Symmetry Reduction of Asymmetric Heavenly Equation and 2+1-dimensional Bi-Hamiltonian Integrable System

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PART:1

1.1 Asymetric Heavenly Equation

Asymmetric heavenly equation was obtained as one of the canonical equations from a classification of second order partial differential equations that possess partner symmetries [1]. Asymmetric heavenly equation in 3+1 dimension is given by [2],

$$u_{tx}u_{ty} - u_{tt}u_{xy} + au_{tz} + bu_{xz} + cu_{xx} = 0 \qquad (1.1)$$

where u is the unknown that depends on the four independent variables t, x, y, z and subscripts denote partial derivatives u, e.g. $u_{tx} = \frac{\partial^2 u}{\partial t \partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ ect. while a,b,c are constant. By choosing $u_t = v$ as the second unkown, we have converted the asymmetric heavenly equation to the two-component evolution system

$$\begin{cases} u_t = v \\ v_t = \dfrac{1}{u_{xy}}(v_x v_y + av_z + bu_{xz} + cu_{xx}) \equiv Q \end{cases} \qquad (1.2)$$

1.2 Dirac's constraints analysis and symplectic sturcture of the asymmetric heavenly equation

We start with the degenerate Lagrangian denstiy for the system (1.2):

$$L = \left(\frac{v^2}{2} - vu_t\right)u_{xy} + \frac{1}{2}\left(au_t u_z + bu_x u_z + cu_x^2\right) \qquad (2.1)$$

In order to get a Hamiltonian formulation, we need to apply Dirac's constraint [3] analysis. Thus, we define the canonical momenta,

$$\Pi_i = \frac{\partial L}{\partial u_t^i}, \qquad (2.2)$$

and using canonical momenta, we get:

$$\pi_u = \frac{\partial L}{\partial u_t} = -vu_{xy} + \frac{a}{2}u_z, \quad \pi_v = \frac{\partial L}{\partial v_t} = 0 \qquad (2.3)$$

we treat definitions (2.3) as the second class constraints

$$\phi_u = \pi_u + v u_{xy} - \frac{a}{2} u_z = 0, \qquad \phi_v = \pi_v = 0 \tag{2.4}$$

and calculate the Poisson bracket of the constraints

$$K_{ij} = \left\{ \phi_i(x, y, z) \, , \, \phi_j(x`, y`, z`) \right\}_{PB} , \qquad i, j = 1, 2 \tag{2.5}$$

Organizing them in the form of a matrix, we find

$$K = \begin{pmatrix} -(v_y D_x + v_x D_y + a D_z + v_{xy}) & u_{xy} \\ -u_{xy} & 0 \end{pmatrix} \tag{2.6}$$

which is an explicitly skew-symmetric symplectic operator. Here the corresponding symplectic two-form is a volume integral $\Omega = \int_V \omega dxdydz$ of the density

$$\omega = \frac{1}{2} du_i \wedge K_{ij} du^j \tag{2.7}$$

$$\omega = u_{xy} du \wedge dv - \frac{1}{2} v_x du \wedge du_y - \frac{1}{2} v_y du \wedge du_x - \frac{a}{2} v_y du \wedge du_z \tag{2.8}$$

which, up to a divergence, can be directly verified to be a closed 2-form, that is, $d\Omega = 0$. Therefore, Ω is indeed a symplectic form and so K, defined by (2.6), is indeed a symplectic operator. Hence, its inverse is a Hamiltonian operator

$$J_0 = K^{-1} = \begin{pmatrix} 0 & -\frac{1}{u_{xy}} \\ \frac{1}{u_{xy}} & J_0^{22} \end{pmatrix} \tag{2.9}$$

where

$$J_0^{22} = -\frac{1}{2}\left(\frac{v_y}{u_{xy}^2} D_x + D_x \frac{v_y}{u_{xy}^2} + \frac{v_x}{u_{xy}^2} D_y + D_y \frac{v_x}{u_{xy}^2}\right) - \frac{a}{u_{xy}} D_z \frac{1}{u_{xy}}$$

The Hamiltonian density, corresponding to J_0, is defined as:

$$H_1 = \pi_u u_t + \pi_v v_t - L$$

with the result

$$H_1 = -\frac{1}{2}(v^2 u_{xy} + b u_x u_z + c u_x^2) \qquad (2.10)$$

One can obtain the flow (1.2) by applying J_0 to variational derivatives of Hamiltonian density H_1

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{u_{xy}}(v_x v_y + a v_z + b u_{xz} + c u_{xx}) \end{pmatrix}. \qquad (2.11)$$

Define a matrix with entries $K_{ij} = \{\Phi_i, \Phi_j\}_{PB}$, the Dirac bracket of two functions on phase space f and g is defined as

$$\{f, g\}_{DP} = \{f, g\}_{PB} - \sum_{i,j} \{f, \Phi_i\}_{PB} K_{ij}^{-1} \{\Phi_j, g\}_{PB} , \qquad (2.12)$$

Where $K_{ij}^{-1} = J_0$. Dirac proved that J_0 is always invertable[11].

