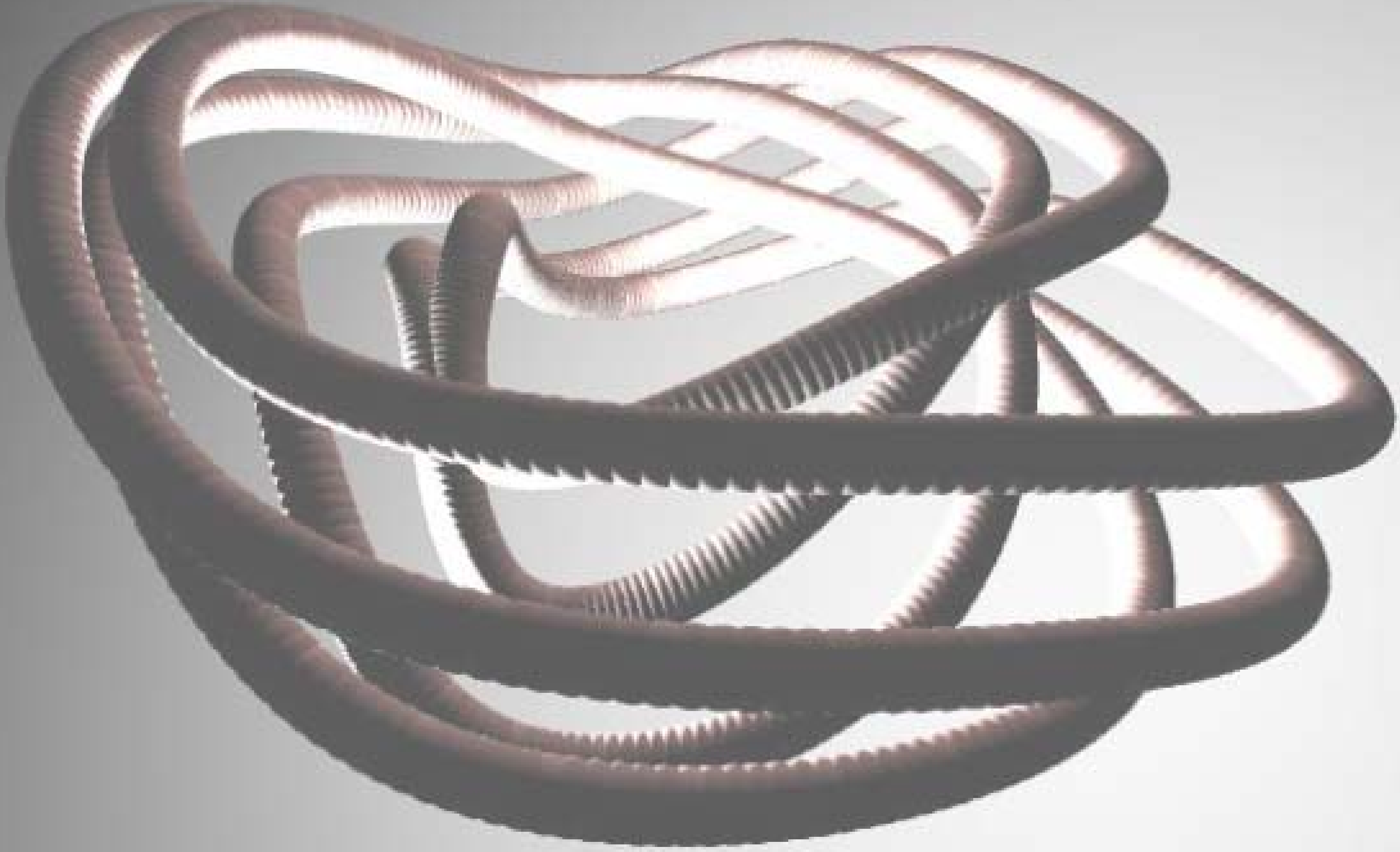
Geometric Invariants for Knots in the Solid Torus



Definition: A knot in $\mathbb{R}^3$ is the image of a piecewise-linear one-to-one mapping f:$S^1 \rightarrow \mathbb{R}^3$.

