Lie and conditional symmetries of nonlinear boundary value problems: definitions, algorithms and applications

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1. Lie symmetries of boundary value problems (BVPs): Bluman's definition, its applicability range and the relevant example
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6. Conditional invariance of BVPs: definition and examples
7. Conclusions

1. R. Cherniha & S.Kovalenko, 2011 Lie symmetries and reductions of multidimensional boundary value problems of the Stefan type. J. Phys. A : Math. Theor. **44** , 485202 (25 pp.).
2. R. Cherniha & S. Kovalenko, 2012 Lie symmetries of nonlinear boundary value problems. Commun. Nonlinear Sci. Numer. Simulat. **17** 71–84.
3. R. Cherniha & J.R. King Lie and conditional symmetries of a class of nonlinear (1+2)-dimensional boundary value problems (in preparation)

A PDE cannot model any real process without additional condition(s) on the unknown function(s) because one reflects only a general physical (biological, chemical etc.) law. Only a boundary-value problem (BVP) based on the given PDE can describe many real processes arising in nature and society.

One may note that the symmetry-based methods were not widely used for solving BVPs. The obvious reason follows from the following observation: the relevant boundary and initial conditions are usually not invariant under any transformations, i.e., they don't admit any symmetry of the governing PDE(s). Nevertheless there are some classes of BVPs which can be solved by means of the Lie symmetry based algorithm. This algorithm uses the notion of Lie's invariance of BVP in question.

The first rigorous definition of Lie's invariance for BVPs was formulated by Bluman in 1970s.[G.W. Bluman, 1971,1974]. This definition and several examples are summarized in his book [Bluman & Anco, 2002] and was used (explicitly or implicitly) in several papers to derive exact solutions of some BVPs. It should be noted that Ibragimov's definition of BVP invariance [N.H. Ibragimov, 1992, 2009, 2011], which was formulated independently, is equivalent to Bluman's. Notably, one may say that BVP invariance became to be widely investigated only since 1990s.

Bluman's definition

In this section, I restrict myself to the case when the basic equation of BVP is a two-dimensional evolution PDE of kth–order ($k \geq 2$). In this case the relevant BVP may be formulated as follows:

$$u_t = F\left(x, u, u_x, \ldots, u_x^{(k)}\right), \ (t, x) \in \Omega \subset \mathbf{R}^2 \tag{1}$$

$$s_a(t, x) = 0 : \ B_a\left(t, x, u, u_x, \ldots, u_x^{(k-1)}\right) = 0, \ a = 1, 2, \ldots, p, \tag{2}$$

where F and B_a are smooth functions in the corresponding domains, Ω is a domain with smooth boundaries and $s_a(t, x)$ are smooth curves. t and x denote differentiation with respect to these variables, $u_x^{(j)} = \frac{\partial^j u}{\partial x^j}, j = 1, 2, \ldots, k$. It is assumed that BVP (1)–(2) has a classical solution (in a usual sense).

Consider the infinitesimal generator

$$X = \xi^0(t,x)\frac{\partial}{\partial t} + \xi^1(t,x)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} \tag{3}$$

(hereafter ξ^0, ξ^1 and η are known smooth functions), which defines a Lie symmetry acting on (t,x,u)–space ! Let $X^{(k)}$ be the kth–prolongation of the generator X calculated by the well-known prolongation formulae.

Definition (Bluman &Kumei, 1989; Bluman & Anco,2002)

The Lie symmetry X (3) is admitted by the boundary value problem (1)–(2) if:

(a) $X^{(k)}\left(F\left(x,u,u_x,\ldots,u_x^{(k)}\right) - u_t\right) = 0$ when u satisfies (1);

(b) $X(s_a(t,x)) = 0$ when $s_a(t,x) = 0,\ a = 1,2,\ldots,p;$

(c) $X^{(k-1)}\left(B_a\left(t,x,u,u_x,\ldots,u_x^{(k-1)}\right)\right) = 0$ when $B_a = 0$ on $s_a(t,x) = 0$ $a = 1,2,\ldots,p.$

Let as consider example [Bluman & Anco,2002]. MAI of the linear heat equation consists of 6 basic operators creating AL_6 [S.Lie, 1881] ???? to check !

$$\begin{array}{l} \partial_t, \partial_x, 2t\partial_t + x\partial_x, \\ I = u\partial_u,\ G = t\partial_x - \frac{1}{2}xu\partial_u, \\ \Pi = t^2\partial_t + tx\partial_x - \frac{1}{2}(\frac{x^2}{2} + t)u\partial_u, \end{array} \tag{4}$$

and the standard (for any linear PDE !) operator X^{∞}. Consider the Cauchy problem

$$\begin{array}{ll} u_t = u_{xx}, & t > 0, x \in \mathbf{R} \\ t = 0: & u = u_0(x) \end{array} \tag{5}$$