Total Hamiltonian density H_T according to Dirac is given by;

$$H_T = H_1 + \lambda^A \phi_A$$

where λ^A are Lagrange multipliers.

$$H_T = -\frac{1}{2}(v^2 u_{xy} + b u_x u_z + c u_x^2) + \lambda^u \phi_u + \lambda^v \phi_v \qquad (2.13)$$

For second-class constraints the Lagrange multipliers are determined from the solution of $\{H_T, \phi_A\}_{PB} = 0$ and we get λ^u and λ^v respectively

$$\lambda^u = v \quad ; \qquad \lambda^v = \frac{1}{u_{xy}}\left(v_x v_y + a v_z + b u_{xz} + c u_{xx}\right) \qquad (2.14)$$

The Dirac bracket is a modification of Poisson bracket designed to vanish on the surface defined by the constraint.

1.3 Recursion operator and Lax pair

We start with the equation determining the symetries of asymmetric heavenly system. We introduce two components for the symmetry charectericties

$$u_\tau = \varphi \qquad \Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \qquad (3.1)$$
$$v_\tau = \psi$$

where τ is the group parameter. The symmetry condition amounts to the compatibility of the Lie equations and equations (1.2):

$$\begin{cases} u_{t\tau} - u_{\tau t} = 0 \\ v_{t\tau} - v_{\tau t} = 0. \end{cases} \qquad (3.2)$$

we obtain the symmetry condition (3.2) in the form of a linear matrix equation

$$\hat{A}(\Phi) = 0$$

where

$$\hat{A} = \begin{pmatrix} D_t & -1 \\ -\frac{c}{u_{xy}}D_x^2 + \frac{Q}{u_{xy}}D_xD_y - \frac{b}{u_{xy}}D_xD_z & D_t - \frac{v_y}{u_{xy}}D_x - \frac{v_x}{u_{xy}}D_y - \frac{a}{u_{xy}}D_z \end{pmatrix} \quad (3.3)$$

so that the first row of (3.3) yields $\varphi_t = \Psi$. Asymetric heavenly equation (1.1)

has the divergence form:

$$\left(-u_{ty}\varphi_t - c\varphi_x + u_{tt}\varphi_y - b\varphi_z\right)_x - \left(-u_{xy}\varphi_t + u_{tx}\varphi_y + a\varphi_z\right)_t = 0$$

$$\tilde{\varphi}_t = -u_{ty}\varphi_t - c\varphi_x + u_{tt}\varphi_y - b\varphi_z$$
$$\tilde{\Psi}_x = -u_{xy}\varphi_t + u_{tx}\varphi_y + a\varphi_z$$

Using the notation $\tilde{\Phi} = \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$, and using $\tilde{\varphi}_t = \tilde{\Psi}$, the recursion relation takes the

matrix form $\tilde{\Phi} = R(\Phi)$ and the recursion operator R obtained as

$$R = \begin{pmatrix} D_x^{-1}(v_x D_y + aD_z) & -D_x^{-1}(u_{xy}) \\ -cD_x + QD_y - bD_z & -v_y \end{pmatrix} \qquad (3.4)$$

For the commutator of the recursion operator R and the operator $\hat{A}$ of the symmetry condition (3.3), computed without using the equation of motion, we obtain

$$[R, \hat{A}] = \begin{pmatrix} -D_x^{-1}(v_t - Q)_x D_y & D_x^{-1}(u_t - v)_{xy} \\ \frac{1}{u_{xy}}[-c(u_t - v)_{xx} + Q(u_t - v)_{xy} & \\ -b(u_t - v)_{xz} - v_y(v_t - Q)_x & \\ -v_x(v_t - Q)_y - a(v_t - Q)_z]D_y & (v_t - Q)_y \end{pmatrix} \qquad (3.5)$$

and as a consequence, the operator R and $\hat{A}$ form a Lax pair of Olver-Ibragimov-Shabat type for the asymmetric heavenly system (1.2), so that R and $\hat{A}$ commute on solutions of this system.