Definition: Two knots K_1 and K_2 in $\mathbb{R}^3$ are said to be equivalent if there exists an orientation-preserving homeomorphism $\varphi:\mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\varphi(K_1) = K_2$, and φ preserves the orientation on the knots.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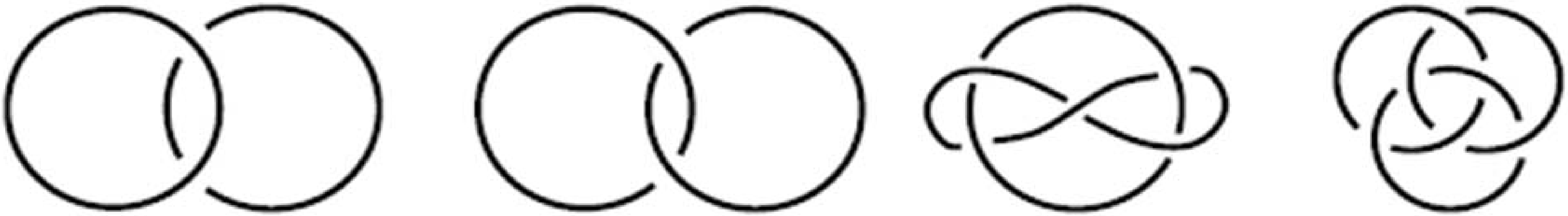
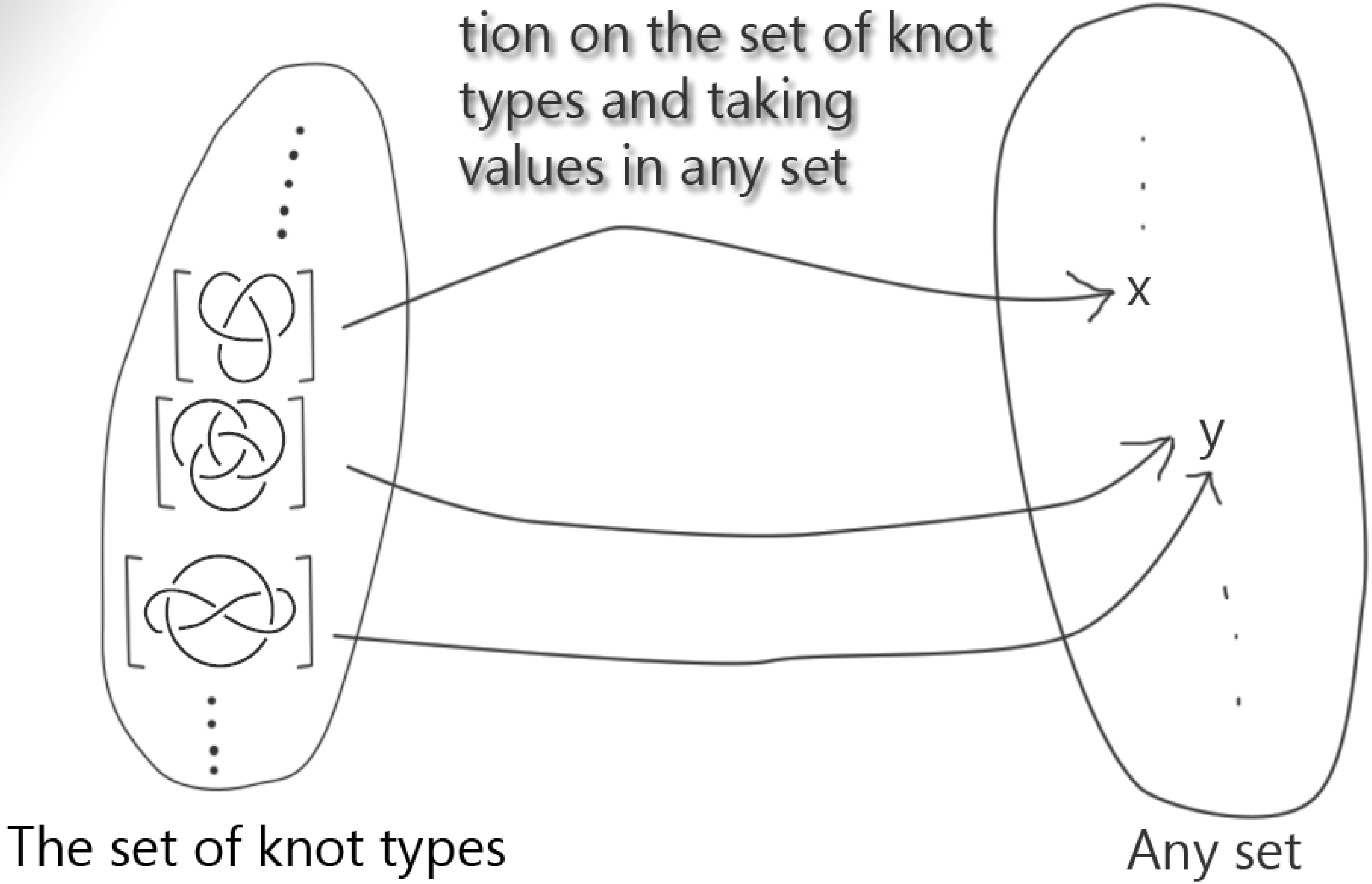
- **Definition**: A link in $\mathbb{R}^3$ is a finite ordered collection of knots, called the components of the link, that do not intersect each other.

- **Definition**: A link type is an equivalence class of links and a link invariant is a function defined on the set of link types and taking values in any set.

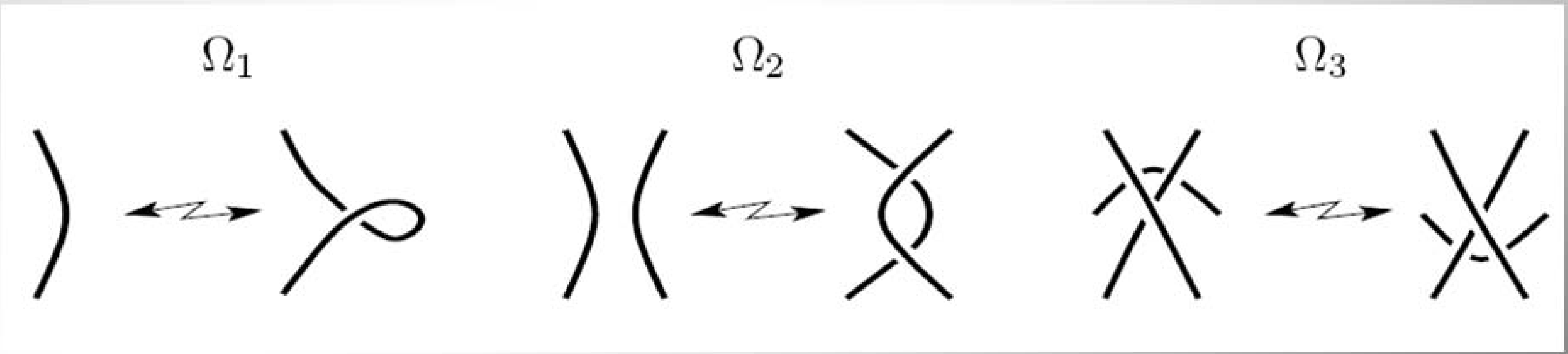


An invariant is simply a well-defined function on the set of knot types and taking values in any set

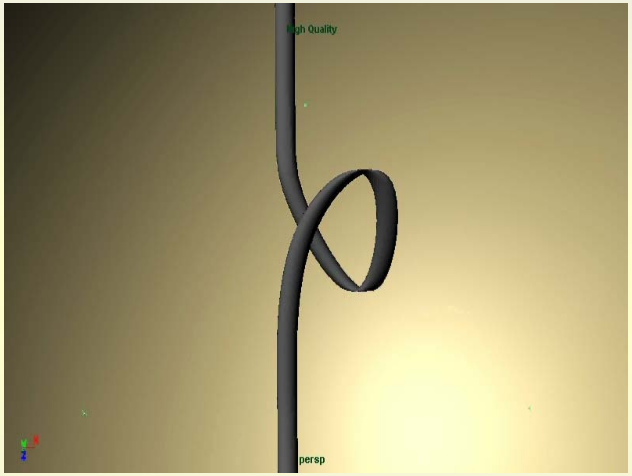


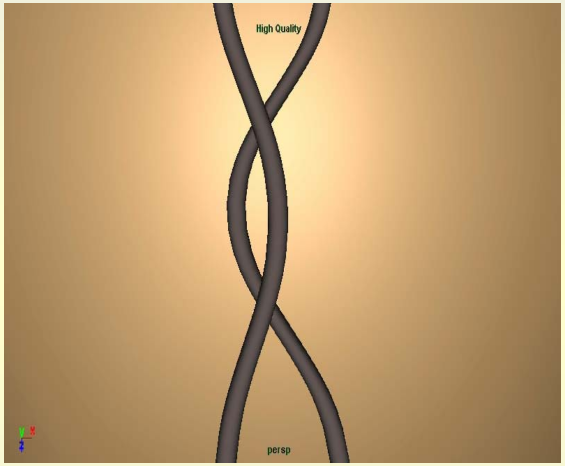
The following theorem is set to simplify the definition of the equivalence of knots.