Consider the most general form of operators from AL_6

$$X = c_i X_i = \xi^0(t,x)\tfrac{\partial}{\partial t} + \xi^1(t,x)\tfrac{\partial}{\partial x} + f(t,x)u\tfrac{\partial}{\partial u}$$
$$\xi^0(t,x) = c_1 + 2c_3 t + c_6 t^2, \quad \xi^1(t,x) = c_2 + c_3 x + c_5 t + c_6 tx \tag{6}$$
$$f(t,x) = c_4 - \tfrac{c_5}{2}x - c_6\tfrac{1}{2}(\tfrac{x^2}{2} + t)$$

Items [b]-[c] of Definition lead to

$$c_1 = 0, \quad f(0,x)u_0(x) = \xi^1(0,x)\frac{du_0}{dx} \tag{7}$$

An interesting case is the Dirac function $u_0 = \delta(x)$, when one arrives at 3-dim Lie algebra

$$G, \quad \Pi, \quad D_1 = 2t\partial_t + x\partial_x - u\partial_u \tag{8}$$

In particular, the well-known solution

$$u(t,x) = \frac{1}{\sqrt{4\pi t}}\exp\Big(-\frac{x^2}{4t}\Big) \tag{9}$$

can be obtained

Bluman's definition can not be directly applied to BVP of more general form:

- with boundary conditions defined on infinity, i.e. manifolds including points (or surfaces) at infinity
- with boundary conditions on the moving surfaces, which are described by unknown functions

Let us consider BVP (1)–(2), which includes also the boundary conditions

$$x = \infty : \ \Gamma\left(t, u, u_x, \ldots, u_x^{(k-1)}\right) = 0,$$

Assume that Eq. (1) and conditions (2) are invariant under the group of translations on the plane (t, x):

$$t' = t + \lambda_1 \varepsilon, \quad x' = x + \lambda_2 \varepsilon, \quad u' = u, \ \lambda_1 \lambda_2 \neq 0, \tag{10}$$

so that the corresponding infinitesimal generator is

$$X = \lambda_1 \partial_t + \lambda_2 \partial_x \tag{11}$$

Nevertheless it is clear that the condition (10) including
$x = \infty$ is invariant under (10)
the Definition is not applicable. Moreover, the problem occurs if one generalizes Definition in the standard way by formulation an additional condition for (11) like those (b) and (c): $x - L = 0$, where $L \to \infty$ because this leads to

$$\lim_{L\to\infty} X(x-L)|_{x=L} = \lim_{L\to\infty} \lambda_2 = \lambda_2 \neq 0.$$

Consider a BVP for a system of n evolution equations ($n \geq 2$) with $m+1$ independent (t,x) (hereafter $x=(x_1,x_2,\ldots,x_m)$) and n dependent $u=(u_1,u_2,\ldots,u_n)$ variables. Let us assume that the basic equations are

$$u_t^i = F^i\left(t,x,u,u_x,\ldots,u_x^{(k)}\right),\ i=1,\ldots,n \tag{12}$$

and are defined on $(0,+\infty)\times\Omega\subset\mathbb{R}^{m+1}$ Hereafter

$$u_x^{(j)} = \frac{\partial^j u}{\partial x_{j_1}\ldots\partial x_{j_n}},\ j=1,2,\ldots,k;\ j_1+\ldots+j_n=j$$

Consider three types of boundary and initial conditions

$$s_a(t,x)=0:\ B_a^j\left(t,x,u,\ldots,u_x^{(k_a^j)}\right)=0, \tag{13}$$

$$S_b(t,x)=0:\ B_b^l\left(t,x,u,\ldots,u_x^{(k_b^l)},S_b^{(1)},\ldots,S_b^{(K_b^l)}\right)=0, \tag{14}$$

$$\gamma_c(t,x)=\infty:\ \Gamma_c^m\left(t,x,u,\ldots,u_x^{(k_c^m)}\right)=0. \tag{15}$$

where $a=1,\ldots,p$, $j=1,\ldots,n_a$; $b=1,\ldots,q$, $l=1,\ldots,n_b$; $c=1,\ldots,q$, $m=1,\ldots,n_c$; $k_a^j<k$, $k_b^l<k$ and $k_c^m<k$ are given numbers.

Consider an N–parameter (local) Lie group G_N of point transformations of variables (t, x, u) in the Euclidean space $\mathbb{R}^{n+m+1}$ (open subset of $\mathbb{R}^{n+m+1}$) defined by the equations

$$t^* = T(t, x, \varepsilon), \quad x_i^* = X_i(t, x, \varepsilon), \quad u_j^* = U_j(t, x, u, \varepsilon),$$

where $i = 1, \ldots, m$, $j = 1, \ldots, n$; $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ — are the group parameters. According to the general Lie group theory, one may construct the corresponding N-dimensional Lie algebra L_N with the basic generators

$$X_\alpha = \xi^0_\alpha \frac{\partial}{\partial t} + \xi^1_\alpha \frac{\partial}{\partial x_1} + \ldots + \xi^m_\alpha \frac{\partial}{\partial x_m} + \eta^1_\alpha \frac{\partial}{\partial u_1} + \ldots + \eta^n_\alpha \frac{\partial}{\partial u_n}, \ \alpha = 1, 2, \ldots, N, \tag{16}$$