1.4 Second Hamiltonian structure and Hamiltonian function

Using the theorem of Magri [9] , one can generate the second Hamiltonian operator by acting with the recursion operator (3.4) on the Hamiltonian operator (2.9):

$$J_1 = RJ_0 = \begin{pmatrix} -D_x^{-1} & \dfrac{v}{u_{xy}} \\ -\dfrac{v}{u_{xy}} & J_1^{22} \end{pmatrix} \tag{4.1}$$

where J_1^{22} is an explicitly skew-symmetric form is defined as

$$J_1^{22} = \frac{1}{2}\left(\frac{c}{u_{xy}} D_x + D_x \frac{c}{u_{xy}}\right) + \frac{1}{2}\left(v_y^2 D_x \frac{1}{u_{xy}^2} + \frac{1}{u_{xy}^2} D_x v_y^2\right)$$

$$+ \frac{1}{2}\left(\frac{v_x}{u_{xy}} D_y \frac{v_y}{u_{xy}} + \frac{v_y}{u_{xy}} D_y \frac{v_x}{u_{xy}}\right) - \frac{1}{2}\left(Q D_y \frac{1}{u_{xy}} + \frac{1}{u_{xy}} D_y Q\right)$$

$$+ \frac{1}{2}\left(\frac{b}{u_{xy}} D_z + D_z \frac{b}{u_{xy}}\right) + \frac{a}{2}\left(\frac{v_y}{u_{xy}} D_z \frac{1}{u_{xy}} + \frac{1}{u_{xy}} D_z \frac{v_y}{u_{xy}}\right)$$

The flow (1.2) can be generated by the Hamiltonian operator J_1 from the Hamiltonian density

$$H_0 = (c_0 - y)vu_{xy} \quad , \tag{4.2}$$

where c_0 is a constant, so that the asymmetric heavenly equation in the two-component form (1.2) admits two Hamiltonian represantation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} = J_1 \begin{pmatrix} \delta_u H_0 \\ \delta_v H_0 \end{pmatrix} \tag{4.3}$$

and thus it is a bi-Hamiltonian system. By repeated applications of the recursion operator to the first Hamiltonian operator J_0 according to Magri's theorem we could generate an infinite sequence of Hamiltonian operators

$$J_n = R^n J_0, \qquad n = 1, 2, 3, \ldots \tag{4.4}$$

which proves that asymmetric heavenly equation considered in a two-component form is a multi-Hamiltonian system in the above sense.

1.5 Jacobi Identity

In this section, we give a proof of Jacobi identity for Hamiltonian operators J_0 and J_1.

Definition 1. *A linear operator* $J : A^q \to A^q$ *is called Hamiltonian if its Poission bracket* $\{P,Q\} = \int \delta P . J \delta Q dxdydz$ *satisfies the skew-symmetry property*

$$\{P,Q\} = -\{Q,P\}, \tag{5.1}$$

and the Jacobi identity

$$\{\{P,Q\},R\} + \{\{R,P\},Q\} + \{\{Q,R\},P\} = 0 \tag{5.2}$$

for all functionals P,Q and R.

However, using this direct verification of Jacobi identity (5.2), even for simplest skew-adjoint operators, appears a hopelessly complicated computational task. For this reason we will use Olver's criterion(theorem 7.8 in this book[11]) which reads as follows.

Let Γ be a skew-adjoint qxq matrix differential operator and $\Theta = \frac{1}{2}\int(\Gamma\omega \wedge \omega)dxdydz$ the corresponding functional bi-vector. Then, Γ is the Hamiltonian if and only if

$$\Pr V_{\Gamma\omega}(\Theta) = 0 \tag{5.3}$$

where $\Pr V_{\Gamma\omega}$ is defined by [11] as

$$\Pr V_{\Gamma\omega} = \sum_{i,J} D_J\left(\sum_j \Gamma_{ij}\omega^j\right)\frac{\partial}{\partial\phi_J^i}, \quad J = 1, x, y, z, xx, xy, xz... \tag{5.4}$$

where $\phi^1 = u$ and $\phi^2 = v$ More precisely, the system is called bi-Hamiltonian if it has the form (4.4), where J_0 and J_1 form a Hamiltonian pair, that is, if every linear combination $\alpha J_0 + \beta J_1$ where α and β arbitrary constants, satisfies the Jacobi identity. If we verify directly the Jacobi identity for $\Gamma = \alpha J_0 + \beta J_1$, then we guaranteed that both J_0 and J_1 also satisfy the Jacobi identity (with the choices , $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$) and that J_0 and J_1 form a Hamiltonian pair.

Thus, we only have to prove that the linear combination $\Gamma = \alpha J_0 + \beta J_1$ satisfies the Jacobi identity. Therefore, we start with

$$\Gamma = \alpha J_0 + \beta J_1 = \begin{pmatrix} -\beta D_x^{-1} & \lambda \\ -\lambda & AD_x + BD_y + cD_z + F \end{pmatrix}. \qquad (5.5)$$

where

$$A = \frac{1}{u_{xy}}(\beta c + \lambda v_y), \qquad B = \frac{1}{u_{xy}}(\lambda v_x - \beta Q),$$

$$C = \frac{1}{u_{xy}}(a\, v_x + b\beta), \qquad \lambda = \frac{\beta v_y - \alpha}{u_{xy}} \qquad (5.6)$$

$$F = -\frac{1}{u_{xy}}(Au_{xxy} + Bu_{xyy} + Cu_{xyz} - \lambda\, v_{xy}),$$

where Q in B was defined in (1.2). Using theorem above, we set the bi-vector as follows:

$$\Theta = \frac{1}{2}\int (\Gamma_{ij}\omega^j \wedge \omega^i)dxdydz, \qquad i,j=1,2 \tag{5.7}$$

