Star products and star exponentials

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In this talk, we give a brief introduction on star products given in the series of papers by H. Omori, Y. Maeda, N. Miyazaki, A. Yoshioka;

Deformation Expression for Elements of Algebras, (I)–(VII) 2011, 2012, 2013, ArXiv. math-ph.

In this series of papers,

- we discuss a family of star products depending on complex matrices.
- Main subjects are star exponentials and their applications.
- Especially, we are interested in star exponential of qudratic polynomials. Different form usual exponential functions, singularities in time variable appear in the star exponentials of quadratic polynomials. These singularities produce certain structures in star product algebras.

Talk

This talk is a brief review on star products. Topics of this talk are:

1. Gereneralizing Moyal product, normal product and anti-normal product, we define a star product on complex polynomials.
2. Using star products, we give a geometric description of the Weyl algebra; an algebra bundle with flat connection.
3. We extend the product on polynomials to the one on certain functions. And then we consider star exponentials.
4. As an application of star exponentials of linear polynomials, we show some concrete examples of star functions.

Based on the joint works with H. Omori, Y. Maeda, N. Miyazaki.

§1. Star product on polynomials

- Star product is an associative product on polynomials given by a power seris of a certain biderivation.
- We start with well-known star product, the Moyal product, which is a typical example of star product.

The Moyal product is given by a power seris of the following biderivation.

Biderivation

For polynomials f, g of variables $(u_1, \ldots, u_m, v_1, \ldots, v_m)$, we consider a biderivation (Poisson bracket)

$$f\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g = \sum_{j=1}^{m} \left(\partial_{v_j} f \ \partial_{u_j} g - \partial_{u_j} f \ \partial_{v_j} g\right)$$

the overleft arrow $\overleftarrow{\partial}$ means that the partial derivative is acting on the polynomial on the left and the overright arrow is similar.

Moyal product

For polynomials f, g of variables $(u_1, \ldots, u_m, v_1, \ldots, v_m)$, the Moyal product $f *_o g$ is given by the power series of the biderivation $\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)$ such that

$$
\begin{aligned}
f *_o g = f \exp \tfrac{i\hbar}{2}\left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g \\
= f \sum_{k=0}^{\infty} \frac{1}{k!} \left(\tfrac{i\hbar}{2}\right)^k \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^k g \\
= fg + \tfrac{i\hbar}{2} f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right) g + \tfrac{1}{2!}\left(\tfrac{i\hbar}{2}\right)^2 f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^2 g \\
+ \cdots + \tfrac{1}{k!}\left(\tfrac{i\hbar}{2}\right)^k f \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} - \overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v}\right)^k g + \cdots \quad (1)
\end{aligned}
$$

where $\hbar$ is a positive number.

Well-definedness and associativity

Remark

When $\hbar$ tends to 0, the product converges to the usual multiplication. In this sense, the Moyal product is regarded as a deformation of the ordinary product.

Since the power series is a finite sum on polynomials we see

Theorem

The Moyal product is well-defined on polynomials, and associative.

Normal product, anti-normal product

Other typical star products are normal product $*_N$, anti-nomal product $*_A$ given similarly by

$$f *_N g = f \exp i\hbar \left(\overleftarrow{\partial_v} \cdot \overrightarrow{\partial_u} \right) g,$$

$$f *_A g = f \exp -i\hbar \left(\overleftarrow{\partial_u} \cdot \overrightarrow{\partial_v} \right) g.$$

These are also well-defined on polynomials and associative.

CCR and Weyl Algebra

By direct calculation we see easily for these star products

Proposition

(i) *The generators* $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ *satisfy the canoical commutation relations*

$$[u_k, v_l]_{*_L} = -i\hbar\delta_{kl}, \; [u_k, u_l]_{*_L} = [v_k, v_l]_{*_L} = 0,$$

$(k, l = 1, 2, \ldots, m)$

where $*_L$ *stands for* $*_O$, $*_N$ *and* $*_A$.