We assume that $S_b(t,x)=0,\ b=1,\dots,q$ are non-degenerating manifolds of the group G_N, i.e., can be expressed via its invariants, which depend only on the independent variables (t,x,u).
In the extended space $\mathbb{R}^{n+m+q+1}$ of the variables (t,x,u,S) (here $S=(S_1,...,S_q)$), the Lie algebra L_N defines the Lie group $\widetilde{G}_N$

$$t^* = T(t,x,\varepsilon),\ x_i^* = X_i(t,x,\varepsilon),\ u_j^* = U_j(t,x,u,\varepsilon),\ S_b^* = S_b(t,x), \qquad (17)$$

where $i=1,\dots,m,\ j=1,\dots,n,\ b=1,\dots,q.$

Definition

A boundary value problem of the form (12)–(15) is called to be invariant with respect to the Lie group $\widetilde{G}_N$ if:

(a) the manifold determined by system (12) in the space of variables $\left(t, x, u, \ldots, u_x^{(k)}\right)$ is invariant with respect to the kth–order prolongation of the group G_N;

(b) each manifold determined by conditions (13) with the any fixed number a is invariant with respect to the k_ath–order prolongation of the group G_N in the space of variables $\left(t, x, u, \ldots, u_x^{(k_a)}\right)$, where $k_a = \max\{k_a^j,\ j = 1, \ldots, n_a\}$;

(c) each manifold determined by conditions (14) with the any fixed number b is invariant with respect to the k_bth–order prolongation of the group $\widetilde{G}_N$ in the space of variables $\left(t, x, u, \ldots, u_x^{(k_b)}, S_b, \ldots, S_b^{(k_b)}\right)$, where $k_b = \max\{k_b^l,\ K_b^l,\ l = 1, \ldots, n_b\}$;

(d) each manifold determined by conditions (15) with the any fixed number c is invariant with respect to the k_cth–order prolongation of the group G_N in the space of variables $\left(t, x, u, \ldots, u_x^{(k_c)}\right)$, where $k_c = \max\{k_c^m,\ m = 1, \ldots, n_c\}$.

Definition

The functions $u_j = \Phi_j(t,x)$, $j = 1,\dots,n$ and $S_b = \Psi_b(t,x)$, $b = 1,\dots,q$ form an invariant solution (u,S) of BVP (12)–(15) corresponding to the Lie group $\widetilde{G}_N$ if:

(i) (u,S) satisfies equations and conditions (12)–(15) ;

(ii) the manifold $\mathcal{M} = \{u_j = \Phi_j(t,x),\ j = 1,\dots,n;\ S_b = \Psi_b(t,x),\ b = 1,\dots,q\}$ is an invariant manifold of this Lie group.

Remark 1. Both definitions can be straightforwardly generalized on BVPs with governing systems of equations of hyperbolic, elliptic and mixed types. However, one should additionally assume that n-component governing system of PDEs are presented in a 'canonical' form (some authors uses the natation 'involution form' in this context), i.e. one possesses a simplest form and there are no any non-trivial differential consequences.

Algorithm for solving the group classification problem for a BVP class

If the system of differential equations contain as coefficients arbitrary functions then the group classification problem springs up. Such kind of problems was formulated and solved for a class of non-linear heat equations in the pioneering Ovsiannikov work in 1959. Ovsiannikov's method is based on the classical Lie scheme and a set of equivalence transformations of the given class of PDEs.

At the present time, more general algorithms for group classification problems were developed, which take into account form-preserving (admissible) transformations [J.Kingston, 1991] and were successfully applied to different classes of PDEs . In particular, the group classification problems were solved for classes of single RDC equations and RD systems using such transformations in [R.Ch. & J.R.King 2001,2005,2006; R.Ch. & M.Serov, 2006; R.Ch., M.Serov & I. Rassokha 2008])

We propose the following algorithm of the group classification

(I) to construct the equivalence group E_{eq} of local transformations, which transform the governing system of equations into itself;

(II) to extend space of E_{eq} action on the variables $S = (S_1, ..., S_q)$ by adding the identity transformations for them, denoting the group obtained as $\widetilde{E}_{eq}$;

(III) to find the equivalence group $\widetilde{E}^{BVP}_{eq}$ of local transformations, which transform the class of BVPs (12)–(15) into itself, one extends space of the $\widetilde{E}_{eq}$ action on the prolonged space, where all arbitrary elements arising in boundary conditions (13)–(15) are treated as new variables.

(IV) to perform the group classification of the governing system (12) up to local transformations generated by the group $\widetilde{E}^{BVP}_{eq}$;

(V) using Definition, to find the principal group of invariance $\widetilde{G}^0$, which is admitted by each BVP belonging to the class in question;

(VI) using Definition and the results obtained at step (IV), to describe all possible $\widetilde{E}^{BVP}_{eq}$-inequivalent BVPs of the form (12)–(15) admitting maximal invariance groups of higher dimensionality than $\widetilde{G}^0$.