The components of the bi-vector ω can be defined as $\omega^1 = \eta$ and $\omega^2 = \theta$ and the wedge products ω^i yield $\theta \wedge \theta = \eta \wedge \eta = 0$ and $\theta \wedge \eta = -\eta \wedge \theta$. Then,formula (5.7) becomes

$$\Theta = \frac{1}{2}\int (2\lambda\eta \wedge \theta + A\theta_x \wedge \theta + B\theta_y \wedge \theta + C\theta_z \wedge \theta)dxdydz. \tag{5.8}$$

We note that Θ in (5.8) does not contain any nonlocal terms since the input of the only nonlocal term in Γ vanishes because of the identity $-\beta D_x^{-1}\eta \wedge \eta = 0$. Therefore, we can apply Olver's criterion without any modification. Substituting (5.8) into the equation

$$\Pr V_{\Gamma\omega}(\Theta) = 0 \tag{5.9}$$

We obtain,

$$\Pr V_{\Gamma\omega}(\Theta) = \frac{1}{2}\int \left\{ \left(\frac{2\beta}{u_{xy}} \Pr V_{\Gamma\omega}(v_y) + 2\lambda u_{xy} \Pr V_{\Gamma\omega}\left(\frac{1}{u_{xy}}\right) \right) \wedge \theta \wedge \eta \right.$$

$$+ \left(\beta c \Pr V_{\Gamma\omega}\left(\frac{1}{u_{xy}}\right) + \frac{2\beta v_y - \alpha}{u_{xy}^2} \Pr V_{\Gamma\omega}(v_y) + \lambda v_y u_{xy} \Pr V_{\Gamma\omega}\left(\frac{1}{u_{xy}^2}\right) \right) \wedge \theta_x \wedge \theta$$

$$+ \left(\frac{\beta v_x}{u_{xy}^2} \Pr V_{\Gamma\omega}(v_y) + \frac{\lambda}{u_{xy}} \Pr V_{\Gamma\omega}(v_x) - \frac{\beta}{u_{xy}} \Pr V_{\Gamma\omega}(Q) + (2\lambda v_x - \beta Q) \Pr V_{\Gamma\omega}\left(\frac{1}{u_{xy}}\right) \right) \wedge \theta y \wedge \theta$$

$$\left. + \left(\frac{a\beta}{u_{xy}^2} \Pr V_{\Gamma\omega}(v_y) + a\lambda u_{xy} \Pr V_{\Gamma\omega}\left(\frac{1}{u_{xy}^2}\right) + b\beta \Pr V_{\Gamma\omega}\left(\frac{1}{u_{xy}}\right) \right) \wedge \theta_z \wedge \theta \right\} dxdydz \qquad (5.10)$$

Using (5.4) with $\phi^1 = u$ and $\phi^2 = v$, we calculate

$$
\begin{aligned}
&Pr\, V_{\Gamma\omega}(v_x) = D_x(-\lambda\eta + A\theta_x + B\theta_y + C\,\theta_z + F\,\theta),\\
&Pr\, V_{\Gamma\omega}(v_y) = D_y(-\lambda\eta + A\theta_x + B\theta_y + C\,\theta_z + F\,\theta),\\
&Pr\, V_{\Gamma\omega}\left(\frac{1}{u_{xy}}\right) = -\frac{1}{u_{xy}^2} D_y(-\eta + \lambda_x\theta + \lambda\theta_x),\\
&Pr\, V_{\Gamma\omega}\left(\frac{1}{u_{xy}^2}\right) = -\frac{2}{u_{xy}^3} D_y(-\eta + \lambda_x\theta + \lambda\theta_x),\\
&Pr\, V_{\Gamma\omega}(Q) = \frac{1}{u_{xy}}(cD_x - bD_z - QD_y)(-\eta + \lambda_x\theta + \lambda\theta_x)\\
&+ \frac{1}{u_{xy}}(v_y D_x + v_x D_y + aD_z)(-\lambda\eta + A\theta_x + B\theta_y + C\,\theta_z + F\,\theta)
\end{aligned}
\tag{5.11}
$$

at the last step, we substitute (5.11) in (5.10). Then, we reconstruct all the terms carefully and set all the total divergence to zero.