(ii) *Then the algebra* $(\mathbb{C}[u, v], *_L) \quad (L = O, N, A)$ *is isomorphic to the Weyl algebra. (Then these are mutually isomorphic).*

Intertwiners

As to the isomorphism, for example, the algebra isomorphism

$$I_O^N : (\mathbb{C}[u,v], *_O) \to (\mathbb{C}[u,v], *_N)$$

is given explicitly by the power series of the differential operator such as

$$I_N^O(f) = \exp\left(-\frac{i\hbar}{2}\partial_u\partial_v\right)(f) = \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{-i\hbar}{2}\right)^l (\partial_u\partial_v)^l \ (f) \qquad (2)$$

The isomorphism is well-defined for polynomials.

And other isomophisms are also given in the similar form.

Remark

We remark here that these isomorphisms are well-known as ordering problem in physics.

§1.2. Star product

Now by generalizing the previous products, we define a star product in the following way.

Biderivation

Let Λ be an arbitrary $n \times n$ complex matrix. We consider a biderivation

$$\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w} = (\overleftarrow{\partial_{w_1}}, \cdots, \overleftarrow{\partial_{w_n}})\Lambda(\overrightarrow{\partial_{w_1}}, \cdots, \overrightarrow{\partial_{w_n}}) = \sum_{k,l=1}^{n} \Lambda_{kl}\overleftarrow{\partial_{w_k}}\overrightarrow{\partial_{w_l}}$$

where $(w_1, \cdots, w_n)$ is a generators of polynomials.

Now we define a star product by the power seires of the above biderivation such that

Definition

$$f *_\Lambda g = f \exp \frac{i\hbar}{2}\left(\overleftarrow{\partial_w}\Lambda\overrightarrow{\partial_w}\right) g \tag{3}$$

Then similary as before we see easily

Theorem

For an arbitrary Λ, the star product $*_\Lambda$ is well-defined on polynomials, and associative.

Remark

(i) *The star product $*_\Lambda$ is a genralization of the previous products. Actually*

- *if we put* $\Lambda = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *then we have the Moyal produt*
- *if* $\Lambda = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$*, we have the normal product*
- *if* $\Lambda = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$ *then the anti-normal product*

(ii) *If Λ is a symmetric matrix, the star product $*_\Lambda$ is commutative.*

(iii) *If $\Lambda = O$ the star product is nothing but an ordinary multiplication of function.*

§2. Geometry of Weyl algebra

Using star products, we obtain a geometric picture of the Weyl algebra.

In this section, we assume Λ has a fixed skew-symmetric part

$$J = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}.$$

Let K be an arbitrary $2m \times 2m$ complex symmetric matrix. We put a complex matrix

$$\Lambda = J + K$$

We consider a star product $*_\Lambda$ on polynomials of variables $(w_1, \cdots, w_{2m}) = (u_1, \cdots, u_m, v_1, \cdots, v_m)$.

Since Λ is determined by the complex symmetric matrix K, we denote the star product by $*_K$ instead of $*_\Lambda$.

Proposition

(i) *For an arbitrary K, the star product $*_K$ satisfies the canonical commutation relation, namely, the generator $(u_1, \ldots, u_m, v_1, \ldots, v_m)$ satisfies*

$$[u_k, v_l]_{*_K} = -i\hbar\delta_{kl},\ [u_k, u_l]_{*_K} = [v_k, v_l]_{*_K} = 0,$$

$(k, l = 1, 2, \ldots, m)$

(ii) *Then the algebra $(\mathbb{C}[u, v], *_K)$ is isomorphic to the Weyl algebra, and the algebra is regarded as a polynomial representation of the Weyl algebra.*

Equialence

Similarly to the case of the Moyal product, normal product and anti-normal product, we have algebra isomorphisms as follows.

Proposition

For arbitrary star product algebras $(\mathbb{C}[u,v], *_{K_1})$ *and* $(\mathbb{C}[u,v], *_{K_2})$ *we have an algebra isomorphism*

$$I_{K_1}^{K_2} : (\mathbb{C}[u,v], *_{K_1}) \to (\mathbb{C}[u,v], *_{K_2})$$

by means of the differential operator $\partial_w(K_2 - K_1)\partial_w$ *such that*

$$I_{K_1}^{K_2}(f) = \exp\left(\frac{i\hbar}{4}\partial_w(K_2 - K_1)\partial_w\right)(f)$$

where $\partial_w(K_2 - K_1)\partial_w = \sum_{kl}(K_2 - K_1)_{kl}\partial_{w_k}\partial_{w_l}$.