Let us consider the class of (1+3)-dim. BVPs describing melting and evaporation of a solid material (see [Lyubov B Ya and Sobol' E N 1983]– linear case; [R.Ch. 2003] – nonlinear case)

$$\frac{\partial u}{\partial t} = \nabla\left(d_1(u)\nabla u\right), \quad (t, \mathbf{x}) \in \Omega_1(t), \tag{18}$$

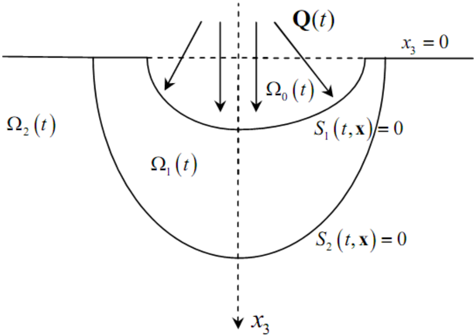
$$\frac{\partial v}{\partial t} = \nabla\left(d_2(v)\nabla v\right), \quad (t, \mathbf{x}) \in \Omega_2(t), \tag{19}$$

$$S_1(t,\mathbf{x}) = 0: \; d_{1v}\frac{\partial u}{\partial \mathbf{n}_1} = H_v \mathbf{V}_1 \cdot \mathbf{n}_1 - \mathbf{Q}(t)\cdot \mathbf{n}_1, \; u = u_v, \tag{20}$$

$$S_2(t,\mathbf{x}) = 0: d_{2m}\frac{\partial v}{\partial \mathbf{n}_2} = d_{1m}\frac{\partial u}{\partial \mathbf{n}_2} + H_m \mathbf{V}_2 \cdot \mathbf{n}_2, u = u_m, v = v_m, \tag{21}$$

$$|\mathbf{x}| = +\infty: \; v = v_\infty, \quad t \in \mathfrak{T}, \tag{22}$$

- $d_1(u)$, $d_2(v)$ are smooth non-negative functions
- d_{1v}, d_{1m}, d_{2m}, H_v, H_m are positive parameters with physical interpretation;
- $\mathbf{Q}(t)\cdot\mathbf{n}_1 \neq 0$, $\mathbf{V}_k \cdot \mathbf{n}_k \neq 0$, $k = 1, 2$; $\nabla \equiv (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$;
- $u_v, u_m, v_m, v_\infty \geq 0$: $u_v \neq u_m$, $v_m \neq v_\infty$; $|\nabla S_k| \neq 0$, $\frac{\partial S_k}{\partial t} \neq 0$, $k = 1, 2$.



$\mathbf{Q}(t)$
$x_3 = 0$
$\Omega_0(t)$
$\Omega_2(t)$
$S_1(t,\mathbf{x}) = 0$
$\Omega_1(t)$
$S_2(t,\mathbf{x}) = 0$
x_3

Figure:

Table 3. The group classification of system (18)–(19) [R.Ch.& J.R.King, 2006]

no	$d_1(u)$	$d_2(v)$	MAI
1.	$\forall$	$\forall$	$AE(1,3) = \langle \partial_t, \partial_{x_a}, x_a\partial_{x_b} - x_b\partial_{x_a}, 2t\partial_t + x_a\partial_{x_a} \rangle$
2.	k_1	$\forall$	$AE(1,3), u\partial_u, \alpha(t,\mathbf{x})\partial_u$
3.	$\forall$	k_2	$AE(1,3), v\partial_v, \beta(t,\mathbf{x})\partial_v$
4.	e^u	e^v	$AE(1,3), x_a\partial_{x_a} + 2\partial_u + 2\partial_v$
5.	e^u	v^m	$AE(1,3), x_a\partial_{x_a} + 2\partial_u + \frac{2}{m}v\partial_v$
6.	u^n	e^v	$AE(1,3), x_a\partial_{x_a} + \frac{2}{n}u\partial_u + 2\partial_v$
7.	u^n	v^m	$AE(1,3), D = x_a\partial_{x_a} + \frac{2}{n}u\partial_u + \frac{2}{m}v\partial_v$
8.	$u^{-\frac{4}{5}}$	$v^{-\frac{4}{5}}$	$AE(1,3), \lvert\mathbf{x}\rvert^2\partial_{x_b} - 2x_bx_a\partial_{x_a} + 5x_bu\partial_u + 5x_bv\partial_v,\ D$
9.	k_1	k_2	$AE(1,3), u\partial_u, v\partial_v, \alpha(t,\mathbf{x})\partial_u, \beta(t,\mathbf{x})\partial_v,$ $G_a = t\partial_{x_a} - x_a\left(\frac{1}{2k_1}u\partial_u + \frac{1}{2k_2}v\partial_v\right),$ $\Pi = t^2\partial_t + tx_a\partial_{x_a} - \frac{1}{4k_1}(\lvert\mathbf{x}\rvert^2 + 6k_1t)u\partial_u -$ $-\frac{1}{4k_2}(\lvert\mathbf{x}\rvert^2 + 6k_2t)v\partial_v$
10.	k_1	k_1	$AE(1,3), u\partial_u, v\partial_v, v\partial_u, u\partial_v,$ $\alpha(t,\mathbf{x})\partial_u, \beta(t,\mathbf{x})\partial_v, G_a,\ \Pi\ (k_2 = k_1)$