1.6 Symmetries and conservation laws

Using the software packages LIEPDE and CRACK by Wolf [13], run under REDUCE 3.8, we have calculated all point symmetries of the asymmetric heavenly system (1.2). The basis generators of one-parameter subgroups of the complete Lie group of point symmetries for the asymmetric heavenly system(1.2) have the form

$$
\begin{aligned}
X_1 &= y\partial_y + u\partial_u + v\partial_v, \\
X_2 &= \left(\frac{akx}{b^2} - kt + aF'(s)\right)\partial_t + bF'(s)\partial_x + ky\partial_y \\
X_d &= \left((bt - ax)d_{yz} - vd_{yy}\right)\partial_t + cd_y\partial_x + bd_z\partial_y + bd_y\partial_z \\
&\quad + \left(-\frac{1}{2}d_{yy}v^2 + \left(\frac{1}{2}a^2x^2 - abtx + \frac{1}{2}b^2t^2\right)d_{zz} - actd_z\right)\partial_u \\
&\quad + \left(-bd_{yz}v + (b^2t - abx)d_{zz} - acd_z\right)\partial_v \\
X_f &= f_y\partial_t + (bt - ax)f_z\partial_u + bf_z\partial_v \\
X_g &= g(y,z)\partial_u
\end{aligned}
\qquad (6.1)
$$

We are interested in the integrals of motion generating all this point symetris. The relation between symmetries and integrals is given by the Hamiltonian form of Noether's theorem

$$\begin{pmatrix} \hat{\eta}_i^u \\ \hat{\eta}_i^v \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_i \\ \delta_v H_i \end{pmatrix} \qquad (6.2)$$

The two component symmetry characteristics for the symmetries given above, $\hat{\eta}_i^u$ and $\hat{\eta}_i^v$ are given by,

$$\hat{\eta}_1^u = u - yu_y, \qquad \hat{\eta}_1^v = v - yv_y,$$

$$\hat{\eta}_2^u = \frac{1}{b}(acq(s) + a^2cyF(s) + 2ackty - bku) - (\frac{akx}{b^2} - kt + aF'(s))v - bF'(s)u_x - kyu_y$$

$$\hat{\eta}_2^v = \frac{2acky}{b} - \left(\frac{akx}{b^2} - \mathrm{kt} + \mathrm{aF'(s)}\right)\mathrm{Q} - \mathrm{bF'(s)}\,\mathrm{v}_x - kyv_y$$

$$\hat{\eta}_d^u = \frac{1}{2} d_{yy} v^2 + \left(\frac{1}{2}\mathrm{a^2x^2} - \mathrm{abtx} + \frac{1}{2}\mathrm{b^2t^2}\right)\mathrm{d}_{zz} - actd_z$$
$$-v(\mathrm{bt} - \mathrm{ax})\,\mathrm{d}_{yz} - cd_y u_x + bd_z u_y - bd_y u_z,$$

$$\hat{\eta}_d^v = -bd_{yz} v + (\mathrm{b^2t} - \mathrm{abx})\,\mathrm{d}_{zz} - acd_z$$
$$-\left[(\mathrm{bt} - \mathrm{ax})\,\mathrm{d}_{yz} - vd_{yy}\right] Q - cd_y v_x + bd_z v_y - bd_y v_z,$$

$$\hat{\eta}_f^u = (\mathrm{bt} - \mathrm{ax})\,\mathrm{f}_z - vf_y, \quad \hat{\eta}_f^v = b\,\mathrm{f}_z - f_y\,\mathrm{Q},$$

$$\hat{\eta}_g^u = g(\mathrm{y},\mathrm{z}), \quad \hat{\eta}_g^v = 0$$

where a, b, c, k are arbitrary constants, d(y,z), f(y,z), g(y,z), q(s), F(s) are arbitrary functions with $s=bx-cz$ and F' denotes the derivative of F with respect to s. Hamiltonian operators provide the natural link between commuting symmetries in evolutionary form [11] and conservation laws (integral of motion) in involution with respect to Poisson brackets.

Where $H = \int_{-\infty}^{+\infty} \mathrm{H}\, dx\,dy\,dz$ is an integral of the motion along flow (1.2), with conserved density H. We determine conserved densities H, corresponding to known symmetry characteristics $\hat{\eta}^u, \hat{\eta}^v$ by inverting relation (6.3) in the form of the inverse Noether's theorem

$$\begin{pmatrix} \hat{\eta}_i^u \\ \hat{\eta}_i^v \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_i \\ \delta_v H_i \end{pmatrix}$$

$$\begin{pmatrix} \delta_u H_i \\ \delta_v H_i \end{pmatrix} = K \begin{pmatrix} \hat{\eta}_i^u \\ \hat{\eta}_i^v \end{pmatrix} = \begin{pmatrix} -(v_y D_x + v_x D_y + a D_z + v_{xy} & u_{xy} \\ -u_{xy} & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta}_i^u \\ \hat{\eta}_i^v \end{pmatrix} \qquad (6.3)$$

where the symplectic operator $K = J_0^{-1}$ is defined (2.7). From above equation, we obtain