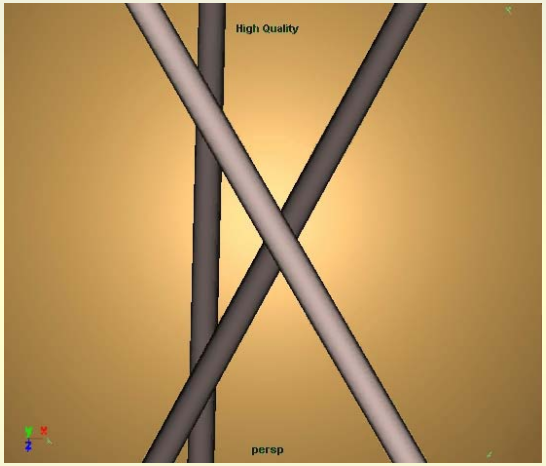
Theorem: Two knots (links) are equivalent if and only if one can get from the diagram of one of them to a diagram of the other by a finite sequence of the three Reidemeister moves.



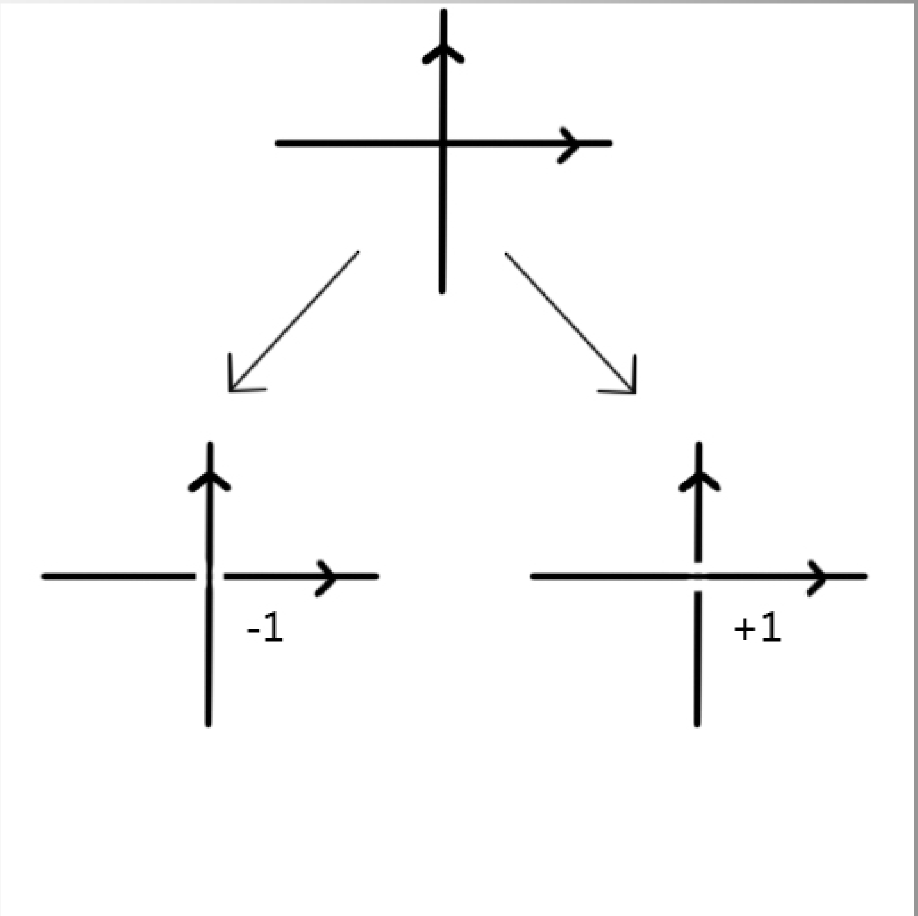
The Three Reidemeister moves







To represent the orientation on a link diagram a little arrow is drawn on each link component to show the direction on that component. At each crossing of an oriented link diagram we get oriented crossing which will be given signs of ±1.



As an important example of a link invariant we give the following definition

Definition: The linking number $lk(K_1,K_2)$ of a 2-component oriented link diagram $L=\{K_1,K_2\}$ is equal to $(1/2)[c(p_1)+c(p_2)+....+c(p_m)]$. Where K_1 and K_2 intersect at p_1, p_2,....., pm (ignoring the crossing points of the projections of K_1 and K_2, which are self-intersections of the knot component)

- **Theorem:** The linking number $lk(K_1,K_2)$ of a 2-component oriented link diagram $L=\{K_1,K_2\}$ is an invariant of L.

- *lk* is not very efficient at telling different links apart.
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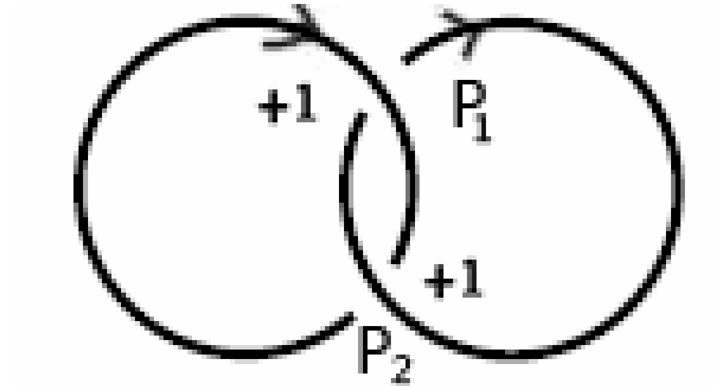
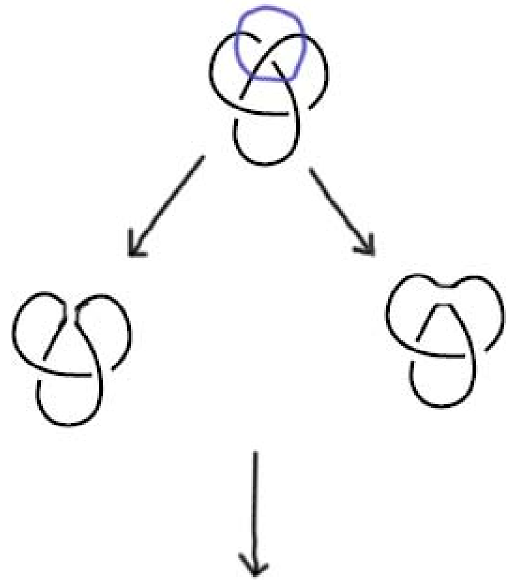
- ***lk*** **is not very efficient at telling different links apart.**

It is clear, at least intuitively that the following two link diagram are different.