Theorem

BVP (18)–(22) with the arbitrary given functions $d_1(u)$, $d_2(v)$ ($d_1(u) \neq d_2(v)$) and $Q_a(t), \ a = 1, 2, 3$ is invariant under the three-dim. Lie algebra (trivial Lie algebra) presented in case 1 of Table 4. The maximal algebra of invariance (MAI) of BVP (18)–(22) doesn't depend on the form of $d_1(u)$ and $d_2(v)$. There are only five BVPs from the class (18)–(22) with the correctly-specified functions $Q_a(t), \ a = 1, 2, 3$ admitting MAI of a higher dimensionality, namely: four- or five-dim. Lie algebras of invariance (up to equivalent representations generated by equivalence transformations from the group $\widetilde{E}^{BVP}_{eq}$). These MAI and the relevant functions $Q_a(t), \ a = 1, 2, 3$ are presented in cases 2–6 of Table 4.

Table 4. Lie invariance of BVP (18)–(22)

no	$Q_1(t)$	$Q_2(t)$	$Q_3(t)$	MAI
1.	$\forall$	$\forall$	$\forall$	P_1, P_2, P_3
2.	0	0	$q(t)$	P_1, P_2, P_3, J_{12}
3.	$\Theta_1(\lambda t)$	$\Theta_2(\lambda t)$	q_3	$P_1, P_2, P_3, P_t + \lambda J_{12}$
4.	$\frac{1}{\sqrt{t}}\,\Theta_1\left(\frac{1}{2}\lambda \log t\right)$	$\frac{1}{\sqrt{t}}\,\Theta_2\left(\frac{1}{2}\lambda \log t\right)$	$\frac{q_3}{\sqrt{t}}$	$P_1, P_2, P_3, D_0 + \lambda J_{12}$
5.	0	0	q	$P_t, P_1, P_2, P_3, J_{12}$
6.	0	0	$\frac{q}{\sqrt{t}}$	$P_1, P_2, P_3, D_0, J_{12}$

where : $P_a = \partial_{x_a}, P_t = \partial_t, J_{12} = x_1\partial_{x_2} - x_2\partial_{x_1}, D_0 = 2t\partial_t + x_a\partial_{x_a}$

$$\Theta_1(\tau) = q_1 \cos\tau + q_2 \sin\tau, \quad \Theta_2(\tau) = -q_1 \sin\tau + q_2 \cos\tau,$$

$q \neq 0$, q_1, q_2, q_3, λ are arbitrary constants, $q(t) \neq 0$ is an arbitrary smooth function.

Now we demonstrate how one can apply the result obtained to construct exact solutions of BVPs from the class under study. Let us consider a nonlinear model of heat transfer processes in metals under the action of intense constant energy flaxes directed perpendicular to the metal surface. This model coincides with the BVP problem (18)–(22), where $\mathbf{Q}(t) = \mathbf{q} \equiv (0, 0, q)$, $q =$ const. According to Table 4 such BVP admits the five-dim. MAI A_5. First of all, we construct the optimal systems of s-dimensional subalgebras ($s \leq 5$) of A_5. This can be done using the well-known Lie-Goursat classification method for the subalgebras of algebraic sums of Lie algebras and the known results of subalgebras classification of low-dimensional real Lie algebras [Pathera J et al 1975; Pathera J & Winternitz P 1977]. Thus, we've constructed the complete list of subalgebras of the algebra A_5. This list can be divided on subalgebras of different dimensionality.

Reduction of BVP (18)–(22) via $\langle J_{12} + \beta\left(P_3 \cos\phi + P_t \sin\phi\right), P_3 \sin\phi - P_t \cos\phi\rangle$

- Ansatz and BVP obtained :

$$u = u(r,z),\ v = v(r,z),\ S_k = S_k(r,z),\ k = 1,2,$$

where $z = x_3 - \mu t - \beta \arctan\frac{x_1}{x_2}$, $r = \sqrt{x_1^2 + x_2^2}$ are invariant variables.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r d_1(u)\frac{\partial u}{\partial r}\right) + \left(\frac{\beta^2}{r^2}+1\right)\frac{\partial}{\partial z}\left(d_1(u)\frac{\partial u}{\partial z}\right) + \mu\frac{\partial u}{\partial z} = 0, \quad (23)$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r d_2(v)\frac{\partial v}{\partial r}\right) + \left(\frac{\beta^2}{r^2}+1\right)\frac{\partial}{\partial z}\left(d_2(v)\frac{\partial v}{\partial z}\right) + \mu\frac{\partial v}{\partial z} = 0, \quad (24)$$

$$S_1(r,z) = 0:\ d_{1v}\nabla' u \cdot \nabla' S_1 = (\mu H_v - q)\frac{\partial S_1}{\partial z},\ u = u_v, \quad (25)$$

$$S_2(r,z) = 0: d_{2m}\nabla' v \cdot \nabla' S_2 = d_{1m}\nabla' u \cdot \nabla' S_2 + \mu H_m \frac{\partial S_2}{\partial z}, u = u_m, v = v_m, \quad (26)$$

$$r^2 + z^2 = +\infty:\ v = v_\infty, \quad (27)$$

Here μ is to-be-determined velocity, $\nabla' \equiv \left(\frac{\partial}{\partial r}, \sqrt{\frac{\beta^2}{r^2}+1}\frac{\partial}{\partial z}\right)$.