$$\delta_u H_i = -(v_y\,\hat{\eta}^u_{ix} + v_x\,\hat{\eta}^u_{iy} + a\,\hat{\eta}^u_{iz} + v_{xy}\,\hat{\eta}^u_i) \qquad (6.4)$$

$$\delta_v H_i = -u_{xy}\hat{\eta}^u_i$$

Using (6.5), one can find Hamiltonians only for symmetries X_f and X_g from the list (6.1) and corresponding conserved densities are

$$H_f = v\,u_{xy}\left(\frac{vf_y}{2} - (bt-ax)f_z\right) + \frac{u_x}{2}(bf_y u_z - bf_z u_y + cf_y u_x) - au(bt-ax)f_{zz}$$

$$H_g = (u_z - vu_{xy})\,g(y,z).$$

Therefore , X_f and X_g are Hamiltonian (variational or Noether) symmetries, whereas X_1, X_2 and X_d are not Hamiltonian symmetries of the flow (1.2)

PART:2

2.1 Symmetry reduction of an asymmetric heavenly equation

We use one of the point symmetries given in (6.1) for reduction of the asymmetric heavenly equation

$$X_d = \left((bt - ax)d_{yz} - vd_{yy}\right)\partial_t + cd_y\partial_x + bd_z\partial_y + bd_y\partial_z + \left(-\frac{1}{2}d_{yy}v^2 + \left(\frac{1}{2}a^2x^2 - abtx + \frac{1}{2}b^2t^2\right)d_{zz} - actd_z\right)\partial_u + \left(-bd_{yz}v + (b^2t - abx)d_{zz} - acd_z\right)\partial_v \quad (7.1)$$

For a particular choice of $d=y$, we obtain

$$X_y = c\partial_x + b\partial_z \quad (7.2)$$

The invariants of X_y are determined by a characteristic system as

$$X = bx - cz,\ Y = y,\ T = t,\ U = u,\ V = v \quad (7.3)$$

The symmetry reduction implies the ansatz:

$$u = U(X,Y,T),\ v = V(X,Y,T) \quad (7.4)$$

Substituting this into the original system (1.2) and

$$D_x = bD_x \ , \ D_y = D_y \quad , \quad D_z = -cD_x \tag{7.5}$$

$$U \to u, V \to v, Y \to y, T \to t$$

we obtain the new 2+1-dimensional **reduced system**

$$\begin{cases} u_t = v \\ v_t = \dfrac{v_x}{u_{xy}}\left(v_y - \dfrac{ac}{b}\right) = Q \quad , b \neq 0 \end{cases} \tag{7.6}$$

2.2 Hamiltonian structure of the reduced system

We start with the denerate Lagrangian denstiy for the system (7.6):

$$L_{red} = (\frac{1}{2} v^2 - v u_t)\, b\, u_{xy} - \frac{ac}{2} (u_t u_x) \quad (8.1)$$

In order to get a Hamiltonian formulation, we need to apply Dirac's constraint [3] analysis. Thus, we define the canonical momenta

$$\Pi_i = \frac{\partial L_{red}}{\partial u^i_t}, \quad (8.2)$$

and using canonical momenta, we get:

$$\pi_u = \frac{\partial L_{red}}{\partial u_t} = -v u_{xy} - \frac{ac}{2} u_x, \quad \pi_v = \frac{\partial L_{red}}{\partial v_t} = 0 \quad (8.3)$$

we treat definitions (8.3) as the second class constraints

$$\phi_u = \pi_u + bvu_{xy} + \frac{ac}{2} u_x = 0, \quad \phi_v = \pi_v = 0 \tag{8.4}$$

and calculate the Poisson bracket of the constraints

$$K_{ij} = \left\{ \phi_i(x, y), \phi_k(x`, y`) \right\}_{PB}, \qquad i, j = 1, 2 \tag{8.5}$$

Organizing them in the form of a matrix, we find

$$K_{red} = \begin{pmatrix} -((\mathrm{b}v_y - ac)D_x + v_x D_y + v_{xy}) & u_{xy} \\ -u_{xy} & 0 \end{pmatrix} \tag{8.6}$$

which is an explicitly skew-symmetric symplectic operator. Here the correcponding symplectic two-form is a volume integral $\Omega = \int_V \omega dxdydz$ of the density

$$\omega = \frac{1}{2} du_i \wedge K_{ij} du^j \qquad (8.7)$$

which gives

$$\omega = bu_{xy} du \wedge dv - \frac{b}{2} v_x du \wedge du_y - \frac{b}{2} v_y du \wedge du_x + \frac{ac}{2} v_y du \wedge du_x \qquad (8.8)$$

which, up to a divergence, can be directly verified to be a closed 2-form, that is, $d\Omega = 0$. Therefore, Ω is indeed a symplectic form and so K, defined by (8.6), is indeed a symplectic operator. Hence, its inverse is a Hamiltonian operator