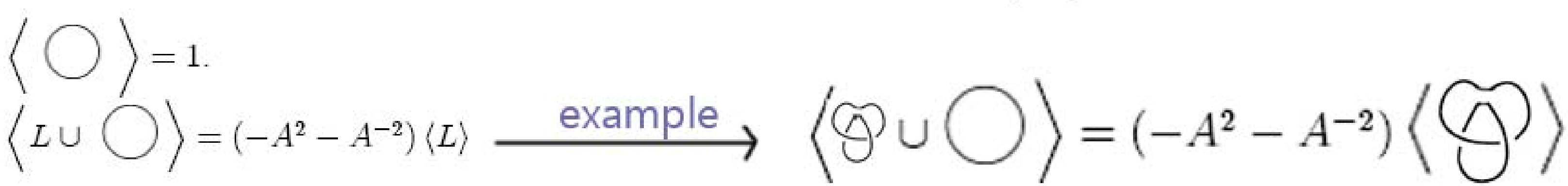
The need for more powerful invariant is clear.

Kauffman associates a polynomial with each knot in the euclidean space



The skein relation

Kauffman added two other rules to define his bracket polynomial



Definition 1 *The Kauffman bracket polynomial is a function from unoriented link diagrams in the oriented plane to Laurent polynomials with integer coefficients in an indeterminate A. It maps a diagram L to $\langle L \rangle \in \mathbb{Z}[A, A^{-1}]$ and it is characterized by the three rules:*

(i) $\left\langle \bigcirc \right\rangle = 1.$

(ii) $\left\langle L \cup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle L \rangle$

(iii) $\langle$  $\rangle = A \langle$  $\rangle + A^{-1} \langle$  $\rangle$

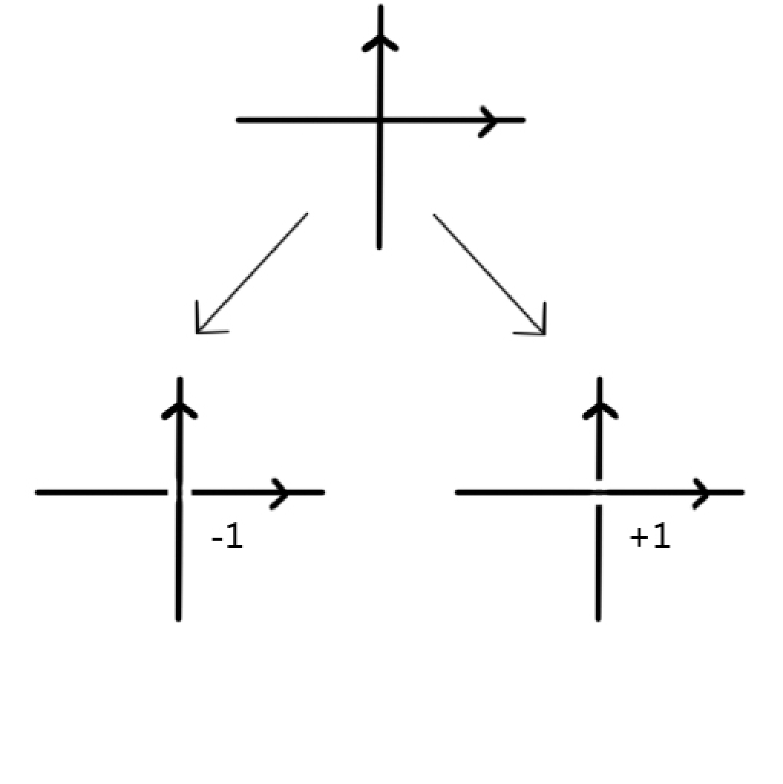
In this definition, $\bigcirc$ is the diagram of the unknot with no crossings, and $L \cup \bigcirc$ is a diagram consisting of the diagram L together with an extra closed curve $\bigcirc$ that contains no crossings at all, neither with itself nor with L. In (iii) the formula refers to three diagrams that are exactly the same except near a point where they differ in the way indicated. In the case when we have an oriented link L and we want to calculate the bracket polynomial for this link, we will denote by $|L|$ the nonoriented diagram that is obtained from L by forgetting the orientations of all components.

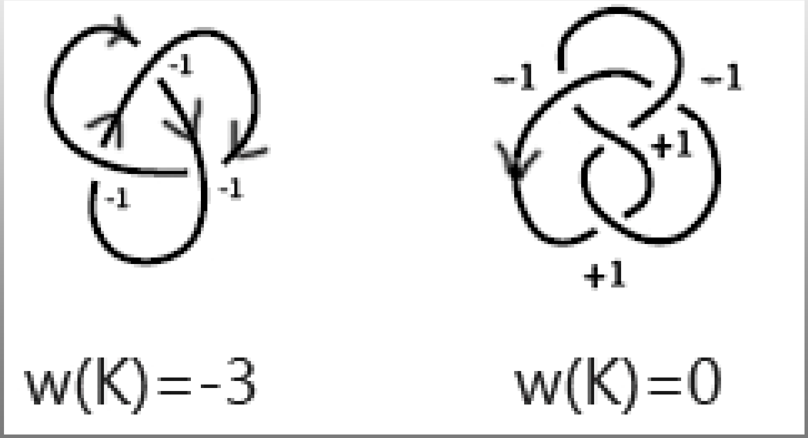
Kauffman's bracket polynomial is invariant under the second and the third Reidemeister moves, but not under the first Reidemeister move, hence the bracket polynomial is not a link invariant.

However, Kauffman normalizes this polynomial by multiplying it by the .*writhe*.

- **Definition:** **The writhe *w(L)* of a diagram *L* of an oriented link is the sum of the signs of the crossings of *L*, where each crossing has sign +1 or -1 as defined in the figure.**

- **The writhe of an oriented link diagram is not an invariant, since it is not an invariant under the first Reidemeister move.**





Theorem *Let L be a an oriented link diagram. Then the X polynomial defined by*

$$X(L) = (-A)^{-3w(L)} \langle |L| \rangle$$

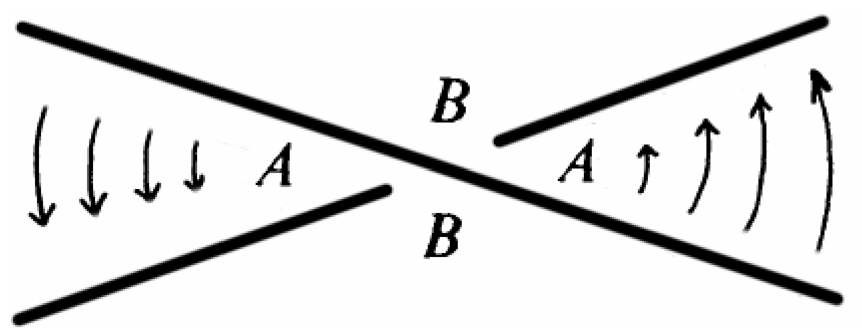
is an invariant of oriented links.