The further reduction of BVP (23)–(27) with $\beta = 0$.
The variables $z = x_3 - \mu t,\ r = \sqrt{x_1^2 + x_2^2}$ admit clear physical meaning: the first one makes the transition to a moving coordinate system (in the direction of the variable x_3) with the origin at the evaporation surface, the second one presents the radial symmetry of the process with respect to the variables x_1 and x_2. Obviously, such situation takes place if the surface bounded by a circle of the radius R is exposed by the flux $\mathbf{Q}(t) = \mathbf{q} \equiv (0, 0, q)$.
Using ad hoc ansatz [Ivantsov GP, 1947]

$$u = u(\omega),\ v = v(\omega), S_k = S_k(\omega),\ \omega = z + \sqrt{z^2 + r^2},\ k = 1, 2,$$

we arrive at BVP for ODEs

$$\frac{d}{d\omega}\left(\omega d_1(u)\frac{du}{d\omega}\right) + \mu\frac{\omega}{2}\frac{du}{d\omega} = 0, \quad 0 < \omega_1 < \omega < \omega_2,$$

$$\frac{d}{d\omega}\left(\omega d_2(v)\frac{dv}{d\omega}\right) + \mu\frac{\omega}{2}\frac{dv}{dz} = 0, \quad \omega > \omega_2,$$

$$\omega = \omega_1 :\ 2d_{1v}\frac{du}{d\omega} = \mu H_v - q,\ u = u_v,$$

$$\omega = \omega_2 :\ 2d_{2m}\frac{dv}{d\omega} = 2d_{1m}\frac{du}{d\omega} + \mu H_m,\ u = u_m,\ v = v_m,$$

$$\omega = +\infty :\ v = v_\infty,$$

where $\omega_k,\ k = 1, 2$ and μ are to-be-determined constants.

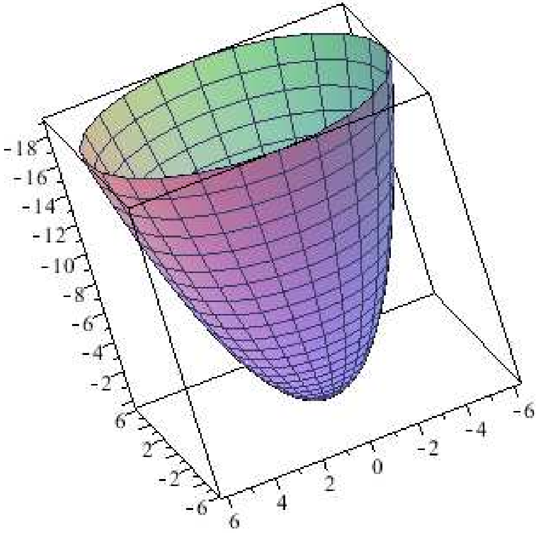


Figure: Paraboloid $\frac{x_1^2+x_2^2}{\omega_k^2}+\frac{2(x_3-\mu t)}{\omega_k}-1=0,\ k=1,2$

Exact solution of (18)–(22) with $d_1(u) = a_1$, $d_2(v) = a_2$, and $\mathbf{Q}(t) = \mathbf{q}$ [Lyubov B & Sobol' E 1983]

$$u = \frac{u_v - u_m}{\Phi_1(R) - \Phi_1(\omega_2)} \Phi_1 \left(\sqrt{x_1^2 + x_2^2 + (x_3 - \mu t)^2} + x_3 - \mu t \right) + u_{mv},$$
$$v = \frac{v_m - v_\infty}{\Phi_2(\omega_2)} \Phi_2 \left(\sqrt{x_1^2 + x_2^2 + (x_3 - \mu t)^2} + x_3 - \mu t \right) + v_\infty,$$
$$S_k \equiv \frac{x_1^2 + x_2^2}{\omega_k^2} + \frac{2(x_3 - \mu t)}{\omega_k} - 1 = 0,\ \ k = 1, 2;\ \ \omega_1 = R.$$

where $\omega_1 = R$, while ω_2 and μ are solutions of the transcendent equations

$$2d_{1v} \frac{u_v - u_m}{\Phi_1(R) - \Phi_1(\omega_2)} R^{-1} e^{-\frac{\mu}{2a_1} R} = \mu H_v - q,$$
$$2d_{2m} \frac{v_m - v_\infty}{\Phi_2(\omega_2)} \omega_2^{-1} e^{-\frac{\mu}{2a_2} \omega_2} = 2d_{1m} \frac{u_v - u_m}{\Phi_1(R) - \Phi_1(\omega_2)} \omega_2^{-1} e^{-\frac{\mu}{2a_1} \omega_2} + \mu H_m,$$

and the notation

$$\Phi_k(\omega) = \int_{\omega}^{+\infty} \omega^{-1} e^{-\frac{\mu}{2a_k}} d\omega,\ \ k = 1, 2; u_{mv} = \frac{u_m \Phi_1(R) - u_v \Phi_1(\omega_2)}{\Phi_1(R) - \Phi_1(\omega_2)}$$