$$J_0^{red} = K_{red}^{-1} = \begin{pmatrix} 0 & -1/\mathrm{b}\, u_{xy} \\ 1/\mathrm{b}\, u_{xy} & J_0^{22} \end{pmatrix} \qquad (8.9)$$

where

$$J_0^{22} = \left(\frac{ac}{b^2}\frac{1}{u_{xy}^2} - \frac{v_y}{bu_{xy}^2}\right) D_x - \left(\frac{v_x}{bu_{xy}^2}\right) D_y \qquad (8.10)$$

$$- \frac{v_{xy}}{bu_{xy}^2} + \frac{u_{xxy}}{bu_{xy}^3} v_y + \frac{u_{xyy}}{bu_{xy}^3} v_x - \frac{ac}{b^2}\frac{u_{xxy}}{u_{xy}^3}$$

The first Hamiltonian density, becomes J_0

$$H_1^{red} = -\frac{b}{2} v^2 u_{xy} \tag{8.11}$$

One can obtain the flow (7.6) by applying J_0 to variational derivatives of Hamiltonian density $\mathrm{H}_1^{\mathrm{red}}$

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0^{red} \begin{pmatrix} \delta_u H_1^{red} \\ \delta_v H_1^{red} \end{pmatrix} = \begin{pmatrix} v \\ \frac{v_x}{u_{xy}}(v_y - \frac{ac}{b}) \end{pmatrix} \tag{8.12}$$

$$\begin{cases} u_t = v \\ v_t = \frac{v_x}{u_{xy}}\left(v_y - \frac{ac}{b}\right) = Q \quad , b \neq 0 \end{cases}$$

2.3 Recursion operator and Lax pair for the reduced system

If we impose the reduction prosudure to $\hat{A}$ and R recpectively.

$$\hat{A}_{red} = \begin{pmatrix} D_t & -1 \\ \frac{Q}{u_{xy}} D_x D_y & D_t - \frac{v_y}{u_{xy}} D_x - \frac{v_x}{u_{xy}} D_y + \frac{ac}{bu_{xy}} D_x \end{pmatrix}$$

$$R_{red} = \begin{pmatrix} D_x^{-1}(v_x D_y - \frac{ac}{b} D_x) & -D_x^{-1}(u_{xy}) \\ QD_y & -v_y \end{pmatrix}$$

For the commutator of the recursion operator R and the operator $\hat{A}$ of the symmetry condition (9.3), computed without using the equation of motion, we obtain

$$[R,\,\hat{A}]_{red} = \begin{pmatrix} -D_x^{-1}(v_t - Q)_x D_y & D_x^{-1}(u_t - v)_{xy} \\ \frac{1}{u_{xy}}[-cb^2(u_t - v)_{xx} + Qb(u_t - v)_{xy} & \\ b^2 c(u_t - v)_{xx} - bv_y(v_t - Q)_x & \\ -bv_x(v_t - Q)_y + ac(v_t - Q)_x]D_y & (v_t - Q)_y \end{pmatrix}$$

and as a consequence, the operator R_{red} and $\hat{A}_{red}$ form a Lax pair of the Olver-Ibragimov-Shabat type for the mixed heavenly system (7.6), so that R_{red} and $\hat{A}_{red}$ commute on solutions of this system.

2.4 Second Hamiltonian structure and Hamiltonian function for reduced system.

Using the theorem of Magri , one can generate the second Hamiltonian operator by acting with the reduced recursion operator on the new Hamiltonian operator (8.9):

$$J_1^{red} = R_{red} J_0^{red} = \begin{pmatrix} -\frac{1}{b} D_x^{-1} & v_y / b\, u_{xy} \\ -v_y / b\, u_{xy} & J_1^{22} \end{pmatrix} \tag{10.1}$$

where J_{11}^{22} is an explicitly skew-symmetric form defined as

$$J_1^{22} = \left(-\frac{ac}{b^2}\frac{v_y}{u_{xy}^2} + \frac{v_y^2}{b u_{xy}^2}\right) D_x + \left(-\frac{Q}{b u_{xy}} + \frac{v_x v_y}{b u_{xy}^2}\right) D_y$$

$$+ \frac{Q}{b}\frac{u_{xyy}}{u_{xy}^2} + \frac{v_{xy} v_y}{b u_{xy}^2} - \frac{u_{xxy} v_y^2}{b u_{xy}^3} - \frac{u_{xyy} v_x v_y}{b u_{xy}^3} + \frac{ac}{b^2}\frac{u_{xxy} v_y}{u_{xy}^3}$$

The flow (7.6) can be generated by the Hamiltonian operator J_1^{red} from the Hamiltonian density

$$H_0^{red} = b(c_0 - y)vu_{xy} \tag{10.2}$$

where c_0 is a constant, so that the asymmetric heavenly equation in the two-component form (7.6) admits two Hamiltonian representation

