 1979), so we shall call them Lenart-type relations or Lenart chains.

One can prove that with the above identification $\mathrm{d}d_{\nu,H} = \mathrm{i}h^{a}_{\nu,H}$ which explains why the functions $h^{a}_{\nu,H}$ are so important in the study of the conservation laws. One can use another type of Wronskian relations in order to get another expression which in fact is none but the Poincaré lemma for closed forms:

$$d_{H,s} = -\mathrm{i}\int\limits_{-\infty}^{+\infty}\int\limits_{0}^{1}\langle q, \Lambda_{\pm}^{s-1}|_{(\zeta q)}\mathrm{ad}_{J}^{-1}[H,\zeta q]\rangle \mathrm{d}\zeta\mathrm{d}x \tag{53}$$

One can observe that in it enters Λ^{s-1} instead of Λ^{s} as in (48). So in the calculation of the conservation laws the above formula can be a real advantage as the expressions become increasingly more complicated when s increases.

4.3 CBC system – locality of the conservation laws

Now we shall treat the questions of the locality of the NLEEs related to the CBC system and their conservation laws, proving also that the conservation laws are in involution with respect to the hierarchy of symplectic structures we introduced earlier. The idea of the proof we present here has been used in (Gerdjikov,Yanovski 1985 JINR) for the case of the classical Zakharov-Shabat system. We start with the following:

Theorem 0.2 *If for arbitrary $H, W \in \mathfrak{h}$ the expression $\langle J, [q_H, \Lambda_-^N q_W]\rangle$ is x-derivative of local function on $q, q_x, \ldots$ then the expression $\Lambda_-^{N+1} q_F$ for every $F \in \mathfrak{h}$ is also a local function.*

Proof. Indeed,

$$\Lambda_- \Lambda_-^N q_F = \mathrm{i}\,\mathrm{ad}_J^{-1} \partial_x \Lambda_-^N q_F + \mathrm{ad}_J^{-1}[q, \pi_0 \Lambda_-^N q_F] + \mathrm{i}\,\mathrm{ad}_J^{-1} \mathrm{ad}_q \mathrm{I}_- \Lambda_-^N q_F$$

The expression in the integrand of $\mathrm{I}_- \Lambda_-^N q_F$ can be cast into the following form:

$$(\mathrm{id} - \pi_0)[q, \Lambda_-^N q_F] = \sum_{s=1}^{r} H_s \langle H^s, [q, \Lambda_-^N q_F]\rangle$$

where $\{H_s\}_{s=1}^r, \{H^s\}_{s=1}^r$ are two bi-orthogonal bases of $\mathfrak{h}$. By assumption $\mathrm{I}_-\Lambda_-^N q_F$ is local so the expression $\Lambda_-^{N+1} q_F$ is also local. Let us remark that that if what is assumed in the theorem is true then of course in all the formulae of this subsection one could put Λ_+ and I_+ instead of I_-. Indeed from the above proof it follows that for any $N = 1, 2, \ldots$ we have $\int_{-\infty}^{+\infty} (\mathrm{id} - \pi_0)\Lambda_-^N q_F \mathrm{d}x = 0$ so since $\mathrm{I}_-\Lambda_-^N q_F = \mathrm{I}_+\Lambda_-^N q_F + \int_{-\infty}^{+\infty} (\mathrm{id} - \pi_0)\Lambda_-^N q_F \mathrm{d}x$ we get $\mathrm{I}_-\Lambda_-^N q_F = \mathrm{I}_+\Lambda_-^N q_F$ for arbitrary natural N and hence for any natural N we have $\Lambda_-^N q_F = \Lambda_+^N q_F$.

Now we have

Lemma 0.1 *For $H, W \in \mathfrak{h}$ the following formulae hold:*

$$\langle J, [q_H, \Lambda_-^N q_W]\rangle - \langle J, [\Lambda_-^N q_H, q_W]\rangle = \\ = \mathrm{i}\partial_x \left\{ \sum_{k=0}^{N-1} \left(\langle \mathrm{I}_-\Lambda_-^k q_H, \mathrm{I}_-\Lambda_-^{N-k-1} q_W\rangle - \langle \Lambda_-^k q_H, \Lambda_-^{N-k-1} q_W\rangle \right) \right\} \qquad (54)$$

Proof. From the differential equation for h_H it is readily seen that for $H, W \in \mathfrak{h}$, $\lambda \in \Omega_\nu$, $\mu \in \Omega_\eta$ we have the following important relations:

$$\mathrm{i}\partial_x \langle h_{\nu,H}(x,\lambda), h_{W,\eta}(x,\mu)\rangle = (\lambda-\mu)\langle J, [h_{H,\nu}(x,\lambda), h_{W,\eta}(x,\mu)]\rangle \tag{55}$$