Exact solution of (18)–(22) with $d_1(u) = u^{-1}$, $d_2(v) = 1$ and $\mathbf{Q}(t) = \mathbf{q}$ [R.Ch. & S.Kovalenko 2011]

$$\int\limits_{Ru_v}^{(|x(t,\mu)|+x_3-\mu t)u} \frac{d\nu}{\nu\left(1+e^{-W(e^{\mathcal{A}})+\mathcal{A}}\right)} = \ln\frac{|x(t,\mu)|+x_3-\mu t}{R}$$

$$v = \frac{v_m - v_\infty}{\Phi(\omega_2)}\Phi\left(|x(t,\mu)|+x_3-\mu t\right)+v_\infty,$$

$$S_k \equiv \frac{x_1^2+x_2^2}{\omega_k^2}+\frac{2(x_3-\mu t)}{\omega_k}-1=0,\ k=1,2;\ \omega_1=R,$$

where $W(x)$ is the Lambert function, $|x(t,\mu)| = \sqrt{x_1^2+x_2^2+(x_3-\mu t)^2}$ and

$$\mathcal{A} = -\frac{\mu}{2}\nu+\ln\left((\mu H_v - q)\frac{R}{2}\right)+(\mu H_v - q)\frac{R}{2}+\frac{\mu}{2}Ru_v$$

Here $\omega_1 = R$, while ω_2 and μ are solutions of the transcendent equations

$$\int\limits_{Ru_v}^{\omega_2 u_m} \frac{d\nu}{\nu\left(1+e^{-W(e^{\mathcal{A}})+\mathcal{A}}\right)} = \ln\frac{\omega_2}{R},$$

$$2\frac{v_m - v_\infty}{\Phi(\omega_2)}e^{-\frac{\mu}{2}\omega_2} = 2e^{-W\left(e^{\mathcal{A}(\omega_2)}\right)+\mathcal{A}(\omega_2)}+\mu\omega_2 H_m.$$

Definition of conditional invariance

Let's assume that the basic equation of BVP in question is a multidimensional evolution PDE of kth–order ($k \geq 2$). In this case the relevant BVP may be formulated as follows:

$$u_t = F\left(t, x, u, u_x, \ldots, u_x^{(k)}\right), \ x \in \Omega \subset \mathbb{R}^n, \ t > 0 \quad (28)$$

$$s_a(t,x) = 0: \ B_a\left(t, x, u, u_x, \ldots, u_x^{(k_a)}\right) = 0, \ a = 1, 2, \ldots, p, \ k_a < k \quad (29)$$

where F and B_a are smooth functions in the corresponding domains, Ω is a domain with smooth boundaries and $s_a(t,x)$ are smooth curves. Hereafter the notations

$$u_x^{(j)} = \frac{\partial^j u}{\partial x_{j_1} \ldots \partial x_{j_n}}, \ j = 1, 2, \ldots, k; \ j_1 + \ldots + j_n = j$$

are used and assumed that BVP (28)–(29) has a classical solution.

Consider a BVP for the evolution equation (28) involving conditions (29) and the boundary conditions at infinity:

$$\gamma_c(t,x) = \infty : \ \Gamma_c\left(t, x, u, u_x, \ldots, u_x^{(k_c)}\right) = 0, \ c = 1, 2, \ldots, p_\infty. \qquad (30)$$

Here $k_l < k, k_c < k, n_1$ and p_∞ are the given numbers, the $\gamma_c(t,x)$ are specified functions by which the domain (t,x). We assume that a classical solution still exists for this BVP.
Let us assume that the operator

$$Q = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^a(t,x,u)\frac{\partial}{\partial x_a} + \eta(t,x,u)\frac{\partial}{\partial u} \qquad (31)$$

is a Q-conditional symmetry of PDE (28), i.e.:

$$\left(u_t - F\left(t, x, u, u_x, \ldots, u_x^{(k)}\right)\right)\Big|_M = 0, \qquad (32)$$

where $\underset{k}{Q}$ is the kth prolongation of Q and the manifold
$M = \{u_t - F\left(t, x, u, u_x, \ldots, u_x^{(k)}\right) = 0, \ Q(u) = 0\}$ with
$Q(u) \equiv \xi^0(t,x,u)u_t + \xi^a(t,x,u)u_{x_a} - \eta(t,x,u)$.

Remark . Rigorously speaking, one needs to reduce the manifold M by adding the differential consequences of equation $Q(u)=0$ up to order k, which leads to huge technical problems in the application of the criterion obtained. However, in the case of evolution equations the resulting symmetries will be still the same provided $\xi^0(t,x,u)\neq 0$ in Q.