Definition *The Jones polynomial $V(L)$ of an oriented link L is the Laurant polynomial in the indeterminate q.With integer coefficients, defined by*

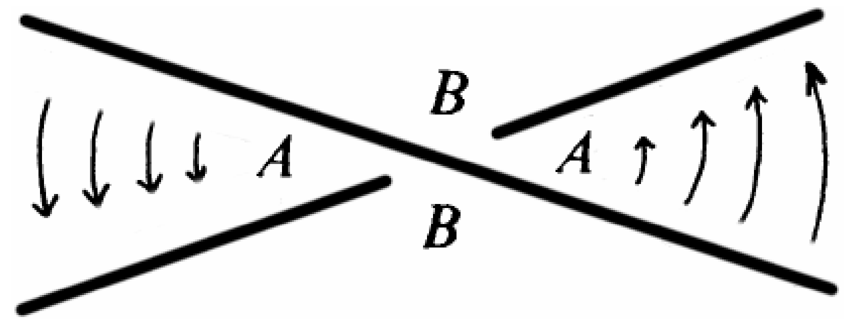
$$V(L) = \left((-A)^{-3w(L)} \langle |L| \rangle\right)_{q^{-1/4}=A} \in \mathbb{Z}[q^{-1}, q]$$

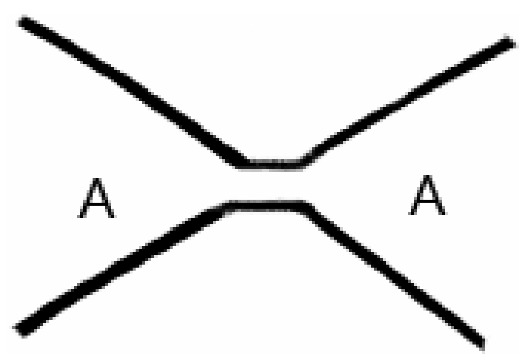
The state sum formula for the Jones polynomial in the Euclidean space

locally each crossing divides the plane into four regions. We label two of them with an *A* and two of them with a *B* by the following rule. The regions labeled by *A* are those swept out when we rotate the upper strand in the crossing counterclockwise, and the regions that are labeled by *B* are the other two regions

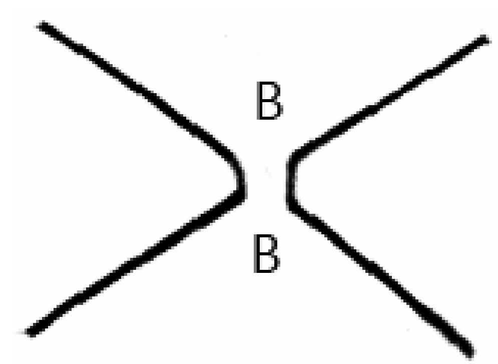


- **We see that *A* regions correspond to a smoothing in the considered crossing that "opens the *A*-channel" while *B* regions correspond to another smoothing in that crossing that opens the *B*-channel. Thus, we can associate at each crossing of the link diagram two types of splits; type *A*-split and type *B*-split.**



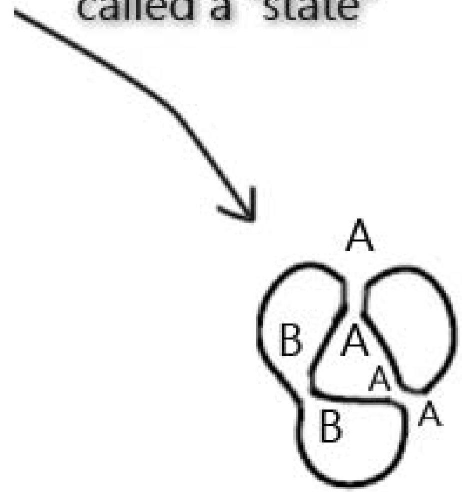


A-split



B-split

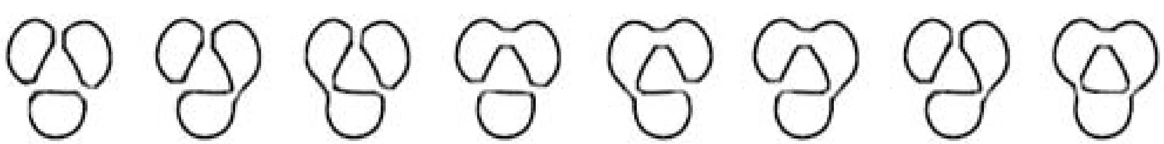
After applying the smoothing operation on all crossings in the knot we get what is called a “state”



A state

A state S of L is a choice of how to smooth all of the n crossings in the link diagram L

Suppse that there are n crossings in a link diagram L, since we have two choices of how to split each crossing, there will be exactly 2 to the power n states



The eight possible states for a diagram of the left trefoil the bracket polynomial of the link diagram L will be the sum over all possible states of these contributions. We write this as

$$< L >= \sum_{S} A^{a(S)-b(S)}(-A^2 - A^2)^{|S|-1}$$

$|S|$ is the total number of unknots in the state S.

$a(S)$ is the number of A-splits in the state S.