For big λ, μ we can insert here the asymptotic formulae we obtained in Lemma 0.1 and as the asymptotic are uniform in x we can differentiate. Comparing then the coefficients of the power series we obtain the following identities:

$$\begin{gathered}\langle J, [\Lambda_-^{k-1} q_H, \Lambda_-^{s} q_W]\rangle - \langle J, [\Lambda_-^{k} q_H, \Lambda_-^{s-1} q_W]\rangle = \\ = \mathrm{i}\partial_x \left(\langle \mathrm{I}_-\Lambda_-^{k-1} q_H, \mathrm{I}_-\Lambda_-^{s-1} q_W\rangle - \langle \Lambda_-^{k-1} q_H, \Lambda_-^{s-1} q_W\rangle\right)\end{gathered} \tag{56}$$

and summing them up for $k = 0, 1, \ldots N-1$, $s = N-k$ we get the result.

Finally, using induction and Lemma 0.1 one proves

Theorem 0.3 *For arbitrary $H, W \in \mathfrak{h}$ and arbitrary natural N the expressions $\langle J, [q_H, \Lambda_\pm^N q_W]\rangle$ are x-derivatives of local functions on $q, q_x, \ldots$.*

Corollary 0.1 *For every $H \in \mathfrak{h}$ and $N = 0, 1, 2, \ldots$ the expressions $\Lambda_\pm^N q_H$ are local function on $q, q_x, \ldots$ and thus all the equations:*

$$\mathrm{ad}_J^{-1}\partial_t q = \Lambda_\pm^N q_H, \quad N = 0, 1, 2, \ldots$$

are local.

Corollary 0.2 *The conservation laws*

$$d_{W,s} = \frac{1}{s} \int\limits_{-\infty}^{+\infty} \mathrm{d}x \int\limits_{-\infty}^{x} \langle [J,q], \Lambda_{\pm}^{s} q_W \rangle \mathrm{d}y, \quad s = 1, 2, \dots, \qquad W \in \mathfrak{h}$$

have local densities.

Corollary 0.3 *The conservation laws $d_{W,s}$ are in involution with respect to the hierarchy of symplectic forms Ω_m.*

Indeed, as it is well-known the NLEEs are Hamiltonian with respect to a hierarchy of symplectic forms:

$$\Omega_m(X,Y) = \int\limits_{-\infty}^{+\infty} \langle X, \Lambda_{\pm}^{m} \mathrm{ad}_J^{-1}(Y) \rangle \mathrm{d}x, \quad m = 0, 1, 2 \dots \tag{57}$$

Taking into account the form of Ω_m the result easily follows.

$\mathbb{Z}_n$ reductions

The general formulae that we have permit to treat easily the case when we have Mikhailov-type reductions. In case we have $\mathbb{Z}_n$ the algebra $\mathfrak{g}$ splits into a direct sum of eigenspaces of an automorphism $\mathcal{K}$ order n of that leaves $\mathfrak{h}$ invariant. As an example let us take $\mathcal{K}$ to be a Coxeter automorphism. We have

$$\mathfrak{g} = \oplus_{s=0}^{n-1} \mathfrak{g}^{[s]} \tag{58}$$

where for each $X \in \mathfrak{g}^{[s]}$ we have $\mathcal{K}X = \omega^s X$ $(\omega^n = 1)$ and the spaces $\mathfrak{g}^{[s]}$, $\mathfrak{g}^{[k]}$ for $k + s \neq 0 \pmod{n}$ are orthogonal with respect to the Killing form. For $\mathfrak{h}$ one has similar splitting but many of the spaces $\mathfrak{h}^{[p]}$ are zero, the numbers p for which $\mathfrak{h}^{[p]} \neq 0$ are called exponents of $\mathfrak{g}$. In particular we have $J \in \mathfrak{h}^{[1]}$, $q(x) \in \mathfrak{g}^{[0]}$. Then one readily sees that if $H \in \mathfrak{h}^{[p]} \neq 0$ from the conservation laws (48) "survive" only those for which $p+s = 0 \bmod (n)$

Needless to say the results about the locality still hold.

As for the symplectic forms Ω_m , again one readily sees for $m \neq 0 \pmod{n}$ they degenerate. However, the remaining still make a hierarchy endowing the space of potentials with Poisson-Nijenhuis structure (Yanovski, 2012 JGSP).

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