Let us consider for each $c=1,2,\dots,p_\infty$ the manifold

$$\mathrm{M}=\{\gamma_c(t,x)=\infty,\ \Gamma_c\left(t,x,u,u_x,\dots,u_x^{(k_c)}\right)=0\} \tag{33}$$

in the extended space of variables $t,x,u,u_x,\dots,u_x^{(k_c)}$ and assume that there exists a such smooth bijective transform of the form

$$\tau=f(t,x),\quad y=g(t,x),\quad w=h(t,x,u), \tag{34}$$

where $y=(y_1,\dots,y_n)$, $f(t,x)$ and $h(t,x,u)$ are smooth functions and $g(t,x)$ is a smooth vector function that maps the manifold M into

$$\mathrm{M}^*=\{\gamma_c^*(t,x)=0,\ \Gamma_c^*\left(\tau,y,u,u_y,\dots,u_y^{(k_c^*)}\right)=0\} \tag{35}$$

of the same dimensionality in the extended space.

Definition

BVP (28)–(29) and (30) is Q-conditionally invariant under operator (31) if:

(a) the criterion (32) is satisfied;

(b) $Q(s_a(t,x)) = 0$ when $s_a(t,x) = 0$, $\quad B_a|_{s_a(t,x)=0} = 0, \quad a = 1,\ldots,p$;

(c) $\underset{k_a}{Q}\left(B_a\left(t,x,u,u_x,\ldots,u_x^{(k_a)}\right)\right) = 0$ when $B_a|_{s_a(t,x)=0} = 0, \quad a = 1,\ldots,p$;

(d) there exists a smooth bijective transform (34) mapping M into M* of the same dimensionality;

(e) $Q^*(\gamma_c^*(\tau,y)) = 0$ when $\gamma_c^*(\tau,y) = 0$, $\ c = 1,2,\ldots,p_\infty$;

(f) $\underset{k_c^*}{Q^*}\left(\Gamma_c^*\left(\tau,y,u,u_y,\ldots,u_y^{(k_c^*)}\right)\right) = 0$ when $\Gamma_c^*|_{\gamma_c^*(\tau,y)=0} = 0$, $c = 1,\ldots,r$,

where Γ_c^* and $\gamma_c^*(\tau,y)$ are the functions Γ_c and $\frac{1}{\gamma_c(t,x)}$, respectively, expressed via the new variables. Moreover, the operator Q_*, i.e (31) in the new variables, is defined on M* (may be, excepting a finite number of points).

Example. Consider the reaction-diffusion-convection equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^m u_x\right) + \lambda_1 u^m u_x + \lambda_2 u^{-m}, \qquad (36)$$

where $\lambda_k, \ k = 1, 2$ and $m \neq -1, 0$ are arbitrary constants in the domain $\Omega = \{(t, x) : t > 0, \ x \in (z_1, z_2), \ z_1 < z_2 \in \mathbf{R}\}$
Supplying the Neumann boundary conditions

$$x = z_a : \ u_x = \varphi_a(t), a = 1, 2, \qquad (37)$$

where $\varphi_1(x), \varphi_2(t)$ are the specified smooth functions, one obtains BVP (36)–(37) is a nonlinear BVP, which is the standard object for investigation. Eq. (36) admits the Q-conditional symmetry (R.Ch. & O.Pliukhin, 2007)

$$Q = \frac{\partial}{\partial t} + \lambda_2 u^{-m}\frac{\partial}{\partial u}, \quad \lambda_2 \neq 0 \qquad (38)$$

Now we apply Definition 2 to BVP (36)– (37) in order to obtain the correctly-specified constraints when this problem is conditionally invariant under operator (38). Obviously, the first item is fulfilled by the correct choice of the operator. Item (b) is satisfied automatically because of the operator structure.

A non-trivial result is obtained by application of item (c) to the boundary conditions (37). In fact, calculating the first prolongation (i.e. $k_a = 1$) of operator (38)

$$\underset{1}{Q} = Q - m\lambda_2 u^{-m-1}\frac{\partial}{\partial u_t} - m\lambda_2 u^{-m-1}\frac{\partial}{\partial u_x} \qquad (39)$$

and acting on (37), one obtains two first-order ODEs

$$x = z_a : \ \dot{\varphi}_a(t) + m\lambda_2\varphi_a(t)u^{-m-1} = 0, \ a = 1, 2 \qquad (40)$$

to find the functions $\varphi_a(t), a = 1, 2$. Thus, BVP (36)– (37) is Q-conditionally invariant under (38) if and only if (40) hold.
One may note that (40) is nothing else but the Dirichet conditions and, generally speaking, they may contradict to the Neumann conditions (37). Happily, there is case when the constraints (40) do not produce any boundary conditions: $\varphi_a(t) = 0, \ a = 1, 2,$ i.e., the problem with the zero Neumann conditions (zero flux on boundaries)

$$x = z_a : \ u_x = 0, a = 1, 2, \qquad (41)$$

is invariant under the Q-conditional symmetry (38).

1. A new definition of BVP invariance and the relevant example are presented
2. Algorithm for solving the group classification problem for a BVP class is worked out
3. The group classification problem (a complete description of Lie symmetries) for the class of (1+3)-dim. BVPs modeling processes of melting and evaporation under a powerful flux of energy is derived
4. Reductions of BVPs of the Stefan type to BVPs for ODE and examples of exact solutions are constructed
5. Definition of conditional invariance for BVPs is worked out and the relevant example is presented

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