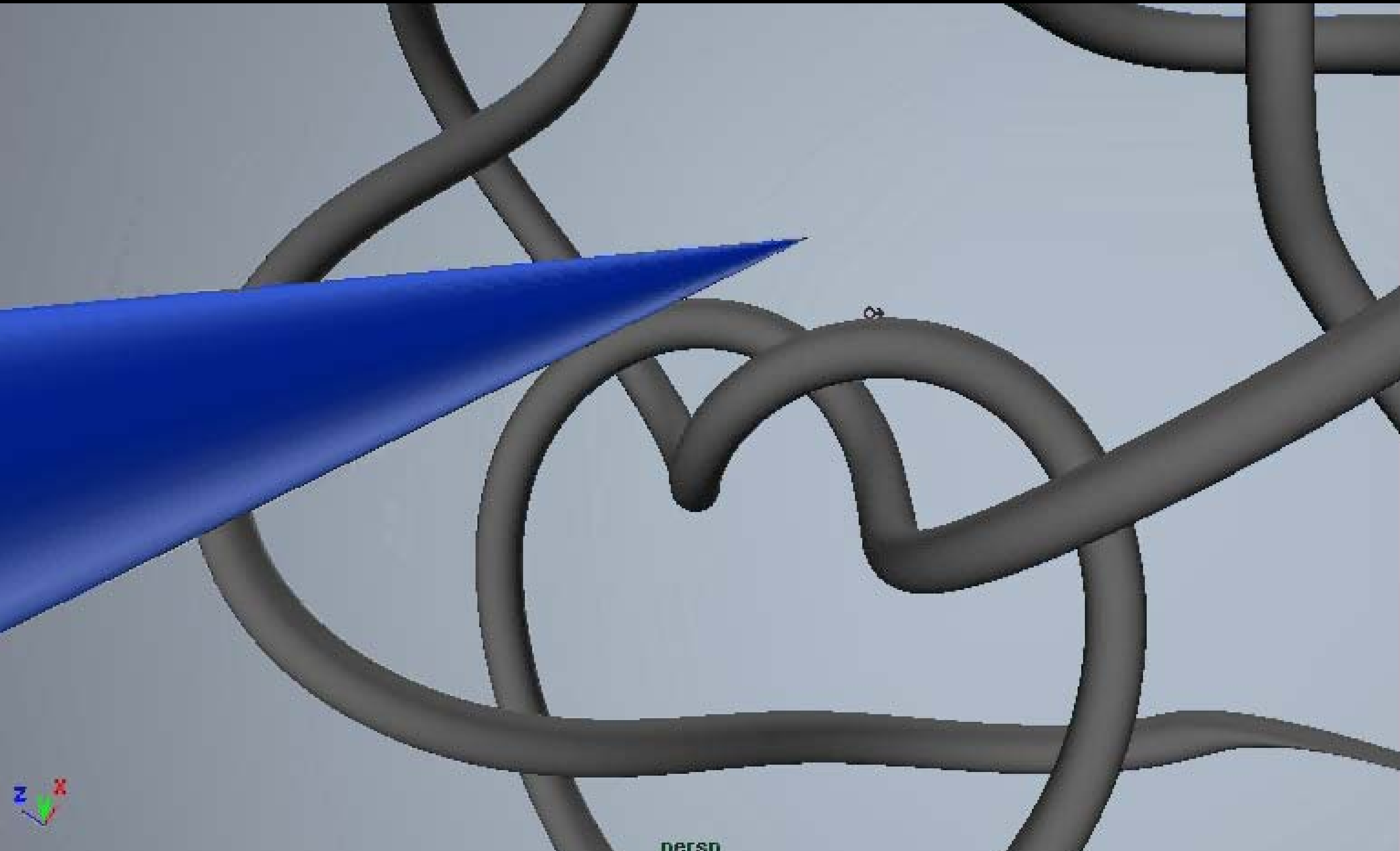
$b(S)$ is the number of B-splits in the state S.

Knots and Links in the Solid Torus and the Handlebody

- **Let HB^n be the closed handlebody with n handles, Let ST be the closed solid torus, then:**

- **Definition : A knot in the handlebody HB^n is the image of piecewise-linear one-to-one mapping $f: S^1 \rightarrow HB^n$ such that $f(S^1) \subseteq int(HB^n)$.**

- **Definition : Two knots K_1 and K_2 in HB^n are said to be isotopy equivalent if there exists an orientaion-preserving homeomorphism $\varphi: HB^n \rightarrow HB^n$ such that $\varphi(K_1)=K_2$, φ is the identity function on the boundary of HB^n, and φ preserves the orientation on the knots.**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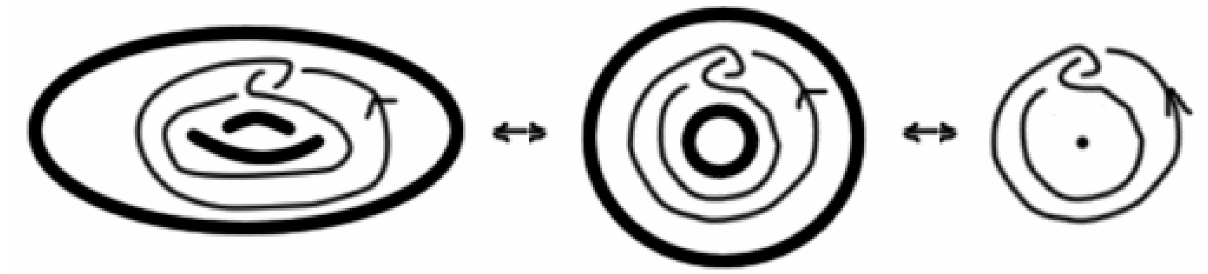
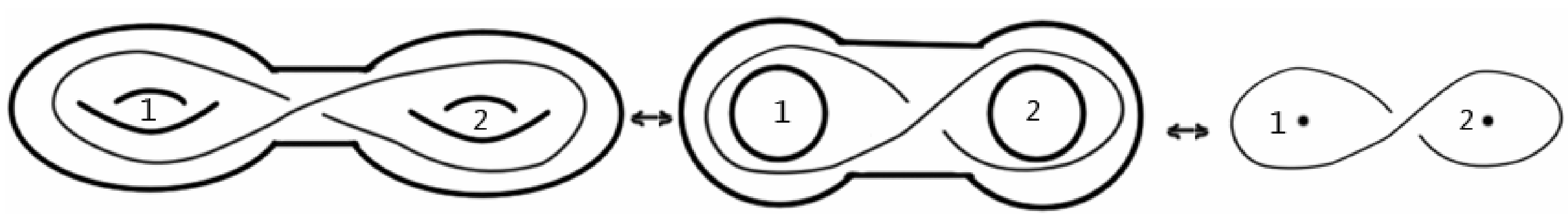


A knot diagram in the annulas and a punctured plane

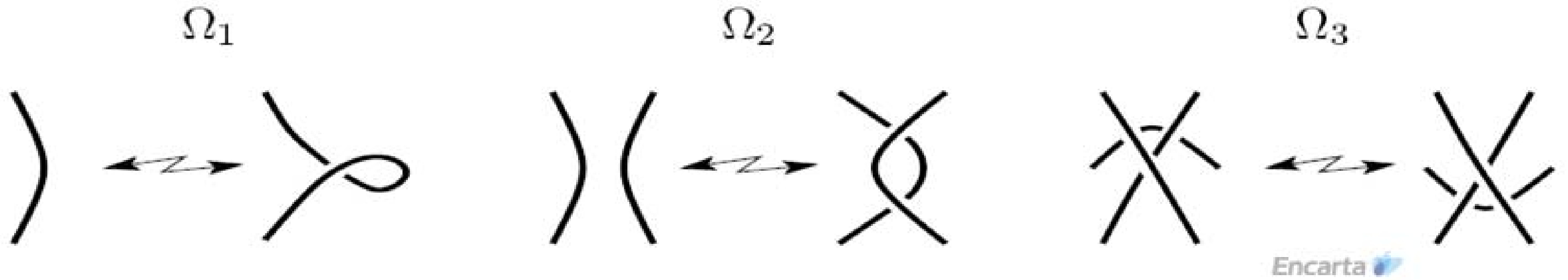


A knot in the handlebody with two handles and double punctured diagram

- **Definition:** A link *L* in the HB^n is a finite ordered collection of knots in HB^n that do not intersect each other.