By a direct calculation we have

Theorem

Then isomorphisms satisfy the following chain rule:

1. $I_{K_3}^{K_1} I_{K_2}^{K_3} I_{K_1}^{K_2} = Id$
2. $\left(I_{K_1}^{K_2}\right)^{-1} = I_{K_2}^{K_1}$

Then using the family $\{(\mathbb{C}[u,v], *_K)\}_K$ and isomorphisms $\{I_{K_1}^{K_2}\}_{K_1,K_2}$, we will introduce an infinite dimensional bundle and a flat connection. By means of the parallel sections of this bundle we will obtain a geometric picture of the Weyl algebra.

Algebra budle

We set $\mathcal{S} = \{K\}$ the space of all $2m \times 2m$ symmetric complex matrices. We consider a trivial bundle over $\mathcal{S}$ with fiber the star product algebras

$$\pi : E = \Pi_{K \in \mathcal{S}}(\mathbb{C}[u, v], *_K) \to \mathcal{S}, \quad \pi^{-1}(K) = (\mathbb{C}[u, v], *_K).$$

Then the previous proposition says that each fiber $(\mathbb{C}[u, v], *_K)$ is mutually isomorphic and isomophic to the Weyl algebra, and the isomprhisms $I_{K_1}^{K_2}$ give isomorphisms between fibers $\pi^{-1}(K_1)$ and $\pi^{-1}(K_2)$ satisfying the chain rule.

Connection

For a curve in the base space $\mathcal{S}$, $C : K = K(t)$ starting from $K(0) = K$, we define a parallel translation of a polynomial $f \in (\mathbb{C}[u, v], *_K)$ by

$$f(t) = \exp \frac{i\hbar}{4}\partial_w(K(t) - K)\partial_w \, (f).$$

By differentiating the paralle translation with respect t, we have a connection of this bundle such that

$$\nabla_X f(K) = \frac{d}{dt} f(t)|_{t=0} - \frac{i\hbar}{4}\partial_w X \partial_w \, f(K)|_{t=0}, \quad X = \dot{K}(t)|_{t=0}$$

where $f(K)$ is a smooth section of the budle E.

Parallel sections and product

We set

$$\mathcal{P} = \{\text{all parallel sections of this bundle}\}$$

The algebra isomorphism $I_{K_1}^{K_2}$ satisfies

$$I_{K_1}^{K_2}(f(K_1) *_{K_1} g(K_1)) = \left(I_{K_1}^{K_2}(f(K_1)\right) *_{K_2} \left(I_{K_1}^{K_2}(g(K_1)\right)$$

and a paralell section $f(K) \in \mathcal{P}$ of this bundle satisfies $I_{K_1}^{K_2} f(K_1) = f(K_2)$. Then we have a star product $*$ for $f, g \in \mathcal{P}$ by

$$f * g\,(K) = f(K) *_K g(K), \quad \forall K \in \mathcal{S}$$

Then we have

Theorem

The space of parallel sections $\mathcal{P}$ is equipped with the star product $*$, and the associative algebra $(\mathcal{P}, *)$ is isomorphic to the Weyl algebra.

Remark

The algebra $(\mathcal{P}, *)$ is regarded as a geometric realization of the Weyl algebra.

§3. Extension to functions

We consider to extend the star products $*_\Lambda$ for an arbitary complex matrix Λ from the space of all polynomials to a certain space of functions.

§2.1. Star product on certain holomorphic function space

We want to consider the star products $*_\Lambda$ from polynomials to functions. However, the power seris of the product is not convergent in general, hence we restrict the product to certain subset of smooth functions given by the following semi-norms.

Semi-norms and the completion

Semi-norms

Let f(w) be a holomorphic function on $\mathbb{C}^n$. For a positive number p, we consider a family of semi-norms $\{|\cdot|_{p,s}\}_{s>0}$ given by

$$|f|_{p,s} = \sup_{w\in\mathbb{C}^n} |f(w)| \exp(-s|w|^p), \quad |w| = \sqrt{|w_1|^2 + \cdots + |w_n|^2}.$$

Space

We put

$$\mathcal{E}_p = \{f : \text{ entire } | \, |f|_{p,s} < \infty, \forall s > 0\}$$

With the system of semi-norms the space $\mathcal{E}_p$ becomes a Fréchet space.

Then we have

Theorem

For an arbitrary matrix Λ

(i) *For* $0 < p \leq 2$, $(\mathcal{E}_p, *_\Lambda)$ *is a Frechét algebra. That is, the product converges for any elements of* $\mathcal{E}_p$, *and the product is continuous with respect to this topology.*

(ii) *Moreover for the same* p, *for any* Λ' *with the common skew symmetric part with* Λ, $I_\Lambda^{\Lambda'} = \exp(\frac{i\hbar}{4}\partial_w(\Lambda' - \Lambda)\partial_w)$ *is well-defined algebra isomorphism from* $(\mathcal{E}_p, *_\Lambda)$ *to* $(\mathcal{E}_p, *_{\Lambda'})$. *That is, the expansion convergies for every element, and the operator is continuous with respect to this topology.*

(iii) *For* $p > 2$, *the multiplication* $*_\Lambda : \mathcal{E}_p \times \mathcal{E}_{p'} \to \mathcal{E}_p$ *is well-defined for* p' *such that* $\frac{1}{p} + \frac{1}{p'} = 2$, *and* $(\mathcal{E}_p, *_\Lambda)$ *is a* $\mathcal{E}_{p'}$*-bimodule.*

§4. Star exponentials and star functions

Since we have a complete topogical algebra, we can consider exponential element in the star product algebra $(\mathcal{E}_p, *_\Lambda)$.

§4.1. Definition

For a polynomial $H_* \in \mathcal{P}$, we want to define a star exponential $e_*^{t\frac{H_*}{i\hbar}}$. However, the expansion $\sum_n \frac{t^n}{n!}\left(\frac{H_*}{i\hbar}\right)^n$ is not convergent in general, so we define by means of the differential equation.

Definition

The star exponential $e_*^{t\frac{H_*}{i\hbar}}$ is given as a solution of the following differential equation

$$\frac{d}{dt}F_t = H_* *_\Lambda F_t, \quad F_0 = 1. \tag{4}$$

§4.1.1. Examples. H linear, quadratic

We are interested in the star exponentials of linear, and quadratic polynomials. For these, we can solve the differential equation and obtain explicit solutions.

In what follows, we consider $2m \times 2m$ complex matices Λ with the fixed skew symmetric part $J = \begin{pmatrix} 0 & 1_m \\ -1_m & 0 \end{pmatrix}$. We write $\Lambda = K + J$ where K is a complex symmetric matrix.

First we remark the following.

For a linear polynomial $l = \sum_{j=1}^{2m} a_j w_j$, we see directly an ordinary exponential function e^l satisfies

$$e^l \notin \mathcal{E}_1, \quad \in \mathcal{E}_{1+\epsilon}, \quad \forall \epsilon > 0.$$

For a quadratic polynomial $Q = \sum_{ij=1}^{2m} A_{ij} w_i w_j$, we see directly an ordinary exponential function e^Q satisfies

$$e^Q \notin \mathcal{E}_2, \quad \in \mathcal{E}_{2+\epsilon}, \quad \forall \epsilon > 0.$$

Then put the Fréchet space for a positive number p

$$\mathcal{E}_{p+} = \cap_{q>p} \mathcal{E}_q$$

Star exponential function for Lienar case with respect to $*_K$

Proposition

For $l = \sum_j a_j w_j =< \boldsymbol{a}, \boldsymbol{w} >$

$$e_*^{t(l/i\hbar)} = e^{t^2 \boldsymbol{a} K \boldsymbol{a}/4i\hbar} e^{t(l/i\hbar)} \in \mathcal{E}_{1+}$$

Star exponential function for Quadratic case

Proposition

For $Q_* = \langle \boldsymbol{w} A, \boldsymbol{w} \rangle_*$ *where* A *is a* $2m \times 2m$ *complex symmetric matrix,*

$$e_*^{t(Q_*/i\hbar)} = \frac{2^m}{\sqrt{\det(I - \kappa + e^{-2t\alpha}(I + \kappa))}} e^{\frac{1}{i\hbar}\langle \boldsymbol{w} \frac{1}{I - \kappa + e^{-2t\alpha}(I+\kappa)}(I - e^{-2t\alpha})J, \boldsymbol{w}\rangle}$$

where $\kappa = KJ$ *and* $\alpha = AJ$.

Remark

The star exponential functions of linear functions are belonging to $\mathcal{E}_{1+}$ *then the star products are convergent and continuous. But it is easy to see*

$$e_*^{t(Q_*/i\hbar)} \in \mathcal{E}_{2+}, \quad \notin \mathcal{E}_2$$

and hence star exponential functions $\{e_*^{t(Q_*/i\hbar)}\}$ *are difficult to treat. Some anormalous phenomena happen. For example, if we describe this set by means of the previous bundle picture, then the bundle gerb structure naturally appears. But today we do not discuss these.*

§4.2. Star functions

There are many application of star exponential functions. Today we show some examples using linear star exponentials.

In what follows, we consider the star product for the simple case where

$$\Lambda = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}, \quad \rho \in \mathbb{C}.$$

Then we see easily that the star product is commutative and explicitly given by $p_1 *_\Lambda p_2 = p_1 \exp\left(\frac{i\hbar\rho}{2}\overleftarrow{\partial_{w_1}}\overrightarrow{\partial_{w_1}}\right) p_2$. This means that the algebra is essentially reduced to space of functions of one varible w_1. Thus, we consider functions $f(w), g(w)$ of one variable $w \in \mathbb{C}$ and we consider a commutative star product $*_\tau$ with complex parameter τ such that

$$f(w) *_\tau g(w) = f(w) e^{\frac{\tau}{2}\overleftarrow{\partial_w}\overrightarrow{\partial_w}} g(w)$$

§4.2.1. Star Hermite function

Recall the identity

$$\exp\left(\sqrt{2}tw - \frac{1}{2}t^2\right) = \sum_{n=0}^{\infty} H_n(w)\frac{t^n}{n!}$$

where $H_n(w)$ is an Hermite polynomial. We remark here that

$$\exp\left(\sqrt{2}tw - \frac{1}{2}t^2\right) = \exp_*(\sqrt{2}tw_*)_{\tau=-1}$$

Since $\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty}(\sqrt{2}tw_*)^n \frac{t^n}{n!}$ we have $H_n(w) = (\sqrt{2}tw_*)^n_{\tau=-1}$.

We define $*$-Hermite function by

$$H_n(w,\tau) = (\sqrt{2}w_*)^n, \quad (n = 0, 1, 2, \cdots)$$

with respect to $*_\tau$ product.
Then we have

$$\exp_*(\sqrt{2}tw_*) = \sum_{n=0}^{\infty} H_n(w,\tau)\,\frac{t^n}{n!}$$

Trivial identity $\frac{d}{dt}\exp_*(\sqrt{2}tw_*) = \sqrt{2}w * \exp_*(\sqrt{2}tw_*)$ yields the identity

$$\frac{\tau}{\sqrt{2}}H_n'(w,\tau) + \sqrt{2}wH_n(w,\tau) = H_{n+1}(w,\tau), \quad (n = 0, 1, 2, \cdots).$$

at every $\tau \in \mathbb{C}$.

The exponential law
$\exp_*(\sqrt{2}sw_*) * \exp_*(\sqrt{2}tw_*) = \exp_*(\sqrt{2}(s+t)w_*)$ yields the identity

$$\sum_{k+l=n} \frac{n!}{k!l!} H_k(w,\tau) *_\tau H_l(w,\tau) = H_n(w,\tau).$$

at every $\tau \in \mathbb{C}$.

§4.2.2. Star theta function

Now we consider the Jacobi's theta functions by using star exponentials.

The formula of star exponential function of linear polynomial gives

$$\exp_{*_\tau} i\,tw = \exp(i\,tw - (\tau/4)t^2)$$

Hence for $\Re\tau > 0$, the star exponential $\exp_{*_\tau} ni\,w = \exp(ni\,w - (\tau/4)n^2)$ is rapidly decreasing with respect to the integer n and then we can consider summations

$$\sum_{n=-\infty}^{\infty} \exp_{*_\tau} 2ni\,w = \sum_{n=-\infty}^{\infty} \exp\left(2ni\,w - \tau\,n^2\right) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2ni\,w}, \quad (q = e^{-\tau})$$

for τ satisfying $\Re\tau > 0$.

This is Jacobi's theta function $\theta_3(w,\tau)$. Then we have an expression of theta functions as

$$\theta_{1*_\tau}(w) = \frac{1}{i}\sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau}(2n{+}1)i\,w, \quad \theta_{2*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau}(2n{+}1)i\,w$$

$$\theta_{3*_\tau}(w) = \sum_{n=-\infty}^{\infty}\exp_{*_\tau} 2ni\,w, \quad \theta_{4*_\tau}(w) = \sum_{n=-\infty}^{\infty}(-1)^n \exp_{*_\tau} 2ni\,w$$

Remark that $\theta_{k*_\tau}(w)$ is the Jacobi's theta function $\theta_k(w,\tau)$, $k = 1,2,3,4$ respectively.

It is obvious by the exponential law

$$\exp_{*_\tau} 2i\,w *_\tau \theta_{k*_\tau}(w) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$\exp_{*_\tau} 2i\,w *_\tau \theta_{k*_\tau}(w) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

Then using $\exp_{*_\tau} 2i\,w = e^{-\tau}e^{2i\,w}$ and the product formula directly we have

$$e^{2i\,w-\tau}\theta_{k*_\tau}(w + i\,\tau) = \theta_{k*_\tau}(w) \quad (k = 2, 3)$$
$$e^{2i\,w-\tau}\theta_{k*_\tau}(w + i\,\tau) = -\theta_{k*_\tau}(w) \quad (k = 1, 4)$$

§4.2.3. $*$-delta functions

Since the $*_\tau$-exponential $\exp_*(itw_*) = \exp(itw - \frac{\tau}{4}t^2)$ is raidly decreasing with respect to t when $\Re\tau > 0$. Then the integral of $*_\tau$-exponential

$$\begin{aligned}\int_{-\infty}^{\infty} \exp_*(it(w-a)_*)\,dt &= \int_{-\infty}^{\infty} \exp_*(it(w-a)_*)dt \\ &= \int_{-\infty}^{\infty} \exp(it(w-a) - \frac{\tau}{4}t^2)dt\end{aligned}$$

converges for any $a \in \mathbb{C}$.

We put a star δ-function

$$\delta_*(w-a) = \int_{-\infty}^{\infty} \exp_*(it(w-a)_*)dt$$

which has a meaning at τ with $\Re\tau > 0$.

It is easy to see for any polynomial $p_*(w) \in \mathcal{P}$,

$$p_*(w) * \delta_*(w-a) = p(a)\delta_*(w-a), \; w_* * \delta_*(w) = 0.$$

Using the Fourier transform we have

Proposition

$$\theta_{1*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w + \tfrac{\pi}{2} + n\pi)$$

$$\theta_{2*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n\delta_*(w + n\pi)$$

$$\theta_{3*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w + n\pi)$$

$$\theta_{4*}(w) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\delta_*(w + \tfrac{\pi}{2} + n\pi).$$

We calcultate the integral and easily obtain

$$\delta_*(w-a) = \frac{2\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w-a)^2\right)$$

Then we have

$$\begin{aligned}
\theta_3(w,\tau) &= \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta_*(w+n\pi) = \sum_{n=-\infty}^{\infty} \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}(w+n\pi)^2\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \sum_{n=-\infty}^{\infty} \exp\left(-2n\frac{1}{\tau}w - \frac{1}{\tau}n^2\tau^2\right) \\
&= \frac{\sqrt{\pi}}{\sqrt{\tau}} \exp\left(-\frac{1}{\tau}\right) \theta_{3*}\left(\frac{2\pi w}{i\tau}, \frac{\pi^2}{\tau}\right).
\end{aligned}$$

We also have similar identities for other $*$-theta functions by the similar way.

§4.2.4. Other applications

- linear case : star special functions, star Eisenstein series.
- quadratic case: group like object, singularities. etc.

Reference

H. Omori, Y. Maeda, N. Miyazaki, A. Y—-, Deformation Expression for Elements of Algebras, (I)–(VII) 2011, 2012, 2013, ArXiv. math-ph