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0^{red} \begin{pmatrix} \delta_u H_1^{red} \\ \delta_v H_1^{red} \end{pmatrix} = J_1^{red} \begin{pmatrix} \delta_u H_0^{red} \\ \delta_v H_0^{red} \end{pmatrix} \tag{10.3}$$

and thus it is a bi-Hamiltonian system. By repeated applications of the recursion operator to the first Hamiltonian operator J_0^{red} according to Magri's theorem we could generate an infinite sequence of Hamiltonian operators

$$J_0^{red} = R_{red}^n J_0^{red}, \qquad n = 1, 2, 3, \dots \tag{10.4}$$

which proves that asymmetric heavenly equation considered in a two-component form is a multi-Hamiltonian system in the above sense.

2.5 Symmetries and Integrals of motion for reduced system

Point symmetries genarator of (7.6)

$$X_1 = f(x)\partial_x$$

$$X_2 = \left(-tg_v + tvh_v - uh_v - w_v + th\right)\partial_t + e(y)\partial_y + \left(-tvg_v + tv^2h_v - uvh_v - vw_v + tg + uh + w\right)\partial_u + g\partial_v \qquad (11.1)$$

where, $f(x), g(y,v), h(y,v), w(x,y,v)$ and $e(y)$ arbitrary functions these point symmetries are generated by some integrals of motion, that is, they are variational symmetries. The relation between symmetries and integrals is given by the Hamiltonian form of Noether's theorem.

$$\begin{pmatrix} \hat{\eta}_i^u \\ \hat{\eta}_i^v \end{pmatrix} = J_0^{red} \begin{pmatrix} \delta_u H_i \\ \delta_v H_i \end{pmatrix} \qquad (11.2)$$

where $H = \int_{-\infty}^{+\infty} \mathrm{H} dxdy$ is an integral of motion along the flow (7.6) with conserved density H which generates the symmetry with the two component characteristic $\hat{\eta}_u, \hat{\eta}_v$.

We chose here the Poisson structure determined by our first Hamiltonian operator since we know its inverse *K* given by (8.9) which is used in the inverse Noether's theorem

$$\begin{pmatrix} \delta_u H_i \\ \delta_v H_i \end{pmatrix} = K \begin{pmatrix} \hat{\eta}_u \\ \hat{\eta}_v \end{pmatrix} = \begin{pmatrix} -(v_y D_x + v_x D_y + a D_z + v_{xy}) & u_{xy} \\ -u_{xy} & 0 \end{pmatrix} \begin{pmatrix} \hat{\eta}_u \\ \hat{\eta}_v \end{pmatrix} \quad (11.3)$$

For symmetry, X_1 :

$$\hat{\eta}_u = -u_x f(x), \; \hat{\eta}_v = -v_x f(x) \quad (11.4)$$

Using equation we obtain

$$H^1 = \left(bvu_x u_{xy} - \frac{ac}{2} u^2{}_x \right) f(x) \qquad (11.5)$$

For X_2 only for particular choice of arbitrary functions we can find an integral of motion. For example if we choose $h = w = 0, g = b \; and \; e = 1$ we obtain

$$X_2 = \partial_y + bt\partial_u + b\partial_v \qquad (11.6)$$

and characteristic

$$\hat{\eta}_u = bt - u_y \,,\; \hat{\eta}_v = b - v_y \qquad (11.7)$$

Equation (11.3) gives

$$H^2 = bvu_{xy}(u_y - bt) + \frac{1}{2}\left(b^2 - ac\right) uu_{xy} \qquad (11.8)$$

It is possible to find different integrals of motion for different choices of arbitrary functions.

CONCLUSION

We have shown that a certain symmetry reduction of the 3+1–dimensional asymmetric heavenly equation, taken in a two-component form yields a two component 2 + 1–dimensional multi-Hamiltonian integrable system. For this system, we have presented explicitly two Hamiltonian operators, a recursion operator for symmetries, a complete set of point symmetries and corresponding integrals of the motion. The first impression of the major part of this work could be that it is an easy and even trivial task to obtain a three-dimensional multi-Hamiltonian system by a symmetry reduction of the original four-dimensional second heavenly system. All the main objects $J_0, J_1, K, \hat{A}, R, H_1, H_0$ *and* L could be obtained by the symmetry reduction. However, a slight change in a symmetry chosen for the reduction, ruins all these properties and creates a difficulty in discovering bi-Hamiltonian structure of the reduced system. If we choose more general symmetries for the reduction, for example from the optimal system of one-dimensional sub algebras from [3], then we shall be unable to discover even a single Hamiltonian structure of reduced systems. The problem of conservation of multi-Hamiltonian structure under symmetry reductions seems to be an important and interesting subject for a future research.

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