- **Definition:** Two links $L=\{K_1,K_2,\ldots,Ku\}$ and $L'=\{K'_1,K'_2,\ldots,K'_v\}$ in HB^n are said to be isotopy equivalent if $u=v$, there exists an orientaion-preserving homeomorphism $\varphi:HB^n\rightarrow HB^n$ such that $\varphi(Ki)=K'i$ for $i=\{1,2,\ldots.,u\}$, φ is the identity function on the boundary of HB^n, and φ preserves the orientation on the knots.

- **Theorem:** Two knots K_1 and K_2 in the handlebody are isotopy equivalent if we can get from a diagram of one to a diagram of the other by a finite sequence of the three Reidemeister moves.

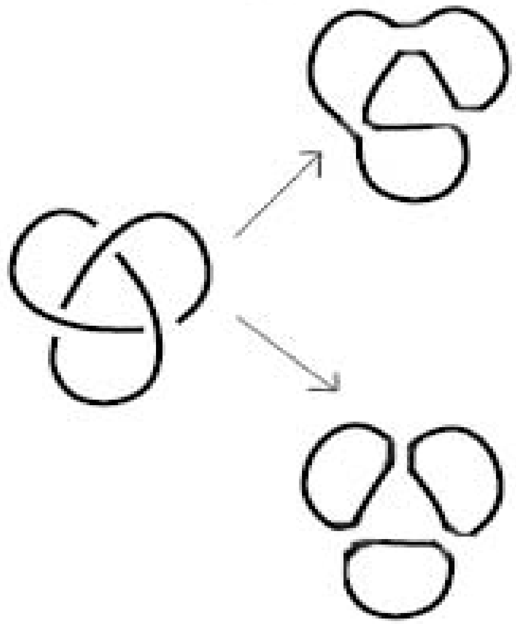


- **As an example of knot Invariants in the Solid Torus we give the following:**

- Aicardi's Invariant [S(K)](t) for Knots in the Solid Torus:

$$\mathrm{s}[K] = \sum_{\mathrm{p}} e(p)\left(t^{i_1(p)} + t^{i_2(p)}\right)$$

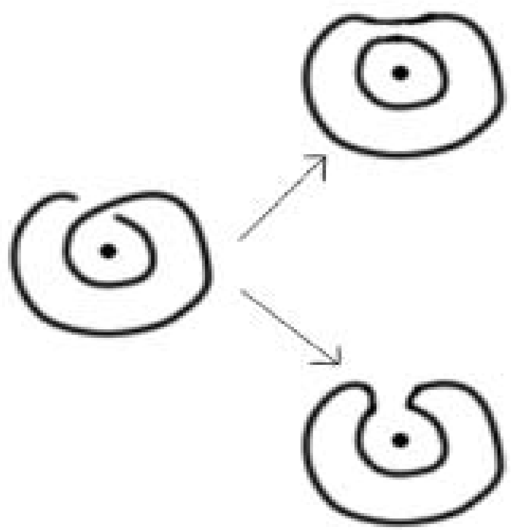
Jones polynomial in the Euclidean space



$$\left\langle |L| \cup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle |L| \rangle$$

$$\left\langle \bigcirc \right\rangle = 1$$

Jones polynomial in the solid torus



$$\left\langle |L| \cup \odot \right\rangle = ? \ \langle |L| \rangle$$

$$\left\langle \odot \right\rangle = ?$$

Jones Polynomial For Oriented Links in the Solid Torus

Theorem *Let L be an oriented link diagram in the solid torus, and let*

$$\langle\ \ \rangle : LST \to \mathbb{Z}[A, A^{-1}, t]$$

be a function defined by

the smoothing formulas

$$(1)\ \langle \times \rangle = A \langle\,)(\, \rangle + A^{-1} \langle \asymp \rangle$$

$$(2)\ \langle \times \rangle = A \langle \asymp \rangle + A^{-1} \langle\,)(\, \rangle$$

the reduction formulas

$$(1)\ \left\langle |L| \cup \bigcirc \right\rangle = (-A^2 - A^{-2}) \langle |L| \rangle$$

$$(2)\ \left\langle |L| \cup \odot \right\rangle = (-A^2 - A^{-2}) t \langle |L| \rangle$$

and the finishing formulas

$$(1)\ \left\langle \bigcirc \right\rangle = 1$$

$$(2)\ \left\langle \odot \right\rangle = t$$

Then $Y(L) = (-A^3)^{-w(L)} \langle |L| \rangle$ is a polynomial invariant for oriented links

solid torus.

The State Sum Formula for the *Y* Invariant

$$< L >= \sum_S A^{a(S)-b(S)}(-A^2 - A^2)^{|S|-1} t^{|D|}$$

$|S|$ is the total number of circles and dotted circles in the state S.

$|D|$ is the number of the dotted circles.

$a(S)$ is the number of A-splits in the state S.

$b(S)$ is the number of B-splits in the state S.

Comparison Between Aicardi's Invariant and the Y Polynomial

B'

$$Y(B') = A^{-16}t^2 + A^{-8}t^2 + A^{-4} + A^{-12} - A^{-8}$$

$$[S(B')](t) = +t^{+1} + t^{-1}$$

C'

$$Y(C') = A^{16}t^2 + A^{8}t^2 + A^{4} + A^{12} - A^{8}$$

$$[S(C')](t) = -t^{+1} - t^{-1}$$

K

$$Y(K) = A^{8} - A^{8}t^2 + t^2$$

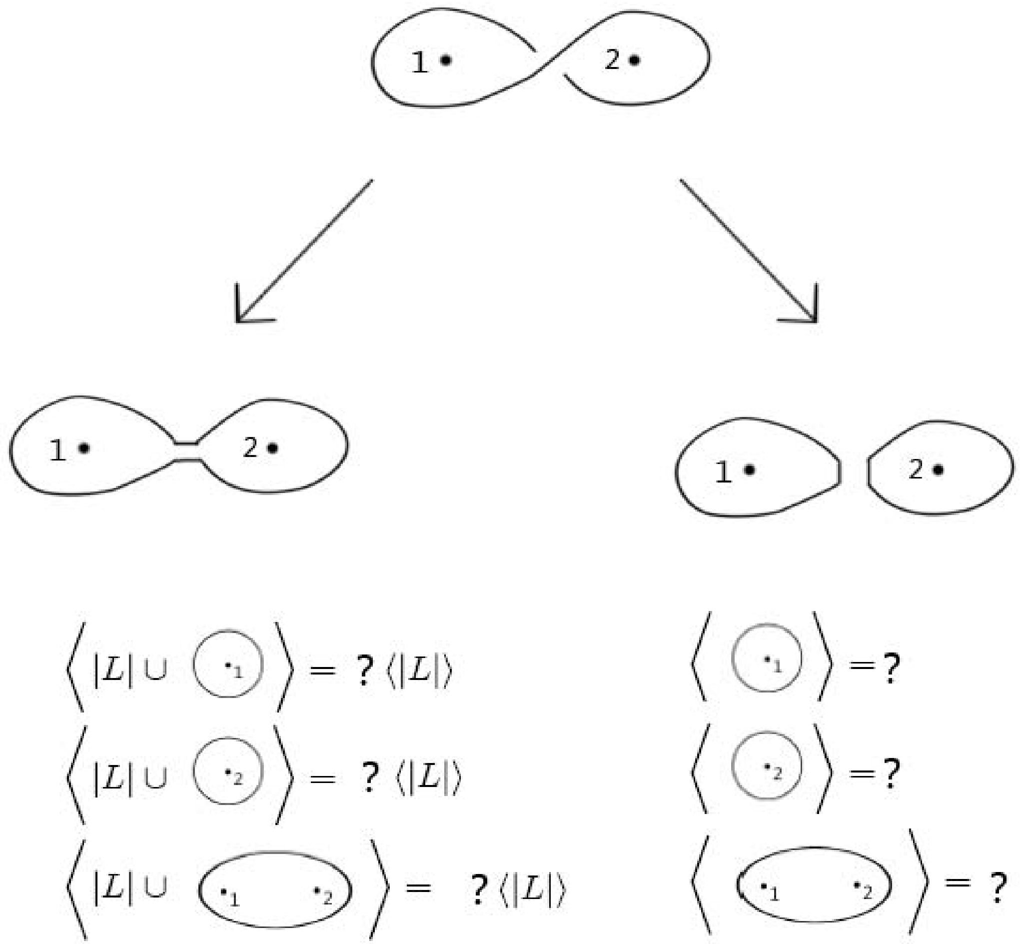
$$[S(K)](t) = -t^{+1} - t^{-1}$$

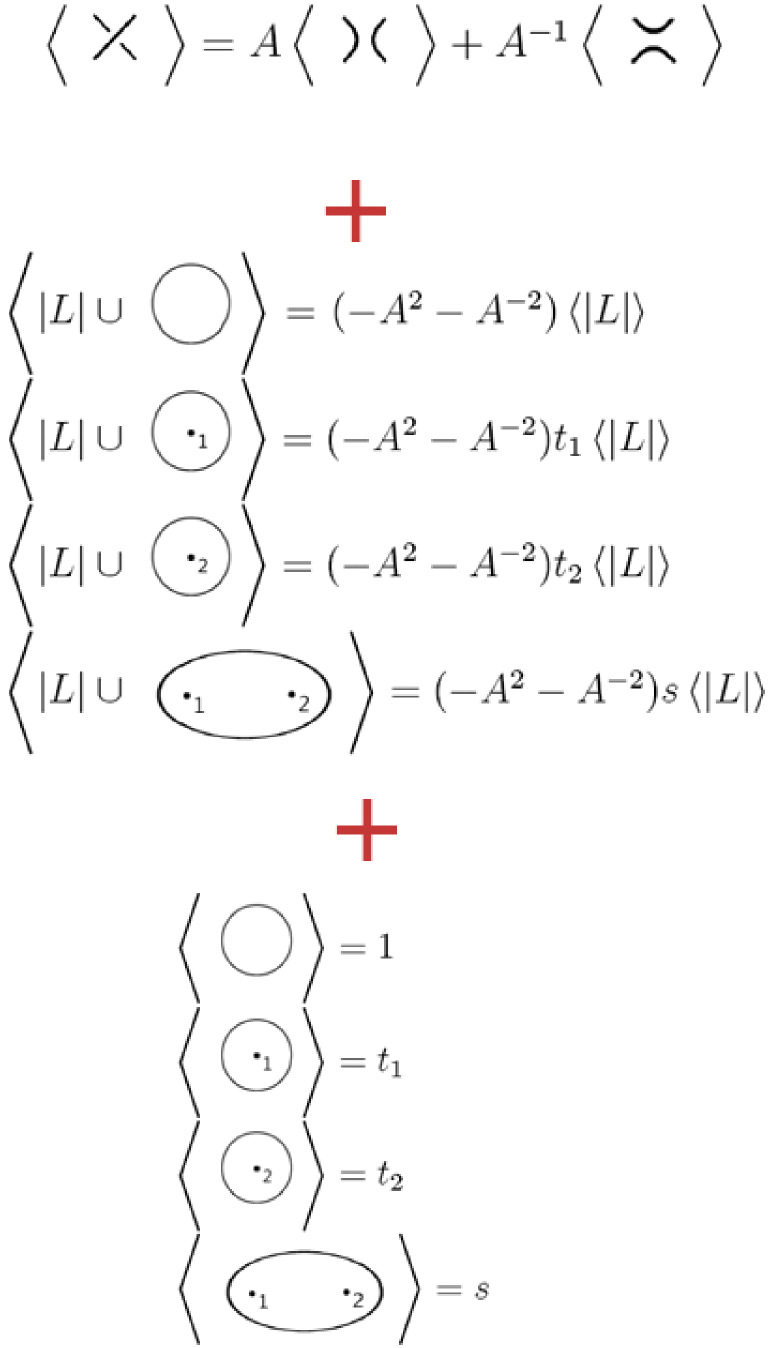
K'

$$Y(K') = A^{-8} - A^{-8}t^2 + t^2$$

$$[S(K')](t) = +t^{+1} + t^{-1}$$

How can we derive Jones polynomial for oriented links in the handlebody?

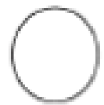
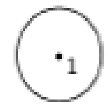


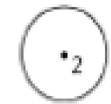


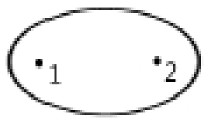
< > associates with every knot in the handlebody a polynomial with four variables with integer coefficients

The State Sum Formula for the *Y* invariant in the handlebody with two handles

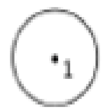
$$< L >= \sum_S A^{a(S)-b(S)}(-A^2 - A^2)^{|S|-1} t_1^{|D_1|}.t_2^{|D_2|}.s^{|H|}$$

$|S|$ is the total number of circles , dottedcircle$_1$ 

, dottedcircle$_2$ 

, and double-dottedcircle 

in the state S.

$|D_1|$ is the number of the dottedcircle$_1$ 

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$|D_2|$ is the number of the dottedcircle$_2$ 

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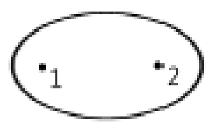
$|H|$ is the number of the double-dottedcircle 

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$a(S)$ is the number of A-split in the state S.

$b(S)$ is the number of B-split in the state S.

Thank